Publ. Math. Debrecen
59 / 1-2 (2001), 203–219

Cesaro means of Vilenkin–Fourier series

By P. SIMON (Budapest)

Abstract. Special Cesaro means with respect to one- and two-parameter Vilenkin systems will be considered. In both cases we shall investigate a class of weighted maximal functions of these $(C, 1)$ means depending on a parameter $0 < \delta \leq 1$ and prove that the corresponding maximal operators are bounded between some Hardy–Lorentz and Lorentz spaces. As special case we get the boundedness of these maximal operators as maps from Hardy space H^p into L^p for some p. Moreover, the weak type $(1, 1)$ of the maximal operators in question follows in the one dimensional case for all δ. By usual density argument our theorems imply some estimations for growth of the Cesaro means mentioned above if the Vilenkin group has an unbounded structure. We remark that in [10] Wade showed similar results but under stronger assumptions. Thus Wade's theorems follow from our statements. Moreover, a question stated in [10] will be also answered.

1. One- and two-parameter Vilenkin systems will be investigated. It is well-known that there is a sharp distance between the so-called bounded and unbounded cases, i.e. when the corresponding Vilenkin groups have a bounded structure or not, resp. For example in the bounded case the one-parameter Fejér means of the Vilenkin–Fourier series of an integrable function converge a.e. to the function (see Pál–Simon [4]), while in the unbounded case there is a continuous function on the group such that its Fejér means diverge at a prescribed point (see Price [5]). Moreover, Price also showed that under suitable assumption on the group there is a continuous function whose Vilenkin–Fourier series is not $(C, 1)$ summable

---

Mathematics Subject Classification: Primary 42C10, 43A75; Secondary 42B30, 60G42.
Key words and phrases: Hardy spaces, p-atom, quasi-local operator, interpolation, Vilenkin functions, Cesaro means.
This research was supported by the Hungarian Research Funds FKFP/0198/1999 and OTKA T032719.

on a set which is non-denumerable. However, in [3] Gát proved that the Vilenkin–Fejér means of a function belonging to L^p for some $p > 1$ converge to the function a.e. even in the unbounded case.

In his paper [10] Wade obtained estimates for the growth of special Vilenkin–Fejér means for integrable functions under some assumptions on Vilenkin groups. A large class of Vilenkin groups with unbounded structure satisfies these assumptions. The main result of Wade's work is a maximal theorem which states that a weighted maximal function $\sigma_{\alpha,\delta} f$ of special Fejér means belongs to Lorentz space $L^{p,q}$ if f is from Hardy–Lorentz $H^{p,q}$ space (for suitable p and q). Moreover, the maximal operator $\sigma_{\alpha,\delta}$ is bounded from $H^{p,q}$ to $L^{p,q}$.

In this work we shall improve these results. The assumption on the generating sequence of the Vilenkin group will be weakened. Furthermore, we modify the definition of the maximal function mentioned above. To this end we apply weights less than in [10] and prove that the boundedness of the corresponding maximal operator between Lorentz- and Hardy–Lorentz spaces remains true. The proof is based (as in [10]) on the atomic structure of some Hardy spaces. An interpolation theorem of Weisz (see e.g. [11]) plays also an important part.

2. In this section we give a short summary of the basic concepts and notations used also in the present work. (For more details we refer to the books Schipp–Wade–Simon and Pál [6] and Weisz [10].)

If $m = (m_0, m_1, \dots, m_k, \dots)$ is a sequence of natural numbers where $m_k \geq 2$ $(k \in \mathbb{N} := \{0, 1, \dots\})$ then for all $k \in \mathbb{N}$ we shall denote by Z_{m_k} the m_kth discrete cyclic group. Let every subset A of Z_{m_k} be measurable with measure α/m_k where α is the cardinality of A. Assume that Z_{m_k} is represented by $\{0, 1, \dots, m_k - 1\}$. G_m will denote the complete direct product of Z_{m_k}'s, then G_m (the so-called *Vilenkin group*) forms a compact Abelian group with Haar measure 1. Therefore the elements of G_m are sequences of the form $(x_0, x_1, \dots, x_k, \dots)$ with $x_k \in Z_{m_k}$ for every $k \in \mathbb{N}$ Hence the group operation $\dot{+}$ in G_m is given by $x \dot{+} y := (x_0 + y_0 \bmod m_0, \dots, x_k + y_k \bmod m_k, \dots,$ where $x = (x_0, \dots, x_k, \dots)$, $y = (y_0, \dots, y_k, \dots) \in G_m$. The inverse of $\dot{+}$ will be denoted by $\dot{-}$. The topology of G_m is completely determined by the *intervals* $I_n(z)$ $(n \in \mathbb{N}, z \in G_m)$ where

$$I_n := I_n(0) := \{(x_0, x_1, \dots, x_k, \dots) \in G_m : x_j = 0 \ (j = 0, \dots, n-1)\}$$

($0 \neq n \in \mathbb{N},\ I_0 := I_0(0) := G_m$) and $I_n(z) := z\dot{+}I_n$. We shall use the following notation: if $I = I_n$ is an interval for some $n \in \mathbb{N}$ and $r = 0, \dots, n$ then let $I^r := I_{n-r}$. Moreover, if $\mathbb{N} \ni r > n$ then define I^r by $I^r := I_0$.

It is well-known [9] that the characters of G_m (the so-called *Vilenkin system*) form a complete orthonormal system $\widehat{G}_m$ in $L^1(G_m)$. If

$$r_n(x) := \exp \frac{2\pi i x_n}{m_n}$$

($n \in \mathbb{N},\ x = (x_0, x_1, \dots) \in G_m,\ i := \sqrt{-1}$), then r_n's and also their finite products are evidently characters. Let these products be ordered in the following sense. We write each $n \in \mathbb{N}$ uniquely in the form $n = \sum_{k=0}^{\infty} n_k M_k$, where $M_0 := 1$, $M_k := \prod_{j=0}^{k-1}$ $(k \geq 1)$ and $n_k \in Z_{m_k}$ $(k \in \mathbb{N})$. It can easily be seen that the elements of $\widehat{G}_m$ are nothing else but the functions $\Psi_n := \prod_{k=0}^{\infty} r_k^{n_k}$.

The so-called *Dirichlet* and *Fejér kernels* with respect to the Vilenkin system will be denoted by D_n and K_j $(0 < n, j \in \mathbb{N})$:

$$D_n = \sum_{k=0}^{n-1} \Psi_k, \qquad K_j := \frac{1}{j} \sum_{k=1}^{j} D_k.$$

If $f \in L^1(G_m)$ and $0 < j \in \mathbb{N}$ then the j-th *Fejér mean* of f is given by

$$\sigma_j f(x) := \int f(t) K_j(x \dot{-} t)\, dt \qquad (x \in G_m).$$

The following well-known statement for D_{M_n} $(n \in \mathbb{N})$ will be often used in the further investigations (see e.g. Vilenkin [9]):

$$D_{M_n}(x) = \begin{cases} M_n & (x \in I_n) \\ 0 & (x \in G_m \setminus I_n). \end{cases} \tag{1}$$

We define a sequence of σ-algebras as follows:

$$\mathcal{F}_n := \sigma\big(\{I_n(x) : x \in G_m\}\big) \qquad (n \in \mathbb{N}),$$

where $\sigma(\mathcal{A})$ denotes the σ-algebra generated by an arbitrary set system $\mathcal{A}$. A sequence $f = (f_n,\ n \in \mathbb{N})$ of integrable functions is said to be a *martin-*

gal if

(i) f_n is $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$;

(ii) $E_n(f_k) = f_n$ if $n, k \in \mathbb{N}$ and $n < k$,

where E_n $(n \in \mathbb{N})$ denotes the *conditional expectation operator* relative to $\mathcal{F}_n$. A non-decraising sequence $\lambda = (\lambda_n,\ n \in \mathbb{N})$ of functions will be called a *predictable sequence* of f if λ_n is $\mathcal{F}_n$-measurable and $|f_{n+1}| \leq \lambda_n$ for every $n \in \mathbb{N}$.

We remark that the definition of Fejér means can be extended in a simple way to martingales (see e.g. Weisz [11]).

Hardy spaces can be defined in various manners. For example let the *conditional quadratic variation* of a martingal $f = (f_n,\ n \in \mathbb{N})$ defined by

$$s(f) := \left(\sum_{n=0}^{\infty} E_{n-1} |f_n - f_{n-1}|^2 \right)^{1/2}$$

$(f_{-1} := 0,\ E_{-1} := E_0)$. We introduce the martingale *Hardy spaces* for $0 < p \leq \infty$ as follows: denote by $H^p(G_m)$ the space of martingales for which

$$\|f\|_{H^p} := \|s(f)\|_p < \infty.$$

The space $\mathcal{H}^p(G_m)$ $(0 < p \leq \infty)$ consists of all L^p-predictable martingales, i.e. of all martingales $f = (f_n,\ n \in \mathbb{N})$ such that $\|f\|_{\mathcal{H}^p} := \inf \|\sup_n \lambda_n\|_p < \infty$ (where the infimum is taken over all predictable sequences λ of f).

The concept of atoms plays an important part also in the further investigations. To their definition let $0 < p < \infty$ then the function $a \in L^\infty(G_m)$ is called an H^p-*atom* if either a is identically equal to 1 or there exists an interval I for which

$$\begin{cases} \text{i)} & \operatorname{supp} a \subset I \\ \text{ii)} & \|a\|_2 \leq |I|^{1/2-1/p} \\ \text{iii)} & \int a = 0, \end{cases}$$

(where $|I|$ stands for the measure of I). The definition of an $\mathcal{H}^p$-*atom* a is the same only with $\|a\|_\infty \leq |I|^{-1/p}$ instead of ii). We say in both cases that a is *supported* on I.

The above concepts can be easily defined also in the two-dimensional case. To this end let us considered two Vilenkin groups (for the sake of simplicity let both groups generated by the same sequence m). Then G_m^2, the Cartesian product of two G_m's is also a compact Abelian group. Let $n \in \mathbb{N}$ and

$$\mathcal{F}_{n,n} := \sigma(\mathcal{F}_n \times \mathcal{F}_n).$$

The conditional expectation operator relative to $\mathcal{F}_{n,n}$ is denoted by $E_{n,n}$. An integrable sequence $f = (f_{n,n};\ n \in \mathbb{N})$ is said to be a (two-parameter) martingal if

(i) $f_{n,n}$ is $\mathcal{F}_{n,n}$-measurable for all $n \in \mathbb{N}$;

(ii) $E_{n,n}(f_{\tilde{n},\tilde{n}}) = f_{n,n}$ for all $n, \tilde{n} \in \mathbb{N}$ such that $n \le \tilde{n}$.

The Hardy spaces $\mathcal{H}^p(G_m^2)$ $(0<p<\infty)$ of two-parameter L^p-predictable martingales will be defined in analogous way as in the one-dimensional case. Furthermore, the definition of a two-dimensional $\mathcal{H}^p$-atom $(0<p<\infty)$ is formally the same as in the one-dimensional case only we assume that in this case $a \in L^\infty(G_m^2)$ is supported on a *square* $I = I_n(x) \times I_n(y)$ with suitable $x, y \in G_m$, $n \in \mathbb{N}$.

The two-parameter Vilenkin system is defined as the Kronecker products of the functions Ψ_j and Ψ_k, i.e. for $j, k \in \mathbb{N}$ let

$$\Psi_{j,k}(x,y) := \Psi_j(x)\Psi_k(y) \qquad (x, y \in G_m).$$

If $0 < k, l \in \mathbb{N}$ then the (k,l)-th Fejér kernel with respect to the two-parameter Vilenkin system is given by

$$K_{k,l}(x,y) := K_k(x)K_l(y) \qquad (x, y \in G_m).$$

Let $f \in L^1(G_m^2)$ then

$$\sigma_{k,l} f(x,y) := \int\int f(u,v) K_{k,l}(x \dot{-} u, y \dot{-} v)\, du\, dv \qquad (x, y \in G_m)$$

is the (k,l)-th (two-parameter) Fejér mean of f.

If $0 < p,\ q \le \infty$ and $X = G_m$ or $X = G_m^2$ then the *Lorentz spaces* $L^{p,q}(X)$ with norms or quasi-norms $\|\,.\,\|_{p,q}$ are defined in a usual way. The *Hardy–Lorentz spaces* $H^{p,q}(G_m)$, $\mathcal{H}^{p,q}(G_m)$ or $\mathcal{H}^{p,q}(G_m^2)$ consist of all one- or two-parameter martingales f such that $\|f\|_{H^{p,q}} := \|s(f)\|_{p,q} < \infty$

or $\|f\|_{\mathcal{H}^{p,q}} := \inf \| \sup_n \lambda_n \|_{p,q} < \infty$, respectively, where the last infimum is taken over all predictable sequences of f again. We remark that in the special cases $p = q$ we get $L^{p,p}(X) = L^p(X)$ and $H^{p,p}(G_m) = H^p(G_m)$, $\mathcal{H}^{p,p}(G_m) = \mathcal{H}^p(G_m)$, $\mathcal{H}^{p,p}(G_m^2) = \mathcal{H}^p(G_m^2)$. (For details see e.g. Weisz [11].)

3. The sequence m will be called *quasi-bounded* if there exists a constant $C > 0$ independent on $n \in \mathbb{N}$ such that

$$\frac{1}{M_{n+1}} \sum_{j=0}^{n-1} M_{j+1} \log m_j \le C \log m_n \qquad (0 < n \in \mathbb{N}). \tag{2}$$

It is clear that every bounded sequence m is also quasi-bounded. Furthermore, there are unbounded sequences m with this property, for example all monotone increasing sequences are trivially quasi-bounded. However, there is m which does not has the property of the quasi-boundedness. Namely, let $m_{2^n-1} := 2$ $(n \in \mathbb{N})$, while for other indices $k \in \mathbb{N}$ let $m_k := k$. Then

$$\frac{1}{M_{n+1}} \sum_{j=0}^{n-1} M_{j+1} \log m_j \ge \frac{M_n \log m_{n-1}}{M_{n+1}} = \frac{\log m_{n-1}}{m_n}.$$

Hence, the property (2) implies that $\log m_{n-1} \le C m_n \log m_n$ $(n = 1, 2, \dots)$ which is obviously not true for m in question.

In [10] Wade considered m's with property

$$\sum_{j=0}^{n-1} \left(\frac{m_j}{m_{j+1} \cdot \ldots \cdot m_n} \right)^{\delta} \le C_\delta \qquad (0 < n \in \mathbb{N}) \tag{3}$$

(and called m in this case δ-quasi-bounded), where $0 < \delta \le 1$ is fixed and C_δ depends only on δ. (From now on C, C_p $(p > 0)$ will denote a positive constants depending at most on p, not always the same in different occurences.) It can be shown in a simple way that (3) with some $0 < \delta < 1$ implies the same property for $\delta = 1$. Furthermore, our assumption (2) follows evidently from Wade's condition (3). On the other hand there is quasi-bounded m such that (3) is not true. Indeed, let $m_{2^n} := n + 2$ $(n \in \mathbb{N})$ and $m_k := k + 2$ for $2^n \ne k \in \mathbb{N}$. Then (3) fails to hold, namely (3) with 2^n $(n \in \mathbb{N})$ instead of n leads to

$$\sum_{j=0}^{2^n-1} \frac{m_j}{m_{j+1} \cdot \ldots \cdot m_{2^n}} \ge \frac{m_{2^n-1}}{m_{2^n}} = \frac{2^n+1}{n+2} \to \infty \qquad (n \to \infty).$$

However, m is quasi-bounded. Indeed, if $1 < n \in \mathbb{N}$ and $n \neq 2^N$ $(N \in \mathbb{N})$ then the left hand side of (2) can be estimated as follows:

$$\sum_{j=0}^{n-1} \frac{\log m_j}{m_{j+1} \cdot \ldots \cdot m_n} \leq C \sum_{j=0}^{n-1} \frac{\log(j+2)}{2^{n-j}} \leq C \log n \leq C \log m_n.$$

Furthermore, when 2^n $(1 < n \in \mathbb{N})$ stands in (2) instead of n then

$$\sum_{j=0}^{2^n-1} \frac{\log m_j}{m_{j+1} \cdot \ldots \cdot m_{2^n}} = \frac{1}{n+2} \sum_{j=0}^{2^n-1} \frac{\log m_j}{m_{j+1} \cdot \ldots \cdot m_{2^n-1}}$$
$$\leq \sum_{j=0}^{2^n-1} \frac{1}{2^{2^n-j-1}} \leq C \leq C \log m_{2^n}.$$

If $0 < \delta \leq 1$ is given then we shall call the sequence m *strong δ-quasi-bounded* if

$$\sup_{0<n} \frac{1}{M_{n+1}^\delta} \sum_{j=0}^{n-1} M_{j+1}^\delta \alpha_j(\delta) < \infty, \tag{4}$$

where the coefficients $\alpha_j(\delta)$ $(j \in \mathbb{N})$ are defined by

$$\alpha_j(\delta) := \begin{cases} \log m_j & (\delta = 1) \\ m_j^{1-\delta} & (\delta < 1). \end{cases}$$

A strong 1-quasi-bounded sequence will be called *strong quasi-bounded*. Let $0 < \delta \leq \mu \leq 1$ and m be strong δ-quasi-bounded. Then m is evidently also strong μ-quasi-bounded. Especially, m is strong quasi-bounded. Of course, if m is strong quasi-bounded then it is also quasi-bounded. We remark that the strong quasi-bounded sequences play an important part also in the work [2] of Gát.

It is clear that for $\delta = 1/2$ the concept of the strong 1/2-quasi-boundedness is the same as Wade's 1/2-quasi-boundedness. On the other hand, if $1/2 < \delta \leq 1$ then (4) is a weaker assumption on m than (3) while for $0 < \delta < 1/2$ the converse is true.

The following estimation for the L^1-norms of special Fejér-kernels is a crucial part of our investigations.

Lemma. *Assume that m is quasi-bounded. Then*

$$\|K_{M_n}\|_1 \le C \log m_{n-1} \qquad (0 < n \in \mathbb{N}),$$

where the constant $C > 0$ is independent on n.

Proof. It is known (see e.g. Pál–Simon [4]) that

$$K_{M_n}(x) = \frac{1}{2}\Big(1 + \frac{1}{M_n}\Big) D_{M_n}(x) - \sum_{j=0}^{n-1} \frac{M_j}{M_n} \sum_{k=1}^{m_j-1} \frac{k}{m_j} \sum_{l=1}^{m_j-1} r_j^{-k}(le_j) D_{M_n}(x \dot{+} le_j) \tag{5}$$

$(n \in \mathbb{N},\ x \in G_m)$, where $le_j \in G_m$ is determined by $(le_j)_j := l$, $(le_j)_k := 0$ $(j \ne k \in \mathbb{N})$. Therefore we have for all $2 \le n \in \mathbb{N}$, $x \in G_m$ that

$$\begin{aligned}|K_{M_n}(x)| &\le D_{M_n}(x) + \sum_{j=0}^{n-1} \frac{M_j}{M_n} \sum_{l=1}^{m_j-1} \frac{1}{m_j} \Big| \sum_{k=1}^{m_j-1} k e^{-\frac{2\pi i l k}{m_j}} \Big| D_{M_n}(x \dot{+} le_j) \\ &= D_{M_n}(x) + \frac{1}{2} \sum_{j=0}^{n-1} \frac{M_j}{M_n} \sum_{l=1}^{m_j-1} \frac{1}{\sin \frac{\pi l}{m_j}} D_{M_n}(x \dot{+} le_j).\end{aligned}$$

Taking into account (1) the last estimation leads to

$$\begin{aligned}\|K_{M_n}\|_1 &\le C\Big(1 + \sum_{j=0}^{n-1} \frac{M_j}{M_n} \sum_{l=1}^{m_j-1} \frac{m_j}{l}\Big) \le C \sum_{j=0}^{n-1} \frac{M_j}{M_n} m_j \log m_j \\ &= C\big(\log m_{n-1} + \frac{1}{M_n} \sum_{j=0}^{n-2} M_{j+1} \log m_j\big) \le C \log m_{n-1},\end{aligned}$$

which was to be proved. □

Let $0 < \delta \le 1$ and define $\lambda_n(\delta)$ $(0 < n \in \mathbb{N})$ as follows:

$$\lambda_n(\delta) := \begin{cases} \max\{\log m_0, \dots, \log m_{n-1}\} & (\delta = 1) \\ \max\{m_0^{1-\delta}, \dots, m_{n-1}^{1-\delta}\} & (\delta < 1). \end{cases}$$

By means of $\lambda_n(\delta)$'s we consider the weighted maximal operator $\sigma_{(\delta)}$ in the following way:

$$\sigma_{(\delta)} f := \sup_{n\geq 1} \frac{|\sigma_{M_n} f|}{\big(\lambda_n(\delta)\big)^{1/\delta}} \qquad \big(f \in L^1(G_m)\big).$$

Therefore if $\delta = 1$ then

$$\sigma_{(1)} f = \sup_{n\geq 1} \frac{|\sigma_{M_n} f|}{\max\{\log m_0, \dots, \log m_{n-1}\}},$$

while for $0 < \delta < 1$ we have

$$\sigma_{(\delta)} f = \sup_{n\geq 1} \frac{|\sigma_{M_n} f|}{\big(\max\{m_0, \dots, m_{n-1}\}\big)^{1/\delta - 1}}.$$

For the maximal operator $\sigma_{(\delta)}$ we prove

Theorem 1. *Assume that m is quasi-bounded and $0 < \delta \leq 1$. Then $\sigma_{(\delta)}$ is of weak type $(1,1)$ and $\sigma_{(\delta)} : H^p(G_m) \to L^p(G_m)$ is bounded for every $\delta \leq p \leq p_0 := 2$. Moreover, $\sigma_{(\delta)} : L^r(G_m) \to L^r(G_m)$ $(1 < r \leq \infty)$ is bounded. If $\delta < p < p_0$ and $0 < q \leq \infty$, then there is a constant $C_{p,q}$ depending at most on p, q such that*

$$\|\sigma_{(\delta)} f\|_{p,q} \leq C_{p,q} \|f\|_{H^{p,q}} \qquad \big(f \in H^{p,q}(G_m)\big).$$

The same statements remain true for $p_0 = \infty$ if we replace the symbol H by $\mathcal{H}$.

Proof. First we remark that by Lemma $\sigma_{(\delta)}$ is trivially bounded from $L^\infty(G_m)$ to $L^\infty(G_m)$. Therefore the (L^r, L^r)-boundedness $(1<r\leq\infty)$ of $\sigma_{(\delta)}$ will follow from its weak type $(1,1)$ by the well-known theorem of Marcinkiewicz on interpolation (see e.g. Schipp–Wade–Simon and Pál [6]).

Now, let $\delta < 1$ and $\delta \leq p \leq 1$. First we shall prove the H^p-quasi-locality of the maximal operator in question. (For the H^p-quasi-locality of operators see e.g. Weisz [13].) This means that for every H^p-atom a supported on the interval I the uniformly boundedness

$$\int_{G_m \setminus I} \big(\sigma_{(\delta)} a\big)^p \leq C_p$$

holds. Here it can be assumed evidently that $I = I_N$ for some $N \in \mathbb{N}$. Since a is a p-atom we have $\hat{a}(k) = 0$ for every $k = 0, \dots, M_N - 1$. Therefore $\sigma_{M_n} a = 0$ when $\mathbb{N} \ni n \le N$. Thus we assume $n > N$ then for every $x \in G_m \setminus I_N$ and $t \in I_N$ it follows that $x \dot{-} t \notin I_N$. Therefore by (1) $\int_{I_N} a(t) D_{M_n}(x \dot{-} t)\, dt = 0$ thus (1) and (5) imply by Hölder's inequality the next estimation:

$$|\sigma_{M_n} a(x)| \le C\|a\|_2 \sum_{j=0}^{n-1} \frac{M_j}{M_n} \sum_{l=1}^{m_j-1} \frac{1}{\sin \frac{\pi l}{m_j}} \Big(\int_{I_N} D_{M_n}^2 (x \dot{+} l e_j \dot{-} t)\, dt \Big)^{1/2}$$

$$\le C M_N^{1/p-1/2} \sum_{j=0}^{n-1} \frac{M_j}{M_n} \sum_{l=1}^{m_j-1} \frac{1}{\sin \frac{\pi l}{m_j}} \Big(\int_{I_N} D_{M_n}^2 (x \dot{+} l e_j \dot{-} t)\, dt \Big)^{1/2}.$$

If $j \ge N$ then by (1) $D_{M_n}(x \dot{+} l e_j \dot{-} t) = 0$ for every $l = 1, \dots, m_j - 1$ and $t \in I_N$. Moreover, applying again (1) we get for $j = 0, \dots, N-1$ and $t \in I_N$ that $D_{M_n}(x \dot{+} l e_j \dot{-} t)$ is non-zero if and only if for some $\nu = 0, \dots, N-1$ and $k = 1, \dots, m_\nu - 1$ the element x has the next form: $x = (0, \dots, 0, m_\nu - k, 0, \dots, 0, x_N, \dots)$ and $j = \nu$, $l = k$. Let us denoted the set of such x's by $J_{\nu,k}^N$. It is clear that the measure of $J_{\nu,k}^N := |J_{\nu,k}^N| = M_N^{-1}$. Hence, for $x \in J_{\nu,k}^N$ ($\nu = 0, \dots, N-1$; $k = 1, \dots, m_\nu - 1$) we have

$$|\sigma_{M_n} a(x)| \le C \frac{M_N^{1/p-1/2} M_\nu}{\sqrt{M_n}} \frac{1}{\sin \frac{\pi k}{m_\nu}} \le C M_N^{1/p-1} M_\nu \frac{1}{\sin \frac{\pi k}{m_\nu}}.$$

Since $\delta < 1$ the last inequality implies that

$$\sigma_{(\delta)} a \le C M_N^{1/p-1} M_\nu \frac{1}{\sin \frac{\pi k}{m_\nu}} \frac{1}{m_\nu^{1/\delta - 1}}.$$

We recall that $p \le 1$ and so

$$\int_{G_m \setminus I_N} \big(\sigma_{(\delta)} a\big)^p = \sum_{\nu=0}^{N-1} \sum_{k=1}^{m_\nu - 1} \int_{J_{\nu,k}^N} \big(\sigma_{(\delta)} a\big)^p \le C_p \frac{1}{M_N^p} \sum_{\nu=0}^{N-1} \sum_{k=1}^{m_\nu-1} \frac{M_\nu^p}{m_\nu^{p/\delta - p}} \frac{m_\nu^p}{k^p}$$

$$\le C_p \frac{1}{M_N^p} \sum_{\nu=0}^{N-1} \frac{M_{\nu+1}^p}{m_\nu^{p/\delta - p}} m_\nu^{1-p} \le C_p \frac{1}{M_N^p} \sum_{\nu=0}^{N-1} M_{\nu+1}^p m_\nu^{1-p/\delta}.$$

Our assumption $\delta \le p$ yields $1 - p/\delta \le 0$ therefore

$$\int_{G_m \setminus I_N} \left(\sigma_{(\delta)} a\right)^p \le C_p \frac{1}{M_N^p} \sum_{\nu=0}^{N-1} M_{\nu+1}^p \le C_p.$$

This means that $\sigma_{(\delta)}$ is H^p-quasi-local if $\delta \le p \le 1$. Thus by the (L^2, L^2)-boundedness of $\sigma_{(\delta)}$ a theorem on interpolation of WEISZ [11] can be applied to conclude that $\sigma_{(\delta)}$ is bounded from $H^r(G_m)$ to $L^r(G_m)$ $(\delta \le r \le 2)$. Moreover, $\sigma_{(\delta)} : H^{s,q}(G_m) \to L^{s,q}(G_m)$ is bounded for every $\delta < s < 2$ and $0 < q \le \infty$.

We prove for $\sigma_{(\delta)}$ $(0 < \delta \le 1)$ a stronger property than H^1-quasi-locality, namely that $\sigma_{(\delta)}$ is strong 1-quasi-local (see SIMON [8]). To this end let $f \in L^1(G_m)$ such that $\operatorname{supp} f \subset I_N$ for some $N \in \mathbb{N}$ and $\int f = 0$. We need to show that

$$\int_{G_m \setminus I_N} \sigma_{(\delta)} f \le C_\delta \|f\|_1.$$

As above, $n > N$ can be assumed and for $x \in J_{\nu,k}^N$ $(\nu = 0, \dots, N-1$, $k = 1, \dots, m_\nu - 1)$ it follows by (5) and (1) that

$$\begin{aligned} |\sigma_{M_n} f(x)| &\le C \frac{M_\nu}{M_n} \frac{1}{\sin \frac{\pi k}{m_\nu}} \int_{I_N} |f(t)| D_{M_n}(x \dotplus k e_\nu \dot{-} t)\, dt \\ &\le C \|f\|_1 M_\nu \frac{1}{\sin \frac{\pi k}{m_\nu}}. \end{aligned}$$

Thus we can write in the case $\delta < 1$ that

$$\begin{aligned} \int_{G_m \setminus I_N} \sigma_{(\delta)} f &\le C \|f\|_1 \frac{1}{M_N} \sum_{\nu=0}^{N-1} \sum_{k=1}^{m_\nu - 1} \frac{M_\nu}{m_\nu^{1/\delta - 1}} \frac{m_\nu}{k} \\ &\le C \|f\|_1 \frac{1}{M_N} \sum_{\nu=0}^{N-1} \frac{M_{\nu+1}}{m_\nu^{1/\delta - 1}} \log m_\nu. \end{aligned}$$

From the assumption $\delta < 1$ it follows that $1/\delta - 1 > 0$ which implies $\log m_\nu \le C_\delta m_\nu^{1/\delta - 1}$ $(\nu = 0, \dots, N-1)$. Therefore

$$\int_{G_m \setminus I_N} \sigma_{(\delta)} f \le C_\delta \|f\|_1 \frac{1}{M_N} \sum_{\nu=0}^{N-1} M_{\nu+1} \le C_\delta \|f\|_1.$$

If $\delta = 1$ then we need only to write $\log m_\nu$ instead of $m_\nu^{1/\delta - 1}$ and the same conclusion follows, i.e. that $\sigma_{(1)}$ is strong 1-quasi-local. Of course, $\sigma_{(\delta)}$ $(0 < \delta \le 1)$ is also H^1-quasi-local. Therefore Weisz's interpolation theorem cited above guaranties the boundedness of $\sigma_{(\delta)}$ stated in our theorem also for $\delta = 1$. Since the strong 1-quasi-locality with (L^∞, L^∞)-boundedness implies the weak type $(1,1)$ (see e.g. Schipp–Wade–Simon and Pál [6] or Simon [8]), $\sigma_{(1)}$ is also of weak type $(1,1)$.

It is clear that the same proof with a little modification leads to the $\mathcal{H}^p$-quasi-locality of $\sigma_{(\delta)}$ from which we get by interpolation the rest statements and the proof of Theorem 1 is complete. □

We remark that by the proof $\sigma_{(\delta)}$ $(0 < \delta \le 1)$ is $H^p(\mathcal{H}^p)$-quasi-local $(\delta \le p \le 1)$ and strong 1-quasi-local for all m independently on the quasi-boundedness of m.

Since the Vilenkin polynomials, i.e. the linear hull of $\widehat{G}_m$ is dense in $L^1(G_m)$, from the weak type $(1,1)$ part of Theorem 1 we get an estimation for growth of special Cesaro means by usual density argument. Namely, the following corollary is true.

Corollary 1. *Assume that m is quasi-bounded and $0 < \delta \le 1$. Then for all $f \in L^1(G_m)$*

$$\sigma_{M_n} f(x) - f(x) = o\Big(\big(\lambda_n(\delta)\big)^{1/\delta}\Big) \qquad (n \to \infty)$$

for almost every $x \in G_m$.

It is obvious that for bounded m Theorem 1 remains true if we omit the factors $\big(\lambda(\delta)_n\big)^{1/\delta}$ $(n \in \mathbb{N})$ in the definition of the maximal operator. In other words, then all statements of Theorem 1 hold for the unweighted maximal operator $f \to \sup_n |\sigma_{M_n} f|$ instead of $\sigma_{(\delta)}$. (In this case we have $H^p(G_m) = L^p(G_m)$ $(p > 1)$.) Moreover, if m is bounded then $f \to \sigma^* f := \sup_n |\sigma_n f|$ is bounded from $H^1(G_m)$ to $L^1(G_m)$ (see Fujii [1] and Simon [7]). Furthermore, in the bounded case this "full" maximal operator is of weak type $(1,1)$ and for every $1 < r \le \infty$ is bounded from $L^r(G_m)$ to $L^r(G_m)$ (see Pál and Simon [4]). Of course, then we can write in Corollary 1 n and 1 instead of M_n and $\big(\lambda_n(\delta)\big)^{1/\delta}$, respectively. The (H^1, L^1)-boundedness of σ^* was generalized to $(H^{p,q}, L^{p,q})$-boundedness (with some p, q) for bounded m by Weisz in [11].

In [10] Wade considered a maximal operator $\tilde{\sigma}_{(\delta)}$ $(0 < \delta \le 1)$ analogous to $\sigma_{(\delta)}$ but with weights $\big(\max\{m_0, \dots, m_{n-1}\}\big)^{1/\delta}$ instead of $\big(\lambda_n(\delta)\big)^{1/\delta}$:

$$\tilde{\sigma}_{(\delta)} f := \sup_{n\ge 1} \frac{|\sigma_{M_n} f|}{\big(\max\{m_0, \dots, m_{n-1}\}\big)^{1/\delta}} \qquad \big(f \in L^1(G_m)\big).$$

He proved that $\tilde{\sigma}_{(\delta)}$ is bounded from $L^\infty(G_m)$ to $L^\infty(G_m)$ and $\mathcal{H}^p$-quasi-local for every $\delta \le p \le 1$ assumed that m satisfies the assumption (3). Therefore Weisz's interpolation theorem cited above implies the boundedness of $\tilde{\sigma}_{(\delta)} : \mathcal{H}^{p,q}(G_m) \to L^{p,q}(G_m)$ if $\delta < p < \infty$, $0 < q \le \infty$. Unfortunatly, from this cannot be deduced that $\tilde{\sigma}_{(\delta)}$ is of weak type $(1,1)$. Theorem 1 is evidently an improving of Wade's result.

To the formulation of the analogous result in the two-dimensional case let $\alpha > 0$ and define $r \in \mathbb{N}$ by $r - 1 < \alpha \le r$. It is clear that $r \ge 1$. Furthermore, let

$$\mu_{k,j} := \max\{m_0, \dots, m_{s-1}\}, \quad \lambda_{k,j} := \max\{\log m_0, \dots, \log m_{s-1}\},$$

where $s := \max\{k, j\}$ $(0 < j, k \in \mathbb{N})$. Now, we consider the two-parameter maximal operator $\sigma_{\alpha,\delta}$ for every $0 < \delta \le 1$ by

$$\sigma_{\alpha,\delta} f := \sup_{|k-j|\le\alpha} \frac{|\sigma_{M_k,M_j} f|}{\lambda_{k,j}\mu_{k+r,j+r}^{2r/\delta}} \qquad \big(f \in L^1(G_m^2)\big).$$

Then the two-dimensional analogue of Theorem 1 reads as follows.

Theorem 2. *Let us supposed that $0 < \delta \le 1$ and m is strong δ-quasi-bounded. Then for every $\delta \le p \le \infty$ the maximal operator $\sigma_{\alpha,\delta} : \mathcal{H}^p(G_m^2) \to L^p(G_m^2)$ is bounded. Moreover, if $\delta < p < \infty$ and $0 < q \le \infty$ then we have*

$$\|\sigma_{\alpha,\delta} f\|_{p,q} \le C_{p,q} \|f\|_{\mathcal{H}^{p,q}} \qquad \big(f \in \mathcal{H}^{p,q}(G_m^2)\big).$$

Proof. As in the one-dimensional case Lemma implies that $\sigma_{\alpha,\delta} : L^\infty(G_m^2) \to L^\infty(G_m^2)$ is bounded. Then, taking into account again Weisz's theorem on interpolation (see the proof of Theorem 1), we need only to show that $\sigma_{\alpha,\delta}$ is $\mathcal{H}^p$-quasi-local for all $\delta \le p \le 1$. To this end let a be a (two-parameter) $\mathcal{H}^p$-atom with support I. Clearly it can be assumed that $I = I_N \times I_N$ for some $N \in \mathbb{N}$. To the $\mathcal{H}^p$-quasi-locality it is enough to prove that

$$\int_{G_m^2 \setminus I^r} (\sigma_{\alpha,\delta} a)^p \leq C_p. \tag{6}$$

Let $A_1 := (G_m \setminus I_{N-r}) \times I_{N-r}$, $A_2 := I_{N-r} \times (G_m \setminus I_{N-r})$ and $A_3 := (G_m \setminus I_{N-r}) \times (G_m \setminus I_{N-r})$. We shall show that

$$\int_{A_l} (\sigma_{\alpha,\delta} a)^p \leq C_p \qquad (l = 1, 2, 3),$$

from which (6) follows evidently.

First we deal with the case $l = 1$. As in the proof of Theorem 1 we get $\sigma_{M_n,M_s} a = 0$ if both n and s are less or equal than N. Thus we may suppose that either n or s is greater than N. If $|n - s| \leq \alpha$ then $n \geq N - r + 1$ or $s \geq N - r + 1$. Therefore

$$\sigma_{\alpha,\delta} f \leq \sup_{n,s \geq N-r+1} \frac{|\sigma_{M_n,M_s} f|}{\lambda_{n,s} \mu_{n+r,s+r}^{2r/\delta}}.$$

Let $(x, y) \in A_1$ and $N - r + 1 \leq n,\ s \in \mathbb{N}$ then

$$\begin{aligned} |\sigma_{M_n,M_s}(x, y)| &= \Big| \int_{I_N} \int_{I_N} a(u, v) K_{M_n}(x \dot{-} u) K_{M_s}(y \dot{-} v)\, du\, dv \Big| \\ &\leq \|a\|_\infty \|K_{M_s}\|_1 \int_{I_N} |K_{M_n}(x \dot{-} u)|\, du. \end{aligned}$$

Similarly to the proof of Theorem 1 we get

$$\begin{aligned} |\sigma_{M_n,M_s}(x, y)| \leq \|a\|_\infty \|K_{M_s}\|_1 & \sum_{j=0}^{N-r-1} \frac{M_j}{M_n} \sum_{l=1}^{m_j-1} \frac{1}{\sin \frac{\pi l}{m_j}} \\ & \times \int_{I_N} D_{M_n}(x \dot{+} l e_j \dot{-} u)\, du. \end{aligned}$$

Analogously to the one-dimensional case we may suppose that $x \in J_{\nu,l}^{N-r}$

for some $\nu = 0, \dots, N-r-1$ and $l = 1, \dots, m_\nu - 1$ in which case by Lemma

$$\begin{aligned}
|\sigma_{M_n,M_s} a(x,y)| &\le \|a\|_\infty \|K_{M_s}\|_1 \frac{M_\nu}{M_n} \frac{1}{\sin \frac{\pi l}{m_\nu}} \le C M_N^{2/p} \log m_{s-1} \frac{M_\nu}{M_n} \frac{1}{\sin \frac{\pi l}{m_\nu}} \\
&\le C M_N^{2/p} \log m_{s-1} \frac{M_\nu}{M_{N-r+1}} \frac{1}{\sin \frac{\pi l}{m_\nu}} \\
&\le C \Big(\frac{M_N}{M_{N-r}} \Big)^{2/p} \log m_{s-1} \frac{M_\nu}{\sin \frac{\pi l}{m_\nu}} \frac{M_{N-r}^{2/p}}{M_{N-r+1}} \\
&\le C \mu_{n+r,s+r}^{2r/p} \lambda_{n,s} \frac{M_\nu}{\sin \frac{\pi l}{m_\nu}} \frac{M_{N-r}^{2/p}}{M_{N-r+1}}.
\end{aligned}$$

Taking into consideration the definition of $\sigma_{\alpha,\delta}$ we have by $2r/p \le 2r/\delta$ that

$$\sigma_{\alpha,\delta} a(x,y) \le C \frac{M_\nu}{\sin \frac{\pi l}{m_\nu}} \frac{M_{N-r}^{2/p}}{M_{N-r+1}},$$

from which by the strong δ-quasi-boundedness of m

$$\begin{aligned}
\int_{A_1} \big(\sigma_{\alpha,\delta} a\big)^p &= \sum_{\nu=0}^{N-r-1} \sum_{l=1}^{m_\nu - 1} \int_{J_{\nu,l}^{N-r}} \int_{I_{N-r}} \big(\sigma_{\alpha,\delta} a\big)^p \\
&\le C_p \frac{1}{M_{N-r}^2} \sum_{\nu=0}^{N-r-1} \sum_{l=1}^{m_\nu - 1} \frac{M_{\nu+1}^p}{l^p} \frac{M_{N-r}^2}{M_{N-r+1}^p} \\
&\le C_p \begin{cases} \dfrac{1}{M_{N-r+1}} \sum\limits_{\nu=0}^{N-r-1} M_{\nu+1} \log m_\nu & (p=1) \\ \dfrac{1}{M_{N-r+1}^p} \sum\limits_{\nu=0}^{N-r-1} M_{\nu+1}^p m_\nu^{1-p} & (p<1) \end{cases} \\
&\le C_p
\end{aligned}$$

follows. (We recall that the strong δ-quasi-boundedness implies the strong p-quasi-boundedness.)

The proof of the estimation $\int_{A_2} \big(\sigma_{\alpha,\delta} a\big)^p \le C_p$ is the same as above for A_1 instead of A_2. Finally, for $(x,y) \in A_3$ and $n, s \in \mathbb{N}$ we get

$$|\sigma_{M_n,M_s} a(x,y)| \le C \|a\|_\infty \int_{I_N} |K_{M_n}(x \dot{-} u)| \, du \int_{I_N} |K_{M_s}(y \dot{-} v)| \, dv.$$

Thus applying the same method as above it follows that

$$\int_{A_3} \left(\sigma_{\alpha,\delta} a\right)^p \le C_p \left(\sum_{\nu=0}^{N-r-1} \sum_{l=1}^{m_\nu - 1} \frac{M_{\nu+1}^p}{l^p} \frac{1}{M_{N-r+1}^p} \right)^2 \le C_p,$$

which completes the proof of Theorem 2. □

We do not know whether $\sigma_{\alpha,\delta}$ is of weak type $(1,1)$ or not. In Wade [10] an analogous statement to Theorem 2 is proved for a maximal operator which is defined as $\sigma_{\alpha,\delta}$ but with $\mu_{k,j}^{1/\delta}$ instead of $\lambda_{k,j}$ and assumed (3). Of course, Wade's theorem follows from Theorem 2.

If m is bounded then – as in Theorem 1 – the weights $\lambda_{k,j}\mu_{k+r,j+r}^{2r/\delta}$ can be omitted in the definition of our maximal operator and all statements of Theorem 2 follow for $f \to \sup_{|n-s|\le\alpha} |\sigma_{M_n,M_s} f|$ (see Weisz [12]). Moreover, Weisz proved analogous theorems also for the maximal operators $f \to \sup_{n,s} |\sigma_{M_n,M_s} f|$, $f \to \sup_{2^{-\alpha}\le n/s\le 2^\alpha} |\sigma_{n,s} f|$, $f \to \sup_{n,s} |\sigma_{n,s} f|$ (for details see Weisz [12]).

References

[1] N. Fujii, A maximal inequality for H^1-functions on a generalized Walsh–Paley group, *Proc. Amer. Math. Soc.* **77** (1979), 111–116.

[2] Gy. Gát, Investigations of certain operators with respect to the Vilenkin system, *Acta Math. Hungar.* **61** (1993), 131–149.

[3] Gy. Gát, Pointwise convergence of the Fejér means of functions on unbounded Vilenkin groups, *Journal of Approx. Theory* **101** (1999), 1–36.

[4] J. Pál and P. Simon, On a generalization of the concept of derivative, *Acta Math. Acad. Sci. Hungar.* **29** (1977), 155–164.

[5] J. Price, Certain groups of orthonormal step functions, *Canadian J. Math.* **9** (1957), 413–425.

[6] F. Schipp, W. R. Wade, P. Simon and J. Pál, Walsh Series: An Introduction to Dyadic Harmonic Analysis, *Akadémiai Kiadó (Budapest), Adam Hilger (Bristol–New York)*, 1990.

[7] P. Simon, Investigations with respect to the Vilenkin system, *Annales Univ. Sci. Budapest Sect. Math.* **28** (1985), 87–101.

[8] P. Simon, Remarks on Walsh–Fourier multipliers, *Publ. Math. Debrecen* **52** (1998), 635–657.

[9] N. Ja. Vilenkin, On a class of complete orthonormal systems, *Izv. Akad. Nauk. SSSR, Ser. Mat.* **11** (1947), 363–400.

[10] W. R. Wade, Growth of Cesaro means of double Vilenkin–Fourier series of unbounded type, *Appl. Numer. Harmon. Anal. Birkhäuser*, *Boston, MA*, 1999, 41–50.

[11] F. Weisz, Martingale Hardy Spaces and their Applications in Fourier Analysis, *Springer*, *Berlin – Heidelberg – New York*, 1994.

[12] F. Weisz, Hardy spaces and Cesaro means of two-dimensional Fourier series, *Bolyai Soc. Math. Studies* **5** (1996), 353–367.

[13] F. Weisz, Cesaro summability of two-dimensional Walsh–Fourier series, *Trans. Amer. Math. Soc.* **348** (1996), 2169–2181.

[14] F. Weisz, Cesaro summability of one- and two-dimensional Walsh–Fourier series, *Analysis Math.* **22** (1996), 229–242.

PÉTER SIMON
DEPARTMENT OF NUMERICAL ANALYSIS
EÖTVÖS L. UNIVERSITY
H–1518 BUDAPEST, P.O. BOX 32
HUNGARY

E-mail: simon@ludens.elte.hu

(Received October 27, 2000; revised February 12, 2001)