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Modules with comparability

By MIGUEL FERRERO (Porto Alegre) and ALVERI SANT'ANA (Porto Alegre)

Dedicated to Prof. A. Bovdi on his 65th birthday

Abstract. Rings with comparability were introduced in [4] as a class of rings which properly contains right distributive rings. The purpose of this paper is to study modules with comparability. We prove here that many results for rings with comparability can be extended to modules. Also there are nice one-to-one correspondences between submodules and right ideals of the base ring which have a good behaviour concerning primeness, semiprimeness and completely primeness.

Introduction

Let R be a ring and let P be a completely prime ideal of R which is contained in the Jacobson radical of R. Then R is said to satisfy comparability with respect to P if for every elements $a, b \in R$, one the following conditions holds: $aR \subseteq bR$, $bR \subseteq aR$ or $(aR)S^{-1} = (bR)S^{-1}$, where $(aR)S^{-1} = \{x \in R : \exists s \in S \text{ with } xs \in aR\}$ and $S = R \setminus P$. Rings with P-comparability were introduced and studied by the authors in [4], and this class of rings is an extension of the class of right distributive rings which contains a completely prime ideal in the Jacobson radical.

The purpose of this paper is to study modules with comparability. We extend here several results obtained in [4] for rings with comparability. In particular, we show that these modules have a submodule which

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is isomorphic to certain factor of an ideal of the base ring. One-to-one correspondences between submodules and right ideals of R are obtained.

In Section 1 we recall some basic definitions and facts that we will use later on. In Section 2 we define the comparability for modules and give some equivalent conditions and examples.

In Section 3 we obtain some general results, mainly concernig with waists. These results, in general, extends results which are known for right distributive rings ([5]–[7]).

The main results of this paper are contained in Section 4. Assume that M is a right R-module with Q-comparability, where Q is a completely prime ideal of R contained in the Jacobson radical which is a waist of R as right ideal and $MQ \neq 0$. Then we prove that for any $x \in M \setminus MQ$, there is a one-to-one correspondence between submodules of MQ and right ideals of R which are contained in Q and contains the annihilator of x. We also show that a submodule L of MQ is a completely prime (resp. prime, semiprime) submodule of M if and only if the correspondent right ideal of R is completely prime (resp. prime, semiprime) for any $x \notin MQ$ (similar result for any $x \notin L$).

Throughout this paper R is always a ring with an identity element and M is a right module over R. The Jacobson radical of R is denoted by $J(R)$ and the set of units of R by $\mathcal{U}(R)$. Also, if L is a submodule of M and $x \in M$ we denote by $(L : x)$ the set of all the elements $a \in R$ with $xa \in L$. The notations $\subset$ and $\supset$ mean strict inclusions. Ideals of R are assumed to be two-sided unless otherwise stated.

1. Pre-requisites

Let R be a ring and M a right R-module. Recall that a submodule L of M is said to be *prime* (*resp. semiprime*) if for every $m \in M$ and $a \in R$ we have that $mRa \subseteq L$ (resp. $maRa \subseteq L$) implies either $m \in L$ or $Ma \subseteq L$ (resp. $ma \in L$). This definition is the natural extension of prime (resp. semiprime) right ideal of R (see [3]).

On the other hand, a submodule L of M is said to be a *completely prime* submodule if for every $m \in M$ and $a \in R$ we have that $ma \in L$ implies either $m \in L$ or $Ma \subseteq L$ [3]. When $M = R_R$ and P is a two-sided ideal of R, then P is completely prime as right submodule if and only if for all $a, b \in R$ we have that $ab \in P$ implies either $a \in P$ or $b \in P$. The definition that we will use here of a completely prime ideal of R is this last one, even for right ideals and also for right multiplicative ideals of R. This will not cause confusion since most of the completely prime ideals we consider here are contained in the Jacobson radical of R and thus are two-sided ideals ([5], Lemma 2.5).

In ([5], Section 2), the authors introduced the definition of right multiplicative ideals and the right associated multiplicative ideal $P_r(I)$, for a right ideal I of R, to study distributive rings (see also [2]). Now we extend this notion for modules. If L is a submodule of M we define the *associated completely prime right multiplicative ideal* of L by

$$P_r(L) = \{a \in R : \exists y \notin L \text{ with } ya \in L\}.$$

It is easy to check that $P_r(L)$ is a completely prime right multiplicative ideal of R (in the sense given above for ideals of R).

Note that if P is a completely prime ideal of R, then $S = R \setminus P$ is a multiplicatively closed subset of R.

If S is a multiplicatively closed subset of R we put

$$LS^{-1} = \{m \in M : \exists s \in S \text{ with } ms \in L\}.$$

Recall that a multiplicatively closed subset S of R is said to be a *right Ore set*, if for every $s \in S$, $a \in R$, there exist $t \in S$ and $b \in R$ such that $at = sb$. We begin with the following.

Lemma 1.1. *Let R be a ring, M a right R-module and assume that $S \subseteq R$ is a right Ore set. Then $(mR)S^{-1}$ is submodule of M, for every $m \in M$. Moreover, if N is a submodule of M, then NS^{-1} is a submodule of M.*

PROOF. Let $x, y \in (mR)S^{-1}$. Then there exist $s, t \in S$ with $xs, yt \in mR$. Also, there exist $u, v \in S$ such that $su = tv$, because S is a right Ore set, and then $(x-y)su = xsu - ytv \in mR$. In a similar way it follows that $xr \in (mR)S^{-1}$, for every $r \in R$. The rest is clear. $\square$

We say that a submodule L of M is a *waist* if for every submodule N of M we have either $L \subseteq N$ or $N \subseteq L$ [1]. Waists in right distributive rings have been studied in [5] and [6] and in rings with comparability in [4].

Note that L is a waist of M if and only if $L \subset xR$, for every $x \in M \setminus L$. Also a waist is always contained in the Jacobson radical of M.

The following lemma is easy to prove.

Lemma 1.2. *Let M be a right R-module, N a submodule of M and $L \supseteq N$ a waist of M. Then we have $N = x(N : x)$, for every $x \notin L$.*

2. The comparability for modules

In [4] we introduced and study rings with comparability. Recall that if P is a completely prime ideal of a ring R contained in the Jacobson radical, then R is said to satisfy P-comparability if for every $a, b \in R$ one of the following conditions holds: $aR \subseteq bR$, $bR \subseteq aR$ or $(aR)S^{-1} = (bR)S^{-1}$. In this section we introduce the comparability for modules.

If R is a ring and P is a completely prime ideal contained in $J(R)$, we say here that P is a (right) *admissible* ideal if $S = R \setminus P$ is a right Ore set. Recall that if R is a ring with P-comparability, then P is an admissible ideal of R ([4], Proposition 1.4).

If M is a right R-module and P is an admissible ideal of R, then $(mR)S^{-1}$ is an R-submodule of M, for all $m \in M$, by Lemma 1.1.

Definition 2.1. Let M be a right R-module and P an admissible ideal of R. We say that M is an R-module with comparability with respect to P (P-comparability, for short), if for every $x, y \in M$ one of the following conditions holds: $xR \subseteq yR$, $yR \subseteq xR$ or $(xR)S^{-1} = (yR)S^{-1}$, where $S = R \setminus P$.

If M is a module with P-comparability, then we also say that M satisfies (or has) P-comparability. The comparability for modules has also several equivalent formulations, as the comparability for rings. The following result extends ([4], Proposition 1.4) and the proof is similar. For this reason will be omitted here.

Proposition 2.2. *Let M be a right R-module, P an admissible ideal of R and $S = R \setminus P$. The following conditions are equivalent:*

(i) *M satisfies P-comparability.*

(ii) *For all $x, y \in M$ we have either $xR \subseteq yR$ or $(yR)S^{-1} \subseteq (xR)S^{-1}$.*

(iii) *For all $x, y \in M$ we have either $xR \subseteq yR$ or $yR \subseteq (xR)S^{-1}$.*

(iv) *For all $x, y \in M$ we have either $xR \subseteq yR$ or $y \in (xR)S^{-1}$.*

(v) *$(xR)S^{-1}$ is an R-submodule and a waist of M, for all $x \in M$.*

Under certain assumption the fact that $S = R \setminus P$ is a right Ore set can be deduced from the comparability condition. We have,

Lemma 2.3. *Let P be a completely prime ideal of R, M an R-module which contains an element whose annihilator is zero, and assume that for all $x, y \in M$ one of the following conditions holds: $xR \subseteq yR$, $yR \subseteq xR$ or $(xR)S^{-1} = (yR)S^{-1}$, where $S = R \setminus P$. Then R has P-comparability. In particular, P is an admissible ideal.*

Proof. Let $x \in M$ be an element whose annihilator is zero and suppose $a, b \in R$. Apply the comparability condition to xa and xb. It is easy to see that either $aR \subseteq bR$ or there exists $s \in S$ such that $bs \in aR$. $\square$

The typical examples of modules in which the comparability holds are distributive modules. However, as next Example 2.5 shows there are modules with comparability which are not distributive.

Proposition 2.4. *Let M be a right distributive module over R and P a completely prime ideal of R contained in $J(R)$. Then M satisfies P-comparability.*

Proof. Let $x, y \in M$. Since M is distributive Theorem 1.6 of [9] implies that $(xR : y) + (yR : x) = R$. Hence there exist $r, s \in R$ such that $r + s = 1$, $yr \in xR$ and $xs \in yR$. If $r \in J(R)$ then $s = 1 - r \in \mathcal{U}(R)$ and so $xR \subseteq yR$. In the other case we have $r \notin P$ and so $y \in (xR)S^{-1}$. $\square$

In [4] we gave examples of rings with comparability. Assume that R is a ring and P is a completely prime ideal contained in the Jacobson radical of R. If R satisfies P-comparability, then the right R-module R_R satisfies P-comparability. Also, any factor R/I has P comparability, where I is a right ideal of R contained in P. For example, we have the following.

Example 2.5 (c.f. [4], Section 2). Assume that T is a right chain ring with maximal ideal M and let D be a domain contained in the skew field $F = T/M$. Consider the canonical mappings $\pi : T \to F$ and $j : D \to F$. We denote by R the pullback of D and T and let P be the set of all the elements $(0, x) \in R$, where $x \in M$. Then P is a completely prime ideal of R contained in $J(R)$. We assume, in addition, that F is a right skew field of fractions of D. Then R has P-comparability ([4], Theorem 2.4). If I is a right ideal of T, then I can be identified with the right ideal of R which consists of all the elements $(0, y) \in R$, with $y \in I$. It is easy to see that the right R-module $M = R/I$ has P-comparability.

3. Some general results

In this section we give some results on modules with P-comparability which extend results which are known for distributive rings ([5]–[7]). We assume that M is a right R-module which has P-comparability, where P is an admissible (completely prime) ideal of R contained in the Jacobson radical, and put $S = R \setminus P$. We begin with the following.

Proposition 3.1. *Let M be a right R-module with P-comparability. Assume that $I \subseteq P$ is a right ideal of R which is a waist. Then xI is a waist of M, for every $x \in M$. In particular, every element of MI is of the form ma, where $m \in M$ and $a \in I$, and MI is also a waist.*

Proof. Let $x, y \in M$ with $y \notin xI$. If $y \in xR$ then we have $y = xr$ for some $r \in R \setminus I$. Thus we have $I \subset rR$ and so $xI \subseteq xrR = yR$. If $x \in (yR)S^{-1}$, then there exists $s \in S$ such that $xs \in yR$. So in this case $xI \subseteq xsR \subseteq yR$. Hence we always have $xI \subset yR$ and it follows that xI is a waist of M.

Now, if $y \in MI$ we can write $y = \sum_{i=1}^{n} m_i a_i$, where $m_i \in M$ and $a_i \in I$, for every $1 \leq i \leq n$. Then by the first part there exists i such that $m_j a_j \in m_i I$, for all $j \neq i$. So for some $a \in I$ we have $y = m_i a$. It follows that $MI = \bigcup_{m \in M} mI$ and thus MI is also a waist. The proof is complete. $\square$

Lemma 3.2. *Let M be a right R-module and P an admissible ideal of R. If M has P-comparability, then M has P'-comparability, for every admissible ideal $P' \subseteq P$.*

Proof. Let P' be an admissible ideal of R contained in P. For $x, y \in M$ with $xR \nsubseteq yR$, we have $yR \subseteq (xR)S^{-1} \subseteq (xR)S'^{-1}$, where $S' = R \setminus P'$. $\square$

Now we give the following

Definition 3.3. Let M be a right R-module. We say that M is a module with comparability if M satisfies P-comparability, for every admissible ideal $P \subseteq J(R)$.

Note that when R is a ring with comparability there exists a largest completely prime ideal $Q \subseteq J(R)$ ([4], Corollary 1.7) and R has Q-comparability. We can prove that in our case this is also true under some additional assumption. First note the following.

Remark 3.4. Assume that M is a right R-module with P-comparability and there exists some element of M whose annihilator is zero. Then P is a waist as right ideal of R. In fact, R has P-comparability, by Lemma 2.3. Hence it is enough to apply Lemma 1.3 of [4].

From Lemma 3.2 and the results in [4] we immediately have the following.

Corollary 3.5. *Let M be a right module over a ring with comparability R. Then M is a module with comparability if and only if M has Q-comparability, where Q is the largest completely prime ideal of R contained in the Jacobson radical.*

In the following results we assume that P is an admissible ideal of R which is a waist as a right ideal. As we pointed out above the assumption holds if R is a ring with comparability.

Proposition 3.6. *Let M be a right R-module with Q-comparability and $P \subseteq Q$ an admissible ideal of R which is a waist as right ideal. If $MP \neq 0$, then $P_r(MP) = P$.*

Proof. First we claim that $P_r(mP) = P$, for every $m \in M$ with $mP \neq 0$. In fact, since $m \notin mP$ the inclusion $P \subseteq P_r(mP)$ is clear. Conversely, if $a \in P_r(mP)$ there exists $y \notin mP$ such that $ya \in mP$. Also either $y = mr$, for some $r \in R$, or $ms = yr$, for some $r \in R$ and $s \notin P$. Note that in the first case $r \notin P$ and so we have again the second case. Hence $mP = msP = yrP \subseteq yP$, because $P = sP$. It follows that there exists $b \in P$ with $ya = yb$. Thus $y(a - b) = 0$ and consequently $a - b \in P$ (if $a - b \notin P$, then $P \subset (a - b)R$ and so $0 = yP \supseteq mP$, a contradiction). The claim follows.

The rest is clear since $0 \neq MP = \bigcup\{mP : m \in M \text{ with } mP \neq 0\}$. $\square$

Corollary 3.7. *Under the same assumption as in Proposition 3.6, if $MP \neq 0$, then MP is a completely prime R-submodule of M and $MP = xP$, for every $x \in M \setminus MP$.*

Proof. Let $x \in M \setminus MP$ and suppose that there exists $c \in R$ with $xc \in MP$. Then by Proposition 3.6 we have $c \in P_r(MP) = P$ and so $Mc \subseteq MP$. Hence MP is a completely prime submodule of M.

Furthermore, by Proposition 3.1 MP is a waist of M and so $MP = x(MP : x)$. Also, since $x \notin MP$ we have $P \subseteq (MP : x) \subseteq P_r(MP) = P$. The result follows. $\square$

Now we consider semiprime R-submodules of M contained in MP.

Proposition 3.8. *Let M be a right R-module with P-comparability, where P is a waist as right ideal of R. If K is a semiprime R-submodule of M such that $K \subseteq MP$, then K is a waist of M. In particular, a prime R-submodule of M contained in MP is a waist.*

Proof. If $K = MP$ is enough to apply Proposition 3.1. Assume that $K \subset MP$ and take $x \in K$ and $y \notin K$.

If $y \in (xR)S^{-1}$, where $S = R \setminus P$, there exists $s \in S$ such that $ys \in xR \subseteq MP$. Then $y \in MP$ since otherwise we would have $s \in P_r(MP) = P$. Hence there exist $m \in M$ and $q \in P$ with $y = mq$. It follows that $mqRq \subseteq yP = ysP \subseteq xP \subseteq K$, and since K is semiprime we obtain $y = mq \in K$, a contradiction. Hence we have $xR \subseteq yR$ and so K is a waist. □

For any right R-module M, we define a submodule $P(M)$ of M by

$$P(M) = \bigcap \{K : K \text{ is a prime submodule of } M\}.$$

It is well-known that $M/P(M)$ is a semiprime R-module.

Corollary 3.9. *Let M be a right R-module with P-comparability such that $MP \neq 0$, where P is a waist as a right ideal of R. Then $P(M)$ is a prime submodule of M which is a waist of M.*

Proof. Since MP is a completely prime submodule of M we have $P(M) \subseteq MP$. By Proposition 3.8 the family of prime submodules of M contained in MP is linearly ordered by inclusion. Then the result easily follows. □

4. Correspondence between submodules and right ideals

In this section we assume that M is a right R-module with comparability and denote by Q the largest completely prime ideal contained in $J(R)$. In addition, we assume that Q is a waist as a right ideals of R and $MQ \neq 0$. We put $S = R \setminus Q$.

Recall that a right multiplicative ideal I of R is said to be a waist if for any right multiplicative ideal K of R we have either $I \subseteq K$ or $K \subseteq I$ ([5], Section 2).

We begin the section with the following.

Proposition 4.1. *Under the above assumption, Q is the largest waist of R and MQ is the largest waist of M. Moreover, for any right multiplicative ideal I which is a waist as a right multiplicative ideal we have $I \subseteq Q$.*

Proof. First note that if I is a right multiplicative ideal of R and $I \nsubseteq Q$ we have $Q \subset I$. In fact, if there exists $a \in I \setminus Q$, then $Q \subset aR \subseteq I$.

Assume that I is a right multiplicative ideal which is waist as a right multiplicative ideal and $Q \subset I$. We show that $P_r(I) \subseteq J$, where J is the Jacobson radical of R. In fact, if there exists $a \in P_r(I) \setminus J$, for some $t \notin I$ we have $ta \in I$. Define $(I:t) = \{b \in R : tb \in I\}$. Thus $(I:t)$ is a right ideal of R and $(I:t) \nsubseteq J$. Take $c \notin (I:t)$ and $b \in (I:t)$. We have $tc \notin I$ and $tb \in I$. Since I is a waist it follows that $tb = tcr$, for some $r \in R$. Thus $t(b - cr) = 0$ and so $b - cr \in Q$, because $t \notin Q$. Furthermore, $c \notin Q$ and then $Q \subseteq cR$. Hence $b \in cR + Q \subseteq cR$ and consequently $(I:t)$ is a waist, a contradiction because $(I:t) \nsubseteq J$.

Now we show that $P_r(I)$ is a multiplicative waist. For if $a \in P_r(I)$ and $b \notin P_r(I)$, there exists $x \notin I$ such that $xa \in I$ and $xb \notin I$. It follows using the same arguments as above that $a \in bR + Q \subseteq bR$.

From the above it follows that $P_r(I) \subseteq bP_r(I)$, for every $b \notin P_r(I)$. In fact, if there exists $c \in P_r(I) \setminus bP_r(I)$ for some $b \notin P_r(I)$ we have $c = br$, $r \in R$. Thus $r \in P_r(P_r(I)) = P_r(I)$ and consequently $c \in bP_r(I)$, a contradiction. Therefore $P_r(I) \subseteq \bigcap_{b \notin P_r(I)} bP_r(I) \subseteq \bigcap_{b \notin P_r(I)} bJ$. Assume that for some $a \in \bigcap_{b \notin P_r(I)} bJ$, $a \notin P_r(I)$. Then $a \in aJ$ and hence $a = 0$. Thus $P_r(I) = \bigcap_{b \notin P_r(I)} bJ$ and so $P_r(I)$ is a right ideal of R. Now Lemma 2.5 of [5] implies that $P_r(I)$ is a two-sided ideal. It follows that $P_r(I) = Q$, since $P_r(I)$ is completely prime. Consequently $I \subseteq Q$, a contradiction. The first part follows.

For the second, let L be a waist of M and assume that $MQ \subset L$. Take $x \notin L$. Then we know that $L = x(L:x)$ and by Corollary 3.7 $MQ = xQ$. Also, $(L:x)$ properly contains Q since otherwise $L \subseteq MQ$. Now we show that $(L:x)$ is a waist of R, which is a contradiction by the first part. Let I be a right ideal R. We have that either $I \subseteq Q$ or $Q \subset I$. In the first case $I \subseteq (L:x)$. In the second we consider the submodule xI of M. If $xI \subseteq L$, then $I \subseteq (L:x)$. In the contrary case we have $L \subset xI$ and so for any

$a \in (L : x)$, $xa \in xI$. It follows that there exists $b \in I$ such that $xa = xb$. Consequently $x(a - b) = 0$. Also, if for some a the difference $a - b \notin Q$ we have $Q \subset (a-b)R$. Thus $MQ = xQ \subseteq x(a-b)R = 0$, a contradiction. So we must have that $a - b \in Q \subseteq I$ and therefore $a \in I$. This shows that $(L : x) \subseteq I$ and the proof is complete. $\square$

Corollary 4.2. *Under the above assumption, if L is a waist of M such that $P(M) \subseteq L$, then $P_r(L) \subseteq Q$.*

Proof. By Proposition 4.1 we may assume that $L \subset MQ$ since otherwise $L = MQ$ and Proposition 3.6 gives the result.

Suppose that $P_r(L) \nsubseteq Q$. Then $Q \subset P_r(L)$. Take $a \in P_r(L)$ and $b \notin P_r(L)$. Then there exists $x \notin L$ such that $xa \in L$. Also $xb \notin L$, thus $xa = xbr$, for some $r \in R$ and so $x(a - br) = 0$. If $a - br \notin Q$ then $Q \subset (a - br)R$ and so $xQ = 0 \subseteq P(M)$. Since $x \notin P(M)$ we obtain $MQ \subseteq P(M)$, a contradiction because $P(M) \subseteq L$. Consequently $a - br \in Q$ and since $Q \subseteq bR$ it follows that $a \in bR$. Thus $P_r(L)$ is a right multiplicative waist which is a contradiction by Proposition 4.1. $\square$

We denote by $\operatorname{Ann}(x)$ the R-annihilator of $x \in M$. We have the following.

Lemma 4.3. *Under the above assumption, for every $x \notin MQ$ we have $MQ \simeq Q/\operatorname{Ann}(x)$ as right R-modules.*

Proof. Define $\varphi : Q \longrightarrow MQ$ by $\varphi(q) = xq$, for all $q \in Q$. Then φ is a surjective R-homomorphism, by Corolary 3.7. Also, note that Proposition 3.6 implies $\operatorname{Ann}(x) \subseteq Q$. Hence $MQ \simeq Q/\operatorname{Ann}(x)$ as right R-modules. $\square$

From Lemma 4.3 there is a one-to-one correspondence between submodules of M contained in MQ and right ideals I of R such that $\operatorname{Ann}(x) \subseteq I \subseteq Q$. Now we study this correspondence more closely.

Theorem 4.4. *Under the above assumption, if $x \notin MQ$, then there exists a one-to-one correspondence between waists of M and right ideals of R which contains $\operatorname{Ann}(x)$ and are comparable with any other right ideal of R containing $\operatorname{Ann}(x)$.*

Proof. Let L be a waist of M. Then by Proposition 4.1, $L \subseteq MQ$ and so $x \notin L$. Thus we have $L = x(L : x)$. If I is a right ideal of R

containing $\mathrm{Ann}(x)$, then either $xI \subseteq L$ or $L \subseteq xI$. In the first case we have $I \subseteq (L : x)$. In the second case, if $a \in (L : x)$, then $xa \in L \subseteq xI$ and it follows that there exists $b \in I$ such that $xa = xb$. Consequently $x(a - b) = 0$ and hence $a - b \in \mathrm{Ann}(x) \subseteq I$. This shows that $a \in I$. Therefore $(L : x) \subseteq I$ and so $(L : x)$ is comparable with I.

Conversely, let H be a right ideal of R with $\mathrm{Ann}(x) \subseteq H$ and such that for any right ideal K of R with $\mathrm{Ann}(x) \subseteq K$ we have either $K \subseteq H$ or $H \subseteq K$. Note first that $H \subseteq Q$ since otherwise H would be a waist of R with $H \supset Q$. We show that xH is a waits of M. In fact, take $y \in M \setminus xH$. If $y \notin MQ$ we have $xH \subseteq MQ = yQ \subseteq yR$. So we may assume that $y \in MQ = xQ$. Hence there exists $b \in Q$ such that $y = xb$ and clearly $b \notin H$. It follows that $H \subset bR + \mathrm{Ann}(x)$ and we obtain $xH \subseteq xbR = yR$.

Finally, the correspondence is one-to-one by Lemma 4.3. □

Now we relate prime (resp. completely prime, semiprime) submodules with prime (resp. completely prime, semiprime) ideals under the correspondence of Theorem 4.4. We begin with the following.

Theorem 4.5. *Under the above assumption, for any submodule L of M which is a waist, the following conditions are equivalent:*

(i) *L is a completely prime submodule.*

(ii) *$(L : x)$ is a completely prime ideal of R, for any $x \notin L$.*

(iii) *$(L : x)$ is a completely prime ideal of R, for any $x \notin MQ$.*

Proof. (i) $\Longrightarrow$ (ii) Suppose $x \notin L$, $ab \in (L : x)$ and $a \notin (L : x)$. Then $xa \notin L$ and $xab \in L$. Since L is completely prime we have $Mb \subseteq L$ and so $b \in (L : x)$. Hence $(L : x)$ is a completely prime right ideal. Also note that $(L : x) \subseteq P_r(L)$ and so by Corollary 4.2 it follows that $(L : x) \subseteq Q$. Thus $(L : x)$ is contained in $J(R)$ and so is a two-sided ideal ([5], Lemma 2.5).

(ii) $\Longrightarrow$ (iii) It is clear.

(iii) $\Longrightarrow$ (i) Assume that $x \notin MQ$. Then $L = x(L : x)$ and as above $(L : x) \subseteq Q$. Also, by Proposition 3.6 and Corollary 3.7 we have that $P_r(L) = P_r(x(L : x)) = (L : x)$. Suppose that $z \notin L$ and $za \in L$. We show that $Ma \subseteq L$, which completes the proof. We have $a \in P_r(L) = (L : x)$ and so $xa \in L$. Take any $m \in M$. If $m \notin MQ$ by the same argument as above we have $ma \in L$. Assume that $m \in MQ = xQ$. In this case there exists $q \in Q$ such that $m = xq$ and $ma = xqa \in L$ because $(L : x)$ is a two-sided ideal. The proof is complete. □

We could not answer the question on whether for some $x \notin L$ we can have that $(L : x)$ is completely prime but $(L : y)$ is not completely prime for some $y \notin L$. In this direction we have the following.

Lemma 4.6. *Under the above assumption, for a submodule L which is a waist and $x \in M \setminus L$ we have that $(L : x)$ is a completely prime ideal of R if and only if $(L : x)$ is the largest element of the family $\{(L : y)\}_{y \notin L}$.*

Proof. Suppose that $(L : x)$ is a completely prime ideal of R. If $y \notin L$ and $a \in (L : y)$, then $ya \in L$ and so $a \in P_r(L) = P_r(x(L : x)) = (L : x)$. Conversely, if $(L : x)$ is the largest member in the family $\{(L : y)\}_{y \notin L}$, $a, b \in R$, $ab \in (L : x)$ and $a \notin (L : x)$, then we have $xa \notin L$ and $xab \in L$. Hence $b \in (L : xa) \subseteq (L : x)$. Thus $(L : x)$ is completely prime. □

From Theorem 4.5 and Lemma 4.6 we have the following.

Corollary 4.7. *Under the above assumption, if L is a completely prime submodule of M which is a waist, then the family $\{(L : y)\}_{y \notin L}$ is an unitary family and $P_r(L) = (L : y)$ for every $y \notin L$.*

Proof. The first part follows easily. The second is immediate since $P_r(L) = \bigcup_{m \notin L}(L : m)$. □

Theorem 4.8. *Under the above assumption, for any submodule L of M which is a waist, the following conditions are equivalent:*

(i) *L is a prime submodule.*

(ii) *$(L : x)$ is a prime right ideal of R, for any $x \notin L$.*

(iii) *$(L : x)$ is a prime right ideal of R, for any $x \notin MQ$.*

Proof. (i) $\Longrightarrow$ (ii) Suppose that L is a prime submodule and $x \notin L$. If $a, b \in R$, $a \notin (L : x)$ and $aRb \subseteq (L : x)$, then $xaRb \subseteq L$ and $xa \notin (L : x)$. Thus $Mb \subseteq L$ and it follows that $xb \in L$, i.e., $b \in (L : x)$.

(ii) $\Longrightarrow$ (iii) Immediate.

(iii) $\Longrightarrow$ (i) Assume that $(L : x)$ is prime, for every $x \notin MQ$ and for $m \in M \setminus L$ and $t \in R$ we have $mRt \subseteq L$. We show that $Mt \subseteq L$. By the way of contradiction, suppose that there exists $y \in M \setminus L$ such that $yt \notin L$. We compare m and y. If $y = mr$, for some $r \in R$, we have $yt = mrt \in L$, a contradiction. Thus there exist $s \in S$ and $b \in R$ such that $ms = yb$. Now we consider two cases.

Case 1: $y \notin MQ$. In this case $(L : y)$ is prime and as $ybRt = msRt \subseteq mRt \subseteq L$ we have $bRt \subseteq (L : y)$. Hence $b \in (L : y)$ because $yt \notin L$. It follows that $ms \in L$, so $s \in P_r(L) \subseteq Q$, which is a contradiction.

Case 2: $y \in MQ$. In this case, choose some $x \notin MQ$. Then we have $y = xq$, for some $q \in Q$. Thus $xqbRt = ybRt = msRt \subseteq L$, hence

$qbRt \subseteq (L : x)$ and by assumption we have either $xqb \in L$ or $xt \in L$. But $xqb = yb \notin L$ and consequently the only possibility is $xt \in L$. Now we compare yt and yb.

If $yt = ybr$, for some $r \in R$, then $xqtRqt = ytRqt = ybrRqt = xqbrRqt \subseteq L$. Consequently $xqt = yt \in L$, a contradiction. The remaining possibility says that $ybu = ytc$, for some $u \in S$ and $c \in R$. It follows that $xqbu = ybu = ytc = xqtc$ and so $xqtcRqtc = xqbuRqtc \subseteq xqbRtc \subseteq L$. This gives $qtc \in (L : x)$, i.e., $xqtc \in L$. Therefore we have $msu = ybu = ytc = xqtc \in L$, which is a contradiction again since $su \notin Q$. The proof is complete. $\square$

Theorem 4.9. *Under the above assumption, for any submodule L of M which is a waist, the following conditions are equivalent:*

(i) *L is a semiprime submodule.*

(ii) *$(L : x)$ is a semiprime right ideal of R, for any $x \notin L$.*

(iii) *$(L : x)$ is a semiprime right ideal of R, for any $x \notin MQ$.*

Proof. (i) $\implies$ (ii) Let $x \notin L$ and $a \in R$ such that $aRa \subseteq (L : x)$. Then $xaRa \subseteq L$ and since L is semiprime it follows that $xa \in L$. Thus $a \in (L : x)$.

(ii) $\implies$ (iii) Obvious.

(iii) $\implies$ (i) Suppose that $m \in M \setminus L$, $a \in R$ and $maRa \subseteq L$. Thus $aRa \subseteq (L : m)$ and by (iii) it follows that if $m \notin MQ$, then $a \in (L : m)$, i.e., $ma \in L$. Assume that $m \in MQ$ and take $x \notin MQ$. Hence $m = xq$ for some $q \in Q$. Consequently $xqaRqa \subseteq maRa \subseteq L$ and so $qaRqa \subseteq (L : x)$. The assumption gives $qa \in (L : x)$ and hence $ma = xqa \in L$. The proof is complete. $\square$

We end the paper with the following.

Corollary 4.10. *Let R be a ring with comparability and Q the largest completely prime ideal of R contained in $J(R)$. If M is a right R-module with comparability and L is a semiprime R-submodule of M such that $L \subseteq MQ$, then L is a prime R-submodule of M.*

Proof. By Proposition 3.8 L is a waist. Now applying ([4], Theorem 4.1) we obtain that $(L : x)$ is a prime right ideal, for every $x \notin MQ$. The result follows from Theorem 4.8. $\square$

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MIGUEL FERRERO
INSTITUTO DE MATEMÁTICA
UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL
91509–900 – PORTO ALEGRE
BRAZIL

E-mail: ferrero@mat.ufrgs.br

ALVERI SANT'ANA
INSTITUTO DE MATEMÁTICA
UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL
91509–900 – PORTO ALEGRE
BRAZIL

E-mail: alveri@mat.ufrgs.br

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