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Normal subgroups of the group of units in group rings of torsion groups

By A. A. BOVDI (Debrecen) and C. POLCINO MILIES (São Paulo)

Dedicated to Professor Kálmán Győry on his 60th birthday

Abstract. Let $\mathbb{F}G$ be the group algebra of a finite p-group G over a field of characteristic p, and $V(\mathbb{F}G)$ the subgroup of units of augmentation 1. We show that $V(\mathbb{F}G)$ and all of its normal subgroups are such that all factors of their upper central series, with the possible exception of the centre, are elementary abelian p-groups. We give an extension of these results for torsion groups G with some restriction on the group ring RG.

1. Introduction

Let RG be the group ring of a group G over an integral domain R. We denote by $U(RG)$ the group of units of the group ring RG and by $V(RG)$ the subgroup of its normalized units; i.e., the set of all units of augmentation 1.

Group algebras have been characterized under different group-theoretical assumption on its group of units, such as being nilpotent, solvable or n-Engel. One of the hardest and most important problems for modular group algebras consists in describing the structure of its group of units.

First Coleman and Passman [6] proved that, if $\mathbb{F}_pG$ is a group algebra of a finite nonabelian p-group G over the field $\mathbb{F}_p$ of p elements, then

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$V(\mathbb{F}_pG)$ is not a p-regular group and has exponent at least p^2. Shalev [13] and Kurdics [10] pointed out that $V(\mathbb{F}_pG)$ is not metabelian for $p > 3$ and Shalev and Mann [11] showed that the wreath product $C_p \wr A$, for $|A| = p^2$, is a section of $V(\mathbb{F}_pG)$ under weak hypotheses on G.

Let $\mathbb{F}$ be a field of characteristic p. Note that $V(\mathbb{F}G) = 1 + A(\mathbb{F}G)$ for a finite p-group G, where $A(\mathbb{F}G)$ is the augmentation ideal of $\mathbb{F}G$ and the adjoint group $(A(\mathbb{F}G), \circ)$ is isomorphic to $V = V(\mathbb{F}G)$. Thus V is a circle group and by Du's Theorem [7], the k-th term $\zeta_k(V)$ of the upper central series of V coincides with $1+C_k$, where C_k is both a subring and a Lie ideal of $\mathbb{F}G$ and, at the same time, the k-th term of the upper central series of the associated Lie algebra of the augmentation ideal $A(\mathbb{F}G)$. As a consequence, we can obtain the following assertion: the factors $\zeta_{i+1}(V)/\zeta_i(V)$ of the upper central series, with the possible exception of the centre of V, are elementary abelian p-groups.

We extend this result for all normal subgroups of the nilpotent group $V(\mathbb{F}G)$. Moreover, this is true for all nilpotent normal subgroups in $V(\mathbb{F}G)$ with weaker assumptions about the group G, namely, for any torsion group G with some restrictions on $\mathbb{F}G$. This generalizes the results proved by Bovdi and Khripta [1], [2] for abelian and solvable normal subgroups.

2. Normal subgroups of the group of units

Let m be the order of an element $g \in G$ and assume that $1 - \alpha^m \in U(R)$ for some $\alpha \in R$. Then, it is easy to see that $g - \alpha \in U(RG)$ and $(g - \alpha)^{-1} = (1 - \alpha^m)^{-1} \sum_{i=0}^{m-1} \alpha^{m-1-i} g^i$.

Definition. We say that a torsion group G is R-favourable, if, for each element $g \in G$, $g \neq 1$, there exists $0 \neq \alpha \in R$ such that $g - \alpha \in U(RG)$.

We note that a torsion group G is R-favourable in the following cases:

1. R is an infinite local domain;
2. G is a p-group and R is an integral domain of characteristic p with a nontrivial group of units;
3. R is a field, $R \neq \mathbb{F}_2$ and, if G has 2-torsion, then $|R| \neq 1 + 2^n$.

We will denote by $\Delta(G) = \{h \in G \mid [G : C_G(h)] < \infty\}$ the *FC-centre* of G and by $\Delta^+(G)$ the maximal torsion subgroup of $\Delta(G)$.

Lemma 1. *Let R be a commutative integral domain of characteristic $p \geq 0$ and let G be a torsion R-favourable group. If A is a normal subgroup in $V(RG)$, such that any two elements of A which are conjugate in $V(RG)$ commute, then A has one of the following properties:*

(i) *If either $p = 0$ or $p > 2$ and p does not divide the order of any element of the subgroup $\Delta(G)$, then A is a central subgroup in $V(RG)$.*

(ii) *If $p = 2$ then A^2 is a central subgroup in $V(RG)$.*

(iii) *If $p > 2$ and A is a noncentral subgroup in $V(RG)$, then A^p and $A \cap G$ are central subgroups in $V(RG)$, p divides the order of some element of $\Delta(G)$.*

(iv) *If A is a noncentral subgroup, then any conjugate of $h \in A$ is of the form ha where a is an element of order p.*

Proof. Choose elements $g \in G$ and $h \in A$ such that the commutator of g, h in G is not 1, i.e. $(g, h) \neq 1$. Since G is R-favourable, we have that $g - c$ is a unit for some $0 \neq c \in R$ and thus $1 - c \in U(R)$. Then $(1-c)^{-1}(g-c) \in V(RG)$ and there exist $h_0, h_c \in A$ such that

$$ghg^{-1} = h_0, \qquad (g-c)h = h_c(g-c). \tag{1}$$

Clearly, $(h_0 - h_c)g = c(h - h_c)$. Since h, h_0 and h_c are elements of A, which are conjugate, they commute. Thus the elements $(h_0 - h_c)g$ and h also commute, so we obtain the equality $(h_0 - h_c)hg = (h_0 - h_c)gh$, which implies

$$(h_0 - h)(h_0 - h_c) = 0. \tag{2}$$

From (1) and (2) we can see that $0 = (h-h_0)(h_0-h_c)g = c(h-h_0)(h-h_c)$. It follows that $(h - h_0)(h - h_c) = 0$ and by (2) we obtain

$$(ghg^{-1} - h)^2 = 0. \tag{3}$$

Consequently, if $p = 2$, then A^2 is a central subgroup in $V(RG)$.

Suppose that $p \neq 2$. By (3) we have that

$$h^2g - 2hgh + gh^2 = 0. \tag{4}$$

Note that, if g commutes with h, the above relation is trivially true, so this identity holds for $g \in G$.

We consider in RG the inner derivation given by $d(x) = hx - xh$, for all $x \in RG$. By (4) we have that $d^2(x) = 0$ for all $x \in RG$. Since

$$d^2(xy) = d^2(x)y + 2d(x)d(y) + xd^2(y),$$

we get that $d(x)d(y) = 0$. If we take $y = zx$, then

$$d(x)(d(z)x + zd(x)) = d(x)d(zx) = 0$$

and we can see that $d(x)zd(x) = 0$ for all $x, z \in RG$ and $RGd(x)RG$ is a nilpotent ideal. By Passman's theorem ([12], Theorem 4.2.13), if the characteristic of R does not divide the order of any element of $\Delta(G)$, then the ring RG is semiprime and, in this case, we have that $RGd(x)RG = 0$. This implies that, if the characteristic of R is either 0 or $p > 2$, then $d(g) = 0$ for all $g \in G$ and A is central in $V(RG)$, which is a contradiction. Hence, (i) follows.

Now assume that p divides the order of some element of $\Delta(G)$. From (3) we conclude that $(h - g^{-1}hg)^p = 0$. Therefore, $h^p = g^{-1}h^pg$ for any $g \in G$ and $h \in A$, which shows that A^p is a central subgroup of $V(RG)$. If $h \in A \cap G$, then from (3) we obtain that $g^{-1}h^2g - 2g^{-1}hgh + h^2 = 0$. Comparing elements in G we get $gh = hg$.

Let $h \in A$ and $u \in V(RG)$. We claim that the commutator (h, u) is of order p. Indeed, since any two conjugate elements in A commute and A^p is central, we see that $(h^{-1}u^{-1}hu)^p = (h^{-1})^p(u^{-1}hu)^p = (h^{-1})^p u^{-1}h^pu = 1$. Consequently, the conjugate of h is of form ha, where $a \in A$ has order p. □

Lemma 2. *Let $\mathbb{F}_2$ be the field of two elements and let G be a locally finite 2-group. If A is a normal subgroup of $V(\mathbb{F}_2G)$, such that any two elements of A which are conjugates in $V(\mathbb{F}_2G)$ commute, then A^4 is central in $V(\mathbb{F}_2G)$.*

Proof. Suppose that A is noncentral and that $gh \neq hg$ for some $g \in G$ and $h \in A$. Let h_2 be a conjugate of h in $V(\mathbb{F}_2G)$. Since G is a locally finite group, the element $g + h_2 + 1$ is a unit. Set $h_0 = ghg^{-1}$ and $h_1 = (g + h_2 + 1)h(g + h_2 + 1)^{-1}$. Then $h_0, h_1 \in A$ and

(5) $$(h_0 + h_1)g = (h_2 + 1)(h + h_1).$$

According to our assumption, the conjugates h, h_1 and h_2 commute, so it follows that the elements $(h_0 + h_1)g$ and h also commute. Thus

$(h_0+h)(h_0+h_1)=0$ and from (5) we have $(h_0+h)(h_2+1)(h+h_1)=0$. From these we can conclude that

$$\begin{aligned}(h_0+h)^2(h_2+1)&=(h+h_0)(h_0+h_1+h_1+h)(h_2+1)\\&=(h+h_0)(h+h_1)(h_2+1)=0.\end{aligned}$$

If we put $h_2=h$ and $h_2=h_0$, it follows that

$$(h_0+h)^2(h_0+1)=0 \quad \text{and} \quad (h_0+h)^2(h+1)=0.$$

As a consequence we have that $(h_0+h)^3=0$ and $h_0^4=h^4$. Therefore, h^4 is a central element in $V(\mathbb{F}_2G)$. □

Theorem. *Let R be a commutative integral domain of characteristic $p\geq 0$ with a nontrivial group of units and let G be a torsion R-favourable group. Then*

- (i) *if $p=0$ or $p>2$ and p does not divide the order of any element of $\Delta(G)$, then every solvable normal subgroup N of $V(RG)$ is central;*
- (ii) *if $p>0$ and N is a nilpotent normal subgroup of class c, then all factors of the upper central series of N, with the possible exception of its centre, are elementary abelian p-groups and $N^{p^{c-1}}$ is a central subgroup in $V(RG)$.*

Proof. Suppose that a solvable normal subgroup N of $V(RG)$ is noncentral. Then there exists a noncentral solvable normal subgroup H in $V(RG)$ with abelian commutator subgroup H'. By Lemma 1, H' is central in $V(RG)$. Thus any two elements from H which are conjugate in $V(RG)$ commute. So, we can also apply Lemma 1 to H and conclude that it is a central subgroup of $V(RG)$, which is a contradiction.

Let N be a nilpotent normal subgroup of $V(RG)$ and let

$$1\subset\zeta_1(N)\subset\zeta_2(N)\subset\cdots\subset\zeta_c(N)=N$$

be the upper central series of N. Clearly, the subgroups $\zeta_i(N)$ are normal in $V(RG)$ and $\zeta_2(N)$ is a nilpotent group of class two with the property that conjugate elements commute. By Lemma 1, the factor group $\zeta_2(N)/\zeta_1(N)$ has exponent p. It follows from [8, Theorem 4.2.13] that any factor of the upper central series also has exponent p. Therefore, the exponent of $N/\zeta_1(N)$ is a divisor of p^{c-1}. □

Remark 1. If G is not R-favourable then the statement of Theorem does not always hold. For example, let L be a normal subgroup of odd order in a finite group G such that G/L is isomorphic to the dihedral group D_6 of order 6. Then $V(\mathbb{F}_2G)$ has a noncentral solvable normal subgroup which is isomorphic to D_6 (see [1]).

Corollary 1. *Let R be the ring of p-adic integers and let G be a p-group. Then every solvable normal subgroup of $V(RG)$ is central.*

Suppose that $\mathbb{F}G$ is infinite. If $\Delta(V)$ is the FC-centre of $V(\mathbb{F}G)$ and $\Delta^+(V)$ is the subgroup of $\Delta(V)$ consisting of all elements of finite order in $\Delta(V)$ then, by [4], all elements of the commutator subgroup of $\Delta^+(V)$ are unipotent and central in $\Delta(V)$. By Theorem we can conlude the following.

Corollary 2. *Let $\mathbb{F}G$ be an infinite group algebra of characteristic p such that G is $\mathbb{F}$-favourable. Then $\Delta^+(V)^p$ is a central subgroup.*

As it is well-known, by Khripta's theorem [9], the group of units of a modular group algebra KG of characteristic p is nilpotent if and only if G is nilpotent and the commutator subgroup G' is a finite p-group. Thus, we obtain the following:

Corollary 3. *Let G be a nilpotent p-group with finite commutator subgroup and let $\mathbb{F}$ be a field of characteristic p. Then, all normal subgroups of $V(\mathbb{F}G)$ are such that all factors of their upper central series, with the possible exception of the centre, are elementary abelian p-groups.*

Remark 2. Let G be a nilpotent p-group with finite commutator subgroup and let $\mathbb{F}_p$ be the field with p elements. Then by Coleman's theorem [5] the intersection of the commutator subgroup V' of $V(\mathbb{F}_pG)$ with G is the commutator subgroup G' of G. Therefore, G/G' is isomorphic to a subgroup of V/V' and the factors of the lower central series of V do not need to be elementary abelian subgroups.

Definition. Let G be a finite p-group. The *depth* of the conjugacy class C_g is the nonnegative integer d such that $C_G(g) = C_G(g^{p^d})$ and that $C_G(g^{p^d})$ is a proper subgroup of $C_G(g^{p^{d+1}})$.

Let G be a finite p-group and C_g be the conjugacy class of G containing $g \in G$. An element of the form $\widehat{C_g} = \sum_{g\in C_g} g$ of $\mathbb{F}G$ is called a *class sum.* We recall that the set of all class sums is a basis of the centre of $\mathbb{F}G$ over $\mathbb{F}$.

Let p^k be the exponent of the centre of a finite p-group G and let r be the greatest depth of the conjugacy classes of G. According to the result in [3], if the depth of the conjugacy class C_g is d, then $\widehat{C_g}^{p^i} = \widehat{C_{g^{p^i}}}$ for $i = 1, \dots, d$ and $\widehat{C_g}^{p^{d+1}} = 0$. From these we can conclude that the exponent p^t of the centre of $V(\mathbb{F}G)$ is equal to $\max\{p^k, p^{r+1}\}$.

Let $L(A)$ be the associated Lie algebra of the augmentation ideal $A(\mathbb{F}G)$ and denote by C_k the k-th term of the upper central series of $L(A)$. Then

$$C_k = \{a \in A(\mathbb{F}G) \mid [a, x] \in C_{k-1} \text{ for all } x \in A(\mathbb{F}G)\}.$$

Du's theorem [7] described the k-th term $\zeta_k(V)$ of the upper central series of $V(\mathbb{F}_pG)$ as

$$\zeta_k(V) = 1 + C_k.$$

It is well-known that $L(A)$ is a restricted Lie algebra, thus it satisfies the identity

$$[x, y, p] = [x, y, y, \dots, y] = [x, y^p]. \tag{1}$$

Thus if $u = 1 + a \in \zeta_k(V)$ and $v = 1 + b \in V$ then $a \in C_k$ and $b \in A(\mathbb{F}_pG)$. As a consequence (1), we obtain the assertion: $[a^p, b] \in C_{k-p}$ for all $b \in A(\mathbb{F}_pG)$ and $a^p \in C_{k-p+1}$. By Du's theorem $v^p = 1 + a^p \in \zeta_{k-p+1}(V)$ and the factor group $\zeta_k(V)/\zeta_{k-p+1}(V)$ has exponent p.

Applying Theorem we obtain the following.

Corollary 4. *Let G be a finite p-group and $\mathbb{F}$ be a field of characteristic p. Denote by c and p^n the nilpotency class and the exponent of the group $V(\mathbb{F}G)$, respectively. Then $p^n \leq p^{[\frac{c-1}{p}]+t}$, where $[\frac{c-1}{p}]$ is the upper integral part of $\frac{c-1}{p}$.*

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A. A. BOVDI
UNIVERSITY OF DEBRECEN
INSTITUTE OF MATHEMATICS AND INFORMATICS
H–4010 DEBRECEN, P.O. BOX 12
HUNGARY

E-mail: bodibela@math.klte.hu

C. POLCINO MILIES
INSTITUTO DE MATEMÁTICA E ESTATÍSTICA
UNIVERSIDADE DE SÃO PAULO
SÃO PAULO
BRASIL

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