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Gysin sequence and Euler class of spherical Lie algebroids

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Abstract. An n-dimensional Lie algebra $\mathfrak{g}$ will be called a *spherical Lie algebra* if it is cohomologically equivalent to the n-sphere, i.e. $H^i(\mathfrak{g}) = 0$ for $1 \leqslant i \leqslant n-1$ and $H^i(\mathfrak{g}) = \mathbb{R}$ for $i = 0, n$. The 1-dimensional abelian Lie algebra $\mathbb{R}$ and the 3-dimensional Lie algebras $sl(2,\mathbb{R})$ and $sk(3,\mathbb{R})$ are the only such Lie algebras. This work deals with the invariantly oriented transitive Lie algebroids having spherical isotropy Lie algebras. The Lie algebroids of some principal bundles and of some TC-foliations are such algebroids. The aim of this work is to construct and investigate the Gysin sequence and the Euler class of such Lie algebroids by generalizing these notions introduced for $\mathbb{R}$-Lie algebroids in [K5].

1. Introduction

A) Let (E, π, M) be an oriented n-sphere bundle. Then the fibre integral $\oint : \Omega^\star(E) \to \Omega^{\star-n}(M)$ has the following important property [G-H-V]: the space of cohomology of its kernel $H(\ker \oint)$ is isomorphic to the space of cohomology of M,

$$\pi^\# : H(M) \xrightarrow{\cong} H\left(\ker \oint\right). \tag{1.1}$$

Due to this fact, the Gysin sequence and the Euler class of E are easily defined, see, for example, [G-H-V].

On the other hand, in the theory of Lie algebroids we can observe an interesting analogy to the theory of sphere bundles, namely, it turns out that, in some sense, flat connections for Lie algebroids play the same role

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as global cross-sections for sphere bundles. This was observed in [K6] for regular Lie algebroids with the trivial one-dimensional adjoint bundle of Lie algebras admitting a non-singular global cross-section invariant with respect to the adjoint representation (for special cases of Lie algebroids of Poisson manifolds over $\mathbb{R}$-Lie foliations see [K5]). Among transitive Lie algebroids fulfilling these properties we find:

- the Lie algebroids of S^1-principal bundles,
- the Lie algebroid $A(G;H)$ of the TC-foliation of left cosets of a non-closed Lie subgroup H in a Lie group G, such that $\dim \bar{H} - \dim H = 1$,
- the Lie algebroid $A(M;\mathcal{F})$ of a TP-foliation $\mathcal{F}$ on a compact and simply connected manifold with 1-dimensional structural Lie algebras.

Among regular non-transitive ones we have:

- the Lie algebroids of Poisson manifolds over $\mathbb{R}$-Lie foliations.

The fact analogous to (1.1) is their fundamental property which enables us to observe the "flat connections – cross-sections" analogy. What follows appears as common ideas for cross-sections of sphere bundles and flat connections for Lie algebroids: the Gysin sequence, the notion of the Euler class as well as the notion of an index at singularity and the theorem of Euler–Poincaré–Hopf.

It turns out that this analogy – for transitive Lie algebroids – takes place in a wider class of the so-called spherical Lie algebroids in which (a) the isotropy Lie algebras $\mathfrak{g}$ are cohomologically equivalent to the n-sphere S^n, i.e. $H^k(\mathfrak{g}) = 0$ for $1 \le k \le n-1$ and $H^k(\mathfrak{g}) = \mathbb{R}$ for $k = 0, n$, (b) the n-exterior power $\bigwedge^n \boldsymbol{g}$ of the adjoint bundle $\boldsymbol{g}$ possesses a non-singular global cross-section $\varepsilon \in \operatorname{Sec} \bigwedge^n \boldsymbol{g}$ invariant with respect to the adjoint representation. Therefore, except for $\mathbb{R}$-Lie algebroids, this analogy holds for some $sl(2,\mathbb{R})$ and $sk(3,\mathbb{R})$-Lie algebroids as well.

The generalization of the Euler class and the Gysin sequence for spherical Lie algebroids is the aim of this paper. The Lie algebroids of G-principal bundles for 3-dimensional Lie groups G such that $\mathfrak{gl}(G) = sk(3,\mathbb{R})$ (for example, $G = SO(3)$, $O(3)$, $\mathrm{Spin}(3)$) or $\mathfrak{gl}(G) = sl(2,\mathbb{R})$ (in the case $G = SL(2)$) are the first examples of spherical Lie algebroids with 3-dimensional structural Lie algebras. The case $G = O(3)$ gives principal bundles with compact and disconnected fibres, whereas the case $G = SL(2)$ – with non compact fibres. Therefore the classical theory of the Gysin sequence and the Euler class for oriented bundles with fibres cohomologically equivalent to a sphere [G-H-V, Vol. II, s. 5.23] does not apply to these two classes of G-principal bundles. In the next paper [K8] the cohomology theory of flat connections in spherical Lie algebroids is developed. If the dimension of the base manifold is equal to $n+1$, where n is the dimension of the isotropy Lie algebras, the index at an isolated singularity is defined. A version of the Euler–Poincaré–Hopf theorem joining the sum of indexes to the Euler class is given. This generalizes the results from [K5].

The paper is organized as follows. Sections 2 and 3 are devoted to the fibre integral for Lie algebroids (introduced in [K4]) and its fundamental properties. In Section 4 we study the Gysin sequence and the Euler classs of a spherical Lie algebroid A. In Section 5 this class is calculated via the Chern–Weil homomorphism of A. In Section 6 we prove how one can calculate the cohomology algebra of A via $H(M)$ and the Euler class of A. We also show that two Lie algebroids on the same manifold, having isomorphic cohomology algebras, can possess different Euler classes.

B) This work (as regards techniques) is based on our work [K4] where the notion of the fibre integral $\fint_A$ in a regular Lie algebroid $(A, [\![\cdot,\cdot]\!], \gamma)$ with respect to a non-singular cross-section ε of $\bigwedge^n \boldsymbol{g}$ is introduced ($\boldsymbol{g} = \ker\gamma$ is the adjoint bundle of Lie algebras), $n = \operatorname{rank}\boldsymbol{g}$. The pair (A,ε) is called a vertically oriented Lie algebroid and, for the transitive case (considered in this work), the fibre integral $\fint_A : \Omega_A^\star(M) \to \Omega^{\star-n}(M)$ is defined as follows: $\fint_A \Phi = 0$ if $\deg\Phi < n$ and $\gamma^\star(\fint_A \Phi) = (-1)^{nk}\iota_\varepsilon\Phi$ if $\deg\Phi = n+k$, $k \geqslant 0$. We recall that $\Omega_A(M)$ denotes the space of A-differential forms, i.e. the space $\operatorname{Sec}\bigwedge A^\star$ of cross-sections of the bundle $\bigwedge A^\star$. In $\Omega_A(M)$ the exterior derivative d_A works and gives a cohomology algebra $H_A(M)$. We also add that by a *homomorphism of vertically oriented Lie algebroids* $(T,t) : (A,\varepsilon) \to (A',\varepsilon')$ of the same rank n we mean a homorphism $T : A \to A'$ (inducing $t : M \to M'$) of Lie algebroids [K3], such that $(\bigwedge^n T^+)(\varepsilon_x) = \varepsilon'_{tx}$, $x \in M$, where $T^+ : \boldsymbol{g} \to \boldsymbol{g}'$ is the restriction of T to adjoint bundles.

The basic properties of $\fint_A$ are given in the following theorems [K4]:

Theorem 1.1. (a) *If* $(T,t) : (A,\varepsilon) \to (A',\varepsilon')$ *is a homomorphism of vertically oriented Lie algebroids, then* $t^\star \circ \fint_{A'} = \fint_A \circ T^\star$,

(b) $\fint_A \circ\gamma^\star = 0$,

(c) $\fint_A \gamma^\star\psi \wedge \Phi = \psi \wedge \fint_A \Phi$ *for arbitrary forms* $\psi \in \Omega(M)$ *and* $\Phi \in \Omega_A(M)$,

(d) $\fint_A \Phi \wedge \gamma^\star\psi = (-1)^{nk}(\fint_A \Phi) \wedge \psi$ *for* $\psi \in \Omega^k(M)$, $\Phi \in \Omega_A^{\geqslant n}(M)$,

(e) $\fint_A$ *is an epimorphism.*

Theorem 1.2. *The operator* $\oint_A$ *commutes with the exterior derivatives* d_A *and* d_M *if and only if*

(a1) *the isotropy Lie algebras* $\boldsymbol{g}_{|x}$ *are unimodular, and*

(a2) *the cross-section* ε *is invariant with respect to the adjoint representation of* A *on* $\bigwedge^n \boldsymbol{g}$.

We recall [K4] that the invariance of ε is equivalent to the following property: for any open subset $U \subset M$ on which $\varepsilon_{|U} = (h_1 \wedge \ldots \wedge h_n)_{|U}$, $h_i \in \operatorname{Sec} \boldsymbol{g}$, the equality $(\sum_{i=1}^n h_i \wedge \ldots \wedge [\![\xi, h_i]\!] \wedge \ldots \wedge h_n)_{|U} = 0$ holds for each $\xi \in \operatorname{Sec} A$.

Theorem 1.3. *The kernel* $\oint_A$ *is a* d_A*-stable space if and only if the* $\boldsymbol{g}_{|x}$ *are unimodular.*

Theorem 1.4. *For a trivial Lie algebroid* $A = TM \times \mathfrak{g}$ *over a connected manifold* M *a cross-section* ε *of the vector bundle* $\bigwedge^n \boldsymbol{g} = M \times \bigwedge^n \mathfrak{g}$ *is invariant if and only if it is a constant one,* $\varepsilon(x) = \varepsilon_0$ *for a* $\bigwedge^n \operatorname{ad}_{\mathfrak{g}}$*-invariant element* $\varepsilon_0 \in \bigwedge^n \mathfrak{g}$.

The transitive Lie algebroid A fulfilling properties (a1) and (a2) from Theorem 1.2 is called a *transitive unimodular invariantly oriented Lie algebroid* (briefly, *TUIO-Lie algebroid*).

In [K4] and [K7] three sources of such Lie algebroids can be found:

- the Lie algebroids of G-principal bundles for a structure Lie group G not necessarily compact or connected but satisfying $\det(\operatorname{ad}_G a) = +1$, $a \in G$,
- the Lie algebroid of the TC-foliation of left cosets of a nonclosed Lie subgroup H in a Lie group G,
- the Lie algebroid $A(M, \mathcal{F})$ of a TP-foliation on a compact and simply connected manifold.

In [K7] one can find the Poincaré duality theorem for TUIO-Lie algebroids over oriented manifolds establishing an isomorphism $H_A^p(M) \cong H_{A,c}^{n+m-p}(M)^\star$ ($n = \operatorname{rank} \boldsymbol{g}$, $m = \dim M$, $H_{A,c}^{n+m-p}(M)$ denotes the $(n+m-p)$-cohomology space of A-forms with compact support). This theorem implies the duality between $\gamma_c^\# : H_c(M) \to H_{A,c}(M)$ and $\oint_A^\# : H_A(M) \to H(M)$ from which we obtain (as a corollary)

Theorem 1.5. *If A is a TUIO-Lie algebroid over an oriented manifold M, then the following conditions are equivalent:*

(a) $\fint_A^{\# n} : H_A^n(M) \to H^0(M)$ *is an epimorphism,*

(b) $\gamma_c^{\# m} : H_c^m(M) \to H_{A,c}^m(M)$ *is a monomorphism.*

C) Let $\mathfrak{g}$ be any n-dimensional unimodular Lie algebra. For a tensor $0 \neq \varepsilon_0 \in \bigwedge^n \mathfrak{g}$, take the substitution operator $i_{\varepsilon_0} : \bigwedge \mathfrak{g}^\star \to \mathbb{R}$. Then $\ker i_{\varepsilon_0} = \bigoplus_{i=0}^{n-1} \bigwedge^i \mathfrak{g}^\star$. Consider the cochain complex $(\ker i_{\varepsilon_0}, \delta)$

$$0 \to \mathbb{R} \xrightarrow{0} \mathfrak{g}^\star \xrightarrow{\delta^1} \bigwedge^2 \mathfrak{g}^\star \xrightarrow{\delta^2} \cdots \xrightarrow{\delta^{n-2}} \bigwedge^{n-1} \mathfrak{g}^\star \xrightarrow{0} 0.$$

The last arrow is also the Chevalley–Eilenberg differential δ^{n-1} since, in a unimodular Lie algebra $\mathfrak{g}$, we have $\delta^{n-1} = 0$ (this can easily be noticed from the definition or Corollary 3.2 in [K4]). This yields that

$$H^p(\ker i_{\varepsilon_0}) = \begin{cases} H^p(\mathfrak{g}) & \text{if } \ 0 \leqslant p < n, \\ 0 & \text{if } \ p = n. \end{cases} \tag{1.2}$$

Take now a trivial unimodular Lie algebroid $A = TM \times \mathfrak{g}$ equipped with a constant cross-section $\varepsilon \in \operatorname{Sec}(M \times \bigwedge^n \mathfrak{g})$, $\varepsilon(x) = \varepsilon_0$, $0 \neq \varepsilon_0 \in \bigwedge^n \mathfrak{g}$. ε is $\bigwedge^n \operatorname{ad}_A$-invariant [K4, Example 3.1]. Consider the integration operator $\fint_A$ corresponding to ε. The Künneth homomorphism $\kappa : \Omega(M) \times \bigwedge \mathfrak{g}^\star \to \Omega_{TM \times \mathfrak{g}}(M)$, $\psi \otimes \varphi \mapsto \gamma^\star \psi \wedge \pi^\star \varphi$, ($\gamma : TM \times \mathfrak{g} \to TM$ and $\pi : TM \times \mathfrak{g} \to \mathfrak{g}$ are projections) is an isomorphism of graded differential algebras, see [K7, Lemma 6.1].

Lemma 1.1. *The following diagram commutes for a uniquely determined isomorphism κ_1 of graded differential algebras:*

$$\begin{tikzcd}
0 \arrow[r] & \Omega(M) \bigotimes \ker \iota_{\varepsilon_0} \arrow[r] \arrow[d, "\kappa_1", "\cong"'] & \Omega(M) \bigotimes \bigwedge \mathfrak{g}^\star \arrow[r, "\mathrm{id} \otimes \iota_{\varepsilon_0}"] \arrow[d, "\kappa", "\cong"'] & \Omega(M) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker \fint_{TM \times \mathfrak{g}} \arrow[r] & \Omega_{TM \times \mathfrak{g}}(M) \arrow[r, "\fint_{TM \times \mathfrak{g}}"] & \Omega(M) \arrow[r] & 0.
\end{tikzcd}$$

Proof. The right square commutes: (a) if $\deg \Phi < n$, $\Phi \in \Omega(M) \otimes \bigwedge \mathfrak{g}^\star$, then $\fint_{TM \times \mathfrak{g}} \circ \kappa(\Phi) = 0 = \mathrm{id} \otimes i_{\varepsilon_0}(\Phi)$. (b) if $\Phi = \psi \otimes \varphi$, $\psi \in \Omega(M)$,

$\varphi \in \bigwedge \mathfrak{g}^\star$ and $\deg \Phi = n+k$, $k \geqslant 0$, we have, for $\varepsilon_0 = h_1 \wedge \ldots \wedge h_n$ ($h_i \in \mathfrak{g}$),

$$\begin{aligned}\Big(\fint_{TM\times\mathfrak{g}} \circ\kappa\Big)&(\psi\otimes\varphi)(v_1\wedge\ldots\wedge v_k)\\ &=\fint_{TM\times\mathfrak{g}}(\gamma^\star\psi\wedge\pi^\star\varphi)(v_1\wedge\ldots\wedge v_k)\\ &=(-1)^{nk}(\gamma^\star\psi\wedge\pi^\star\varphi)(h_1\wedge\ldots\wedge h_n\wedge v_1\wedge\ldots\wedge v_k)\\ &=(\pi^\star\varphi\wedge\gamma^\star\psi)(h_1\wedge\ldots\wedge h_n\wedge v_1\wedge\ldots\wedge v_k)\\ &\overset{\star}{=}\iota_{\varepsilon_0}\varphi\cdot\psi(v_1\wedge\ldots\wedge v_k)\\ &=(\mathrm{id}\otimes\iota_{\varepsilon_0})(\psi\otimes\varphi)(v_1\wedge\ldots\wedge v_k).\end{aligned}$$

The equality "$\overset{\star}{=}$" holds independently of the degree of φ (if $\deg\varphi < n$, then both sides of "$\overset{\star}{=}$" are 0). Clearly, there exists a uniquely determined homomorphism κ_1 of graded differential algebras for which the above diagram commutes, and κ_1 is an isomorphism. □

Corollary 1.1. *The mapping*

$$\kappa_{1\#} : H(M)\bigotimes H(\ker\iota_{\varepsilon_0}) \xrightarrow{\cong} H\Big(\ker\fint_{TM\times\mathfrak{g}}\Big)$$

induced by κ_1 on cohomology (after its composition with the algebraic Künneth isomorphism) is an isomorphism of graded algebras.

It is easy to see that the mapping

$$\rho : \Omega(M) \longrightarrow \Omega(M)\bigotimes \ker\iota_{\varepsilon_0}, \quad \psi \longmapsto \psi\otimes 1,$$

is a homomorphism of graded differential algebras for which the diagram

$$\begin{array}{ccc}\Omega(M) & \xrightarrow{\ \rho\ } & \Omega(M)\bigotimes\ker\,\iota_{\varepsilon_0}\\ \Big\| & & \cong\Big\downarrow\kappa_1\\ \Omega(M) & \xrightarrow{\ \gamma^\star\ } & \ker(\fint_{TM\times\mathfrak{g}}).\end{array}\tag{1.3}$$

commutes. [The fact that $\operatorname{Im}\gamma^\star\subset\ker\fint_{TM\times\mathfrak{g}}$ follows from Theorem 1.1 (b)].

Finally, consider a transitive Lie algebroid A over a manifold M of dimension m, with the Atiyah sequence $0 \to \boldsymbol{g} \hookrightarrow A \xrightarrow{\gamma} TM \to 0$. Let

$\varepsilon \in \mathrm{Sec} \bigwedge^n \boldsymbol{g}$, $n = \mathrm{rank}\, \boldsymbol{g}$, be a non-singular cross-section. Denote by $\fint_A : \Omega^\star_A(M) \to \Omega^{\star-n}(M)$ the fibre integral with respect to ε. In the present work we examine the short exact sequence of graded spaces

$$0 \longrightarrow \ker \fint_A \overset{i}{\hookrightarrow} \Omega_A(M) \xrightarrow{\fint_A} \Omega(M) \longrightarrow 0. \tag{1.4}$$

The homomorphism i is of degree 0 and $\fint_A$ of degree $-n$. We assume that (A, ε) is a TUIO-Lie algebroid, i.e. that Lie algebras $\boldsymbol{g}_{|x}$ are unimodular and ε is $\bigwedge^n \mathrm{ad}_A$-invariant. Due to Theorems 1.2–3, the sequence (1.4) is a sequence of differential spaces.

2. $H(\ker \fint_A)$ for spherical Lie algebroids

Definition 2.1. An n-dimensional real Lie algebra $\mathfrak{g}$ is called a *spherical Lie algebra* (*s-Lie algebra* for short) if it is cohomologically equivalent to the n-sphere S^n, i.e. if

$$H^k(\mathfrak{g}) = \begin{cases} 0 & \text{for } 1 \leqslant k \leqslant n-1, \\ \mathbb{R} & \text{for } k = 0, n. \end{cases}$$

Every s-Lie algebra is unimodular.

Example 2.1. The following are examples of s-Lie algebras:

- $\mathbb{R}$ – the 1-dimensional abelian Lie algebra,
- $sl(2,\mathbb{R})$ – the Lie algebra of real 2×2-matrices with trace zero,
- $sk(3,\mathbb{R})$ – the Lie algebra of real 3×3 skew symmetric matrices.

The last two Lie algebras are 3-dimensional.

Proposition 2.1. *The above are the only s-Lie algebras.*

Proof. It is sufficient to check that

• *for an arbitrarily taken real Lie algebra* $\mathfrak{g}$ *if* $H^1(\mathfrak{g}) = 0$ then $H^3(\mathfrak{g}) \neq 0$.

The assumption $H^1(\mathfrak{g}) = \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] = 0$ implies that the Levi factor $\mathfrak{l}$ of $\mathfrak{g}$ is nontrivial. Let $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$ be the Levi decomposition of $\mathfrak{g}$. The rest trivially follows from the facts that $H^3(\mathfrak{l}) \neq 0$ and $H^0(\mathfrak{m})^{\mathfrak{g}} = \mathbb{R}$ and the Hochschild–Serre formula [H–S, Theorem 13] (for $a = 3$):

$$H^a(\mathfrak{g}) = \bigoplus^{r} H^r(\mathfrak{l}) \otimes H^{a-r}(\mathfrak{m})^{\mathfrak{g}}. \qquad \square$$

Definition 2.2. A TUIO-Lie algebroid is called a *spherical Lie algebroid* (*s-Lie algebroid* for short) if its isotropy Lie algebras $\boldsymbol{g}_{|x}$ are s-Lie algebras.

Example 2.2. (1) Lie algebroids of S^1-, $SL(2)$-, $SO(3)$-, $O(3)$- or Spin(3)-principal bundles are s-Lie algebroids. *(2)* The Lie algebroid of the TC-foliation of left-cosets of a nonclosed Lie subgroup H in a Lie group G, such that $\dim \bar{H} - \dim H = 1$, is an s-Lie algebroid ([K4, Theorem 3.5]).

Theorem 2.1. *For any s-Lie algebroid (A, ε) over a manifold M,*

$$\gamma^{\#} : H(M) \longrightarrow H\Big(\ker \int_A\Big)$$

is an isomorphism of graded algebras.

Proof. M (paracompact and not necessarily connected) can be covered by a finite family of open (not necessarily connected) subsets $\{U_1, \ldots \ldots, U_k\}$ such that $A_{|U_i}$ is isomorphic to a trivial Lie algebroid. Indeed, each point has a neighbourhood on which A is isomorphic to a trivial Lie algebroid [A–M], [M], next we need to use Th.I from [G-H-V, Vol. I, p. 17].

Our theorem will be proved inductively with respect to k.

Step 1. $k = 1$. In this situation, $A \cong TM \times \mathfrak{g}$. According to (1.3), the theorem follows from the assumption that $\mathfrak{g}$ is an s-Lie algebra: $H(\ker \iota_{\varepsilon_0}) = \mathbb{R}$.

Step 2. Assume the theorem is true for positive numbers $\leqslant k$. We prove this theorem for the number $k+1$. Let $M = U_1 \cup \ldots \cup U_{k+1}$ (the Lie algebroid A over U_i is isomorphic to a trivial one). Put $U = U_1$, $V = \bigcup_{i=2}^{k+1} U_i$. Then $U \cup V = M$, $U \cap V = \bigcup_{i=2}^{k+1} U_1 \cap U_i$. Of course, $A_{|U_1 \cap U_i}$ too is isomorphic to a trivial one. Consider the Mayer–Vietoris sequence for the covering $\{U, V\}$ [K7, Section 3]

$$0 \to \Omega_A(M) \xrightarrow{\alpha} \Omega_{A_{|U}}(U) \bigoplus \Omega_{A_{|V}}(V) \xrightarrow{\beta} \Omega_{A_{|U\cap V}}(U \cap V) \to 0$$

and the following diagram in which $\Omega_1 := \Omega_{A_{|U}}$, $\Omega_2 := \Omega_{A_{|V}}$, $\Omega_{12} := \Omega_{A_{|U\cap V}}$ and the integration operators are determined with respect to the restrictions of ε:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker \fint_A \arrow[r] \arrow[d] & \ker \fint_{A_{|U}} \bigoplus \ker \fint_{A_{|V}} \arrow[r] \arrow[d] & \ker \fint_{A_{|U\cap V}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_A(M) \arrow[r, "\alpha"] \arrow[d, "\fint_A"] & \Omega_1(U) \bigoplus \Omega_2(V) \arrow[r, "\beta"] \arrow[d, "\fint_{A_{|U}} \otimes \fint_{A_{|V}}"] & \Omega_{12}(U\cap V) \arrow[r] \arrow[d, "\fint_{A_{|U\cap V}}"] & 0 \\
0 \arrow[r] & \Omega(M) \arrow[r, "\alpha"] \arrow[d] & \Omega(U) \bigoplus \Omega(V) \arrow[r, "\beta"] \arrow[d] & \Omega(U\cap V) \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The bottom rows and all columns are exact. Therefore, by the nine-lemma, the top row is exact, too. This yields the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Omega(M) \arrow[r] \arrow[d, "\gamma_M^\star"] & \Omega(U) \bigoplus \Omega(V) \arrow[r] \arrow[d, "\gamma_U^\star \oplus \gamma_V^\star"] & \Omega(U \cup V) \arrow[r] \arrow[d, "\gamma_{U\cap V}^\star"] & 0 \\
0 \arrow[r] & \ker \fint_A \arrow[r] & \ker \fint_{A_{|U}} \bigoplus \ker \fint_{A_{|V}} \arrow[r] & \ker \fint_{A_{|U\cap V}} \arrow[r] & 0\,.
\end{tikzcd}$$

By the induction assumptions, $\gamma_U^\#$, $\gamma_V^\#$ and $\gamma_{U\cap V}^\#$ are isomorphisms. Writing the long cohomology sequences and using the five-lemma, we assert that $\gamma_M^\#$ is an isomorphism, too. $\square$

3. Gysin sequence and Euler class of s-Lie algebroids

Let us come back to the sequence (1.4) of graded differential spaces for an arbitrary s-Lie algebroid (A, ε) and consider the corresponding canonical long exact sequence in cohomology

$$\cdots \longrightarrow H_A(M) \xrightarrow{\fint_A{}^\#} H(M) \xrightarrow{\partial} H\Big(\ker \int_A\Big) \xrightarrow{i_\#} H_A(M) \longrightarrow \cdots$$

with the connecting homomorphism ∂ of degree $n+1$. In the following diagram

$$\begin{array}{ccc} H(M) & \xrightarrow{\gamma^{\#}} & H_A(M) \\ \cong \Big\uparrow \gamma^{\#-1} & & \Big\| \\ H(\ker \textstyle\oint_A) & \xrightarrow{i_{\#}} & H_A(M) \\ \Big\uparrow \partial & & \Big\| \\ H(M) & \xleftarrow{\oint_A{}^{\#}} & H_A(M), \end{array}$$

the bottom square commutes. Define the operator $D : H(M) \to H(M)$ as the composition

$$D : H(M) \xrightarrow{\tilde{\omega}} H(M) \xrightarrow{\partial} H(\ker \textstyle\oint_A) \xrightarrow{\gamma^{\#-1}} H(M)$$

where $\tilde{\omega}$ is the involution $\tilde{\omega}(\psi) = (-1)^{\deg \psi + 1}\psi$. D is a homomorphism of degree $n+1$, giving the exactness of the triangle

$$\begin{array}{ccc} H(M) & & \\ & \searrow \gamma^{\#} & \\ D \Big\uparrow & & H_A(M). \\ & \swarrow \oint_A{}^{\#} & \\ H(M) & & \end{array} \tag{3.1}$$

Definition 3.1. D is called the *Gysin homomorphism of the s-Lie algebroid* (A, ε). The long exact sequence

$$\cdots \xrightarrow{D=0} H^k(M) \xrightarrow{\gamma^{\#k}} H_A^k(M) \xrightarrow{\oint_A{}^{\#k}=0} 0 \longrightarrow \cdots \qquad (0 \leqslant k < n)$$

$$\cdots\ 0 \longrightarrow H^n(M) \xrightarrow{\gamma^{\#n}} H_A^n(M) \xrightarrow{\oint_A^n} H^0(M) \xrightarrow{D^0} \cdots$$

$$\cdots H^p(M) \xrightarrow{D^p} H^{p+n+1}(M) \xrightarrow{\gamma^{\#}} H_A^{p+n+1}(M) \xrightarrow{\oint_A^{\#}} H^{p+1}(M) \xrightarrow{D^{p+1}} \cdots$$

$$(p \geqslant 0)$$

corresponding to (3.1) is called the *Gysin sequence* of (A, ε). The cohomology class

$$\chi_A := D^0(1) = \gamma^{\#-1}(\partial(-1)) \in H^{n+1}(M)$$

is called the *Euler class* of (A, ε).

Remark 3.1. The construction of the Gysin homomorphism and the Euler class. Take a closed form $\psi \in \Omega(M)$. There exists a form $\Phi \in \Omega_A(M)$ such that $\fint_A \Phi = \psi$ (see Theorem 1.1 (e)). Then, by Theorem 1.2, $d_A\Phi \in \ker \fint_A$ and $\partial[\psi] = [d_A\Phi]_{\ker \fint_A}$. Therefore

$$D[\psi] = (-1)^{\deg \psi + 1}\gamma^{\#-1}[d_A\Phi]_{\ker \fint_A}.$$

In particular, for $\psi = 1 \in H^0(M)$ and an n-form $\tilde{\Phi}$ such that $\fint_A \tilde{\Phi} = -\psi = -1$, we have $[d_A\tilde{\Phi}]_{\ker \fint_A} = \gamma^{\#}\chi_A$. For any representative Ψ of χ_A, $\chi_A = [\Psi]$, there exists a form $\Phi \in \Omega^n_A(M)$ such that

$$\fint_A \Phi = -1 \quad \text{and} \quad d_A\Phi = \gamma^*\Psi.$$

Indeed, $[d_A\tilde{\Phi}]_{\ker \fint_A} = [\gamma^*\Psi]$, so $d_A\tilde{\Phi} - \gamma^\star\Psi = d_A\Theta$ for a form $\Theta \in \ker \fint_A$. Thus $\gamma^\star\Psi = d_A(\tilde{\Phi} - \Theta)$, and $\Phi := \tilde{\Phi} - \Theta$ is the looked-for form.

Proposition 3.1. *$D(\alpha \wedge \beta) = \alpha \wedge D(\beta)$ for arbitrary classes $\alpha, \beta \in H(M)$.*

Proof. Take a form $\Omega \in \Omega^n_A(M)$ such that $\fint_A \Omega = 1$. Let $\alpha = [\varphi]$, $\beta = [\psi]$. By Theorem 1.1 (c), we have $\fint_A(\gamma^\star(\varphi \wedge \psi)) \wedge \Omega = \varphi \wedge \psi$, so this equality gives

$$\begin{aligned} D(\alpha \wedge \beta) &= (-1)^{\deg \varphi + \deg \psi + 1}\gamma^{\#-1}(\partial[\varphi \wedge \psi]) \\ &= (-1)^{\deg \varphi + \deg \psi + 1}\gamma^{\#-1}[d_A(\gamma^\star(\varphi \wedge \psi) \wedge \Omega)]_{\ker \fint_A} \\ &= (-1)^{\deg \psi + 1}\gamma^{\#-1}\left(\gamma^{\#}[\varphi] \wedge [d_A(\gamma^\star\psi \wedge \Omega)]_{\ker \fint_A}\right) \\ &= [\varphi] \wedge (-1)^{\deg \psi + 1}\gamma^{\#-1}(\partial[\psi]) \\ &= [\varphi] \wedge D[\psi] = \alpha \wedge D\beta. \end{aligned}$$

$\square$

Corollary 3.1. $D(\alpha) = \alpha \wedge \chi_A$.

From this and the Gysin sequence we easily obtain

Proposition 3.2. *For an s-Lie algebroid (A, ε), the following conditions are equivalent:*

(1) $\fint_A^{\#n} : H_A^n(M) \to H^0(M)$ *is an epimorphism,*

(2) $\chi_A = 0$,

(3) $\fint_A^{\#} : H_A(M) \to H(M)$ *is an epimorphism,*

(4) $\gamma^{\#n+1} : H^{n+1}(M) \to H_A^{n+1}(M)$ *is a monomorphism,*

(5) $\gamma^{\#} : H(M) \to H_A(M)$ *is a monomorphism.*

In particular, if $H^{n+1}(M) = 0$, then $\chi_A = 0$.

If additionally M is oriented, then according to Theorem 1.5 the above conditions are equivalent to

(6) $\gamma_c^{\#m}$ *is a monomorphism.*

Corollary 3.2. *If an s-Lie algebroid (A, ε) is flat, then $\chi_A = 0$.*

Proof. Let $\lambda : TM \to A$ be a flat connection. Then $\gamma \circ \lambda = \mathrm{id}$ and λ is a homomorphism of Lie algebroids. This implies the relation $\lambda^{\#} \circ \gamma^{\#} = \mathrm{id}^{\#}$, which yields the monomorphy of $\gamma^{\#}$. The proposition above gives the equality $\chi_A = 0$. □

In particular, the trivial Lie algebroid $TM \times \mathfrak{g}$ has the vanishing Euler class.

From the Gysin sequence we immediately have

Corollary 3.3. *If $\chi_A = 0$, then the following short sequences are exact:*

$$0 \longrightarrow H^{n+p}(M) \xrightarrow{\partial^{\#}} H_A^{n+p}(M) \xrightarrow{\fint_A^{\#}} H_A^p(M) \longrightarrow 0, \quad p \geqslant 0. \tag{3.2}$$

In particular,

$$H_A^{n+p}(M) \cong H^{n+p}(M) \bigoplus H^p(M), \quad p \geqslant 0.$$

Corollary 3.4. *The Euler class χ_A vanishes for any $\mathbb{R}$-s-Lie algebroid A over a noncompact 2-manifold M and for any $sl(2,\mathbb{R})$ or $sk(3,\mathbb{R})$-s-Lie algebroid A over a noncompact 4-manifold M.*

The comparison with principal bundles having a structure Lie group cohomologically equivalent to a sphere is given in the following theorem (we recall that each bundle with the fibre cohomologically equivalent to a sphere possesses the Euler class defined analogously as for sphere bundles [G-H-V, Vol. II, 5.23]).

Theorem 3.1. *Let P be a principal bundle with the structure Lie group G cohomologically equivalent to a sphere, invariantly oriented by $\varepsilon_0 \in \bigwedge^n \mathfrak{g}$ ($\mathfrak{g}$ is the Lie algebra of G, $n = \dim G$), such that $\int_G \Delta_R = 1$ and Δ_R is the right-invariant n-form on G equalling ε_0 at the unit. Then the Euler classes of the bundle P and the invariantly oriented Lie algebroid $A(P)$ are equal to each other:*

$$\chi_{A(P)} = \chi_P.$$

The Gysin homomorphisms are identical and the Gysin sequences are "equivalent":

$$
\begin{tikzcd}
\cdots H(M) \arrow[r, "D"] \arrow[d, equal] & H(M) \arrow[r, "\gamma^{\#}"] \arrow[d, equal] & H_A(M) \arrow[r, "\fint_A^{\#}"] \arrow[d, "\cong"', "\tau_P^{\#}"] & H(M)\cdots \arrow[d, equal] \\
\cdots H(M) \arrow[r, "D"] & H(M) \arrow[r, "\pi^{\#}"] & H(P) \arrow[r, "\fint_P^{\#}"] & H(M)\cdots
\end{tikzcd}
\tag{3.3}
$$

Proof. Let $\chi_{A(P)} = [\Psi]$. Take an n-form $\Phi \in \Omega^n_A(M)$ such that $\fint_A \Phi = -1$ and $d_A\Phi = \gamma^\star\Psi$. Denote by $\tilde{\Phi}$ the right-invariant n-form on P corresponding to Φ, $\tilde{\Phi} = \tau_P(\Phi)$, where $\tau_P : \Omega_{A(P)}(M) \cong \Omega^R(P)$ is the canonical isomorphism (see [K2], [K4, (1.6)]). Then, by the definition of fibre integrals, $\fint_P \tilde{\Phi} = -1$ and $\gamma^\star\Psi = d_A\tau_P^{-1}\tilde{\Phi} = \tau_P^{-1}d\tilde{\Phi}$. Therefore $\pi^\star\Psi = \tau_P\gamma^\star\Psi = d\tilde{\Phi}$, which implies that $\chi_P = [\Psi] = \chi_{A(P)}$ and the commutativity of diagram (3.3) follows. $\square$

Remark 3.2. The last theorem does not refer to the non-compact case $G = SL(2)$.

Theorem 3.2 (Naturality of the Gysin sequence). *A homomorphism $(T,t):(A,\varepsilon)\to(A',\varepsilon')$ of s-Lie algebroids induces a homomorphism of the long exact Gysin sequences*

$$\begin{tikzcd}
\cdots H^k(M') \arrow[r, "\gamma'^{\#}"] \arrow[d, "t^{\#}"] \arrow[dr, phantom, "(1)"] & H^k_{A'}(M') \arrow[r, "\oint^{\#}_{A'}"] \arrow[d, "T^{\#}"] \arrow[dr, phantom, "(2)"] & H^{k-n}(M') \arrow[r, "D_{A'}"] \arrow[d, "t^{\#}"] \arrow[dr, phantom, "(3)"] & H^{k+1}(M')\cdots \arrow[d, "t^{\#}"] \\
\cdots H^k(M) \arrow[r, "\gamma^{\#}"] & H^k_A(M) \arrow[r, "\oint^{\#}_A"] & H^{k-n}(M) \arrow[r, "D_A"] & H^{k+1}(M)\cdots
\end{tikzcd}$$

Proof. The commutativity of (1) follows immediately from the equality $t_\star\circ\gamma=\gamma'\circ T$.

(2) – immediately from the equality $t^\star\circ\oint_{A'}=\oint_A\circ T^\star$ in Theorem 1.1 (a) (passing to cohomology).

(3) From the general fact [G-H-V, Vol. I] we have the commuting diagram

$$\begin{tikzcd}
H(M') \arrow[r, "\partial'"] \arrow[d, "t^{\#}"] & H(\ker\oint'_A) \arrow[d, "\tilde{T}^{\#}"] \\
H(M) \arrow[r, "\partial"] & H(\ker\oint_A)
\end{tikzcd}$$

where $\tilde{T}:\ker\oint_{A'}\to H(\ker\oint_A)$ is the restriction of $T^\star$. Hence it appears that in the diagram below all small squares commute:

$$\begin{tikzcd}
D_{A'}: H(M') \arrow[r, "\tilde{\omega}"] \arrow[d, "t^{\#}"] & H(M') \arrow[r, "\partial'"] \arrow[d, "t^{\#}"] & H(\ker\oint_{A'}) \arrow[r, "(\gamma'^{\#})^{-1}"] \arrow[d, "\tilde{T}^{\#}"] & H(M') \arrow[d, "t^{\#}"] \\
D_A: H(M) \arrow[r, "\tilde{\omega}"] & H(M) \arrow[r, "\partial"] & H(\ker\oint_A) \arrow[r, "(\gamma^{\#})^{-1}"] & H(M),
\end{tikzcd}$$

which easily implies the commutativity of (3). □

As a consequence we obtain

Corollary 3.5 (Naturality of the Euler class). *If $(T,t):(A,\varepsilon)\to(A',\varepsilon')$ is a homomorphism of s-Lie algebroids over manifolds M, M' then $\chi_A=t^{\#}\chi_{A'}$.*

4. Euler class via the Chern–Weil homomorphism

In [K3], the Chern–Weil homomorphism h_A for any regular Lie algebroid A is constructed. We use this homomorphism for an s-Lie algebroid (A, ε) (therefore for a transitive one):

$$h_A : \bigoplus^k \left(\operatorname{Sec} \bigvee^k \boldsymbol{g}^\star \right)_{I^0} \longrightarrow H(M)$$

$$\Gamma \longmapsto \frac{1}{k!}[\langle \Gamma, \underbrace{\Omega_b \vee \ldots \vee \Omega_b}_{k \text{ times}} \rangle]$$

where $(\operatorname{Sec} \bigvee^k \boldsymbol{g}^\star)_{I^0}$ denotes the space of invariant cross-sections of the bundle $\bigvee^k \boldsymbol{g}^\star$ with respect to the adjoint representation $\operatorname{ad}_A^\vee$ of A on $\bigvee^k \boldsymbol{g}^\star$ and $\Omega_b \in \Omega^2(M; \boldsymbol{g})$ is the curvature form of any connection $\lambda : TM \to A$ in the Lie algebroid A, $\Omega_b(X,Y) = -\omega([\![\lambda X, \lambda Y]\!])$, $X, Y \in \mathfrak{X}(M)$ ($\omega : A \to \boldsymbol{g}$ is the connection form of λ). We recall that $\Gamma \in \operatorname{Sec} \bigvee^k \boldsymbol{g}^\star$ is invariant if and only if for any $\xi \in \operatorname{Sec} A$ and $h_1, \ldots, h_k \in \operatorname{Sec} \boldsymbol{g}$,

$$(\gamma \circ \xi)(\langle \Gamma, h_1 \vee \ldots \vee h_k \rangle) = \sum_{i=1}^k \langle \Gamma, h_1 \vee \ldots \vee [\![\xi, h_i]\!] \vee \ldots \vee h_k \rangle.$$

Theorem 4.1. *Let* $0 \to \boldsymbol{g} \hookrightarrow A \xrightarrow{\gamma} TM \to 0$ *be the Atiyah sequence of* A. *The form* $\gamma^\star \langle \Gamma, \Omega_b \vee \ldots \vee \Omega_b \rangle$ *for invariant* $\Gamma \in (\operatorname{Sec} \bigvee^k \boldsymbol{g}^\star)_{I^0}$ *is closed, more precisely,*

$$\gamma^\star \langle \Gamma, \underbrace{\Omega_b \vee \ldots \vee \Omega_b}_{k \text{ times}} \rangle = d_A \Phi$$

for

$$\Phi = k! \sum_{i+j=k-1} \frac{1}{k+j} \left\langle \Gamma, \omega \vee \frac{1}{i!}(d_A \omega)^i \vee \frac{1}{j!} \left(-\frac{1}{2} [\![\omega, \omega]\!] \right)^j \right\rangle. \tag{4.1}$$

[*Remark:* This is a generalization of the standard formula [G-H-V, Vol. II]. In the standard formula concerning principal bundles, the sign "–" does not occur because there is a left Lie algebra of a structure Lie group there, not a right one].

Proof. Each point $x \in M$ has a connected neighbourhood U such that $A_{|U}$ is the Lie algebroid of a trivial principal bundle $\pi : U \times G \to U$

for some connected Lie group G, $A_{|U} = A(U\times G)$. Let $\bar{H}\subset T(U\times G)$ be the connection in $U\times G$ corresponding to the restriction of λ to U, $\lambda_{|U} : TU \to A_{|U}$, and let $\bar{\Omega}_U \in \Omega^2(U,\mathfrak{g})$ be the curvature form of $\bar{H}$. Clearly, $\Gamma_{|U} \in \operatorname{Sec}\bigvee^k \boldsymbol{g}^\star_{|U}$ is $\mathrm{ad}^\vee_{A_{|U}}$-invariant. By the connectedness of $U\times G$, we have the isomorphism of vector spaces

$$\nu : \Big(\bigvee^{k}\mathfrak{g}^\star\Big)_I \longrightarrow \Big(\operatorname{Sec}\bigvee^{k}\boldsymbol{g}^\star_{|U}\Big)_{I^0}$$

[K3, p. 43], where $(\bigvee^k \mathfrak{g}^\star)_I$ denotes the space of Ad_G-invariant vectors from $\bigvee^k\mathfrak{g}^\star$. ν is defined by the formula $\nu(w)(x) = \bigvee^k(\hat{z})^{-1\star}(w)$, $z\in(U\times G)_{|x}$, $\hat{z} : \mathfrak{g}\to\boldsymbol{g}_{|x}$ is an isomorphism of the right Lie algebra $\mathfrak{g}$ of G onto the Lie isotropy Lie algebra $\boldsymbol{g}_{|x}$ at x [K2; p. 11]. Therefore $\Gamma_{|U} = \nu(w)$ for some Ad_G-invariant element $w\in(\bigvee^k\mathfrak{g}^\star)_I$. Let $\bar{\omega}$ be the connection form of $\bar{H}$. By equality (15) from [K3] and the standard formula [G-H-V, Vol. II], we have

$$\pi^\star\langle\Gamma_{|U},\Omega_{b|U}\vee\ldots\vee\Omega_{b|U}\rangle = \langle w,\bar{\Omega}_U\vee\ldots\vee\bar{\Omega}_U\rangle = d\bar{\Phi}$$

for

$$\begin{aligned}\bar{\Phi} &= k!\sum_{i+j=k-1}\frac{1}{k+j}\left\langle w,\bar{\omega}\vee\frac{1}{i!}(d\bar{\omega})^i\vee\frac{1}{j!}\left(\frac{1}{2}[\![\bar{\omega},\bar{\omega}]\!]^L\right)^j\right\rangle\\ &= k!\sum_{i+j=k-1}\frac{1}{k+j}\left\langle w,\bar{\omega}\vee\frac{1}{i!}(d\bar{\omega})^i\vee\frac{1}{j!}\left(-\frac{1}{2}[\![\bar{\omega},\bar{\omega}]\!]^R\right)^j\right\rangle\end{aligned}$$

($[\,\cdot\,,\,\cdot\,]^L$ and $[\,\cdot\,,\,\cdot\,]^R$ denote the left and right structures of a Lie algebra $\mathfrak{g}$ of the Lie group G). On the other hand, the real form $\pi^\star\langle\Gamma_{|U},\Omega_{b|U}\vee\ldots\vee\Omega_{b|U}\rangle$ on $P = U\times G$ is right-invariant and

$$\gamma^\star_{|U}\left\langle\Gamma_{|U},\Omega_{b|U}\vee\ldots\vee\Omega_{b|U}\right\rangle = \tau_P^{-1}\pi^\star\left\langle\Gamma_{|U},\Omega_{b|U}\vee\ldots\vee\Omega_{b|U}\right\rangle$$

for the canonical isomorphism $\tau_P : \Omega_{A(P)}(U)\xrightarrow{\cong}\Omega^R(P)$ [K2]. Since τ_P commutes with exterior derivatives, we have

$$\gamma^\star_{|U}\left\langle\Gamma_{|U},\Omega_{b|U}\vee\ldots\vee\Omega_{b|U}\right\rangle = \tau_P^{-1}(d\bar{\Phi}) = d_{A_{|U}}(\tau_P^{-1}\bar{\Phi}).$$

$\bar{\omega}$ is Ad_G-equivariant, therefore $d\bar{\omega}$ and $[\bar{\omega},\bar{\omega}]^R$ are Ad_G-equivariant too. Considering the adjoint representations $\mathrm{ad}_{A(P)}$ and Ad_G, we have the analogous isomorphism

$$\tau_P^{\vee} : \Omega^{\mathrm{ad}}_{A(P)}\Big(M; \bigvee^{s} \boldsymbol{g}\Big) \xrightarrow{\cong} \Omega^{\mathrm{Ad}}\Big(P; \bigvee^{s} \mathfrak{g}\Big)$$

between the spaces of equivariant forms [M, App.A, Proposition 4.12] (see also [K1] for the language of differential groupoids). The following observations hold:

(1) $\tau_P^{\vee}(\omega) = \bar{\omega}$ (i.e. $\hat{z}(\bar{\omega}_{|z}(v)) = \omega_{|x}[v]$), see, for example [K2, p. 39],

(2) $\tau_P^{\vee}$ commutes with exterior derivatives, in particular, $\tau_P^{\vee}(d_A\omega) = d\bar{\omega}$,

(3) $\tau_P^{\vee}[\![\omega,\omega]\!] = [\bar{\omega},\bar{\omega}]^R$,

(4) $\tau_P^{\vee}\langle \nu(w), \varphi\rangle = \langle w, \tau_P^{\vee}(\varphi)\rangle$ for any invariant vector $w \in \left(\bigvee^s \mathfrak{g}\right)_I$ and equivariant form $\varphi \in \Omega^{\mathrm{ad}}_{A(P)}(M; \bigvee^s \boldsymbol{g})$.

From this we have

$$\begin{aligned}\gamma^{\star}_{|U}\langle \Gamma_{|U}, \Omega_{b|U} \vee \ldots \vee \Omega_{b|U}\rangle &= d_{A_{|U}}(\tau_P^{-1}\bar{\Phi}) \\ &= d_{A_{|U}}\left(k! \sum_{i+j=k-1} \frac{1}{k+j}\left\langle \Gamma_{|U}, \omega \vee \frac{1}{i!}(d_A\omega)^i \vee \frac{1}{j!}\left(-\frac{1}{2}[\![\omega,\omega]\!]\right)^j\right\rangle\right).\end{aligned}$$

The arbitrariness of x yields the assertion of our theorem. □

As a corollary we get

$$\gamma^{\star}\langle \Gamma, \Omega_b\rangle = d_A\langle \Gamma, \omega\rangle, \qquad \Gamma \in (\mathrm{Sec}\,\boldsymbol{g}^{\star})_{I^0}, \tag{4.2}$$

$$\gamma^{\star}\langle \Gamma, \Omega_b \vee \Omega_b\rangle = d_A\left(\langle \Gamma, \omega \vee d_A\omega\rangle - \frac{1}{3}\langle \Gamma, \omega \vee [\![\omega,\omega]\!]\rangle\right), \tag{4.3}$$

$$\Gamma \in \Big(\mathrm{Sec} \bigvee^{2} \boldsymbol{g}^{\star}\Big)_{I^0}.$$

Problem: Prove formula (4.1) using only the category of Lie algebroids.

4.1. s-Lie algebroids of rank 1

Consider an s-Lie algebroid (A, ε) of rank 1. Then $\boldsymbol{g}$ is a trivial rank 1 LAB with global ad_A-invariant cross-section $\varepsilon \in \mathrm{Sec}\,\boldsymbol{g}$, i.e. one satisfying the relation $[\![\xi,\varepsilon]\!]=0$ for each $\xi\in\operatorname{Sec}A$. Take

$$\varepsilon^\star\in\operatorname{Sec}\boldsymbol{g}^\star \tag{4.4}$$

such that

$$\iota_\varepsilon\varepsilon^\star=1.$$

Then $\varepsilon^\star$ is $\mathrm{ad}_A^\natural$-invariant, i.e. $\mathrm{ad}_A^\natural(\xi)(\varepsilon^\star)=0$ for $\xi\in\operatorname{Sec}A$. Indeed, since $\operatorname{rank}\boldsymbol{g}=1$ and $\varepsilon_x\neq 0$ for each $x\in M$, it is enough to notice the equality

$$\mathrm{ad}_A^\natural(\xi)(\varepsilon^\star)(\varepsilon)=(\gamma\circ\xi)\langle\varepsilon^\star,\varepsilon\rangle-\langle\varepsilon^\star,[\![\xi,\varepsilon]\!]\rangle=0.$$

Therefore $\varepsilon^\star$ belongs to the domain of the Chern–Weil homomorphism h_A of A.

Theorem 4.2. *Under the above assumptions,*

$$\chi_A=h_A^{(2)}(-\varepsilon^\star).$$

Proof. According to the definition of h_A, we have $h_A(-\varepsilon^\star)=[\langle-\varepsilon^\star,\Omega_b\rangle]$. To prove our theorem, let us take the 1-form $\Phi=\langle-\varepsilon^\star,\omega\rangle\in\Omega_A^1(M)$. According to (4.2), $d_A\Phi=\gamma^\star\langle-\varepsilon^\star,\Omega_b\rangle$, therefore it is sufficient to check that $⨍_A\Phi=-1$. Clearly, this follows from the equalities

$$\gamma^\star\Big(⨍_A\Phi\Big)=\iota_\varepsilon\Phi=\langle-\varepsilon^\star,\omega(\varepsilon)\rangle=\langle-\varepsilon^\star,\varepsilon\rangle=-1. \qquad\square$$

Corollary 4.1. *There exist nonintegrable s-Lie algebroids (A,ε) of rank 1 having nonzero Euler class χ_A.*

Indeed, take the s-Lie algebroid of rank 1 of the TC-foliation of left cosets of a nonclosed Lie subgroup H in a Lie group G, such that $\dim\bar H-\dim H=1$, (Example 2.2 and [K4, Theorem 3.5.3]). The invariant cross-section $\varepsilon\in\operatorname{Sec}\boldsymbol{g}$ corresponds via the isomorphism φ to an element $0\neq\varepsilon_0=[w]\in\bar{\mathfrak h}/\mathfrak h$. Assume G to be a compact, connected and semisimple Lie group. According to the proof of Theorem 7.4.3 from [K3], $h_A^{(2)}\neq 0$. Since the domain of $h_A^{(2)}$, $(\operatorname{Sec}\boldsymbol{g}^\star)_{I^0}$, is a 1-dimensional vector space (more exactly, $(\operatorname{Sec}\boldsymbol{g}^\star)_{I^0}$ is canonically isomorphic to $(\bar{\mathfrak h}/\mathfrak h)^\star$, see [K3, Proposition 7.4.1 and the next remarks], and via this isomorphism the invariant cross-section $\varepsilon^\star\in(\operatorname{Sec}\boldsymbol{g}^\star)_{I^0}$ is mapped into an element $\varepsilon_0^\star\in(\bar{\mathfrak h}/\mathfrak h)^\star$ such that $\iota_{\varepsilon_0}\varepsilon_0^\star=-1$), we have $h_A^{(2)}(\varepsilon^\star)\neq 0$. Adding the simple connectedness to the properties of G, we obtain the looked-for objects.

4.2. s-Lie algebroids of rank 3.

Consider an s-Lie algebroid (A,ε) of rank 3, $\varepsilon \in (\operatorname{Sec}\bigwedge^3 \boldsymbol{g})_{I^0}$. According to the remark after Theorem 1.2, after writting $\varepsilon = h_1 \wedge h_2 \wedge h_3$ (locally), $h_i \in \operatorname{Sec}\boldsymbol{g}$, we have

$$[\![\xi, h_1]\!] \wedge h_2 \wedge h_3 + h_1 \wedge [\![\xi, h_2]\!] \wedge h_3 + h_1 \wedge h_2 \wedge [\![\xi, h_3]\!] = 0, \quad \xi \in \operatorname{Sec} A.$$

Analogously as in the previous section, take $\varepsilon^\star \in \operatorname{Sec}\bigwedge^3 \boldsymbol{g}^\star$ such that $\iota_\varepsilon \varepsilon^\star = 1$. Then $\varepsilon^\star$ is $\bigwedge^3 \operatorname{ad}_A^\natural$-invariant. Indeed, it is enough to notice that

$$\begin{aligned}(\gamma \circ \xi)\langle \varepsilon^\star, h_1 \wedge h_2 \wedge h_3\rangle - \langle \varepsilon^\star, [\![\xi, h_1]\!] \wedge h_2 \wedge h_3 \\ + h_1 \wedge [\![\xi, h_2]\!] \wedge h_3 + h_1 \wedge h_2 \wedge [\![\xi, h_3]\!]\rangle = 0.\end{aligned}$$

Since the first and the second cohomology groups of the Lie algebra $\boldsymbol{g}_{|x}$ are 0, we have the canonical isomorphism [G-H-V, Vol. III]

$$\rho_x : \Big(\bigvee^2 \boldsymbol{g}^\star_{|x}\Big)_I \xrightarrow{\cong} \bigwedge^3 \boldsymbol{g}^\star_{|x}$$

(each element of $\bigwedge^3 \boldsymbol{g}^\star_{|x}$ is invariant here) given by the formula

$$\langle \rho_x(\Psi), x \wedge y \wedge z\rangle = \langle \Psi, [x,y] \vee z\rangle.$$

We recall that $\Psi \in \bigvee^2 \boldsymbol{g}^\star_{|x}$ is invariant if and only if $\langle \Psi, [x,y]\vee z\rangle + \langle \Psi, [x,z] \vee y\rangle = 0$ for any $x,y,z \in \boldsymbol{g}_{|x}$. By this and the fact that $\boldsymbol{g}$ is an LAB, the sum $(\bigvee^2 \boldsymbol{g}^\star)_I := \bigcup_{x\in M} (\bigvee^2 \boldsymbol{g}^\star_{|x})_I \subset \bigvee^2 \boldsymbol{g}^\star$ is a smooth subbundle of $\bigvee^2 \boldsymbol{g}^\star$. Of course, any $\bigvee^2 \operatorname{ad}_A^\natural$-invariant cross-section $\Gamma \in (\operatorname{Sec}\bigvee^2 \boldsymbol{g}^\star)_{I^0}$ is a cross-section of this subbundle. All the isomorphisms ρ_x induce the linear smooth homomorphism

$$\rho : \Big(\bigvee^2 \boldsymbol{g}^\star\Big)_I \longrightarrow \bigwedge^3 \boldsymbol{g}^\star$$

of vector bundles.

Proposition 4.1. *The homomorphism* $\operatorname{Sec}\rho$ *induced by* ρ *on invariant crosssections is an isomorphism*

$$\operatorname{Sec}\rho : \Big(\operatorname{Sec}\bigvee^2 \boldsymbol{g}^\star\Big)_{I^0} \xrightarrow{\cong} \Big(\operatorname{Sec}\bigwedge^3 \boldsymbol{g}^\star\Big)_{I^0}.$$

Proof. Let $\Gamma \in \operatorname{Sec} \bigvee^2 \boldsymbol{g}^\star$. Then $\rho \circ \Gamma$ is invariant if and only if Γ is. Indeed, since $\boldsymbol{g}$ is an LAB and $\boldsymbol{g}_{|x} = \boldsymbol{g}^2_{|x}$, each cross-section $h \in \operatorname{Sec} \boldsymbol{g}$ is locally a sum of cross-sections of the form $[h_\alpha, h_\beta]$. The rest follows trivially from the Jacobi identity in $\operatorname{Sec} A$. $\square$

Take $\Gamma \in (\operatorname{Sec} \bigvee^2 \boldsymbol{g}^\star)_{I^0}$ such that

$$\rho(\Gamma) = \varepsilon^\star.$$

Theorem 4.3. *Under the above assumptions,*

$$\chi_A = h_A^{(4)}(-2\Gamma).$$

Proof. According to the definition of h_A we have

$$h_A^{(4)}(-2\Gamma) = [\langle -\Gamma, \Omega_b \vee \Omega_b \rangle].$$

To prove our theorem, let us take the 3-form $\Phi \in \Omega^3_A(M)$ defined by

$$\Phi = \langle -\Gamma, \omega \vee d_A\omega \rangle - \frac{1}{3}\langle -\Gamma, \omega \vee [\![\omega, \omega]\!] \rangle.$$

According to (4.3),

$$\gamma^\star \langle -\Gamma, \Omega_b \vee \Omega_b \rangle = d_A \Phi.$$

Therefore it suffices to check that $⨏_A \Phi = -1$, i.e. $\gamma^\star(⨏_A \Phi) = \iota_\varepsilon \Phi = -1$. Since $d_A\omega(h_i, h_j) = [h_i, h_j]$ and $[\![\omega, \omega]\!](h_i, h_j) = 2[h_i, h_j]$ and $\omega(h_i) = h_i$, we obtain

$$\begin{aligned}\iota_\varepsilon \left(\omega \vee d_A\omega - \frac{1}{3}\omega \vee [\![\omega, \omega]\!] \right) &= \left(\omega \vee d_A\omega - \frac{1}{3}\omega \vee [\![\omega, \omega]\!] \right)(h_1 \wedge h_2 \wedge h_3) \\ &= \frac{1}{3}\big(h_1 \vee [h_2, h_3] - h_2 \vee [h_1, h_3] + h_3 \vee [h_1, h_2]\big).\end{aligned}$$

Therefore

$$\begin{aligned}\iota_\varepsilon \Phi &= \left\langle -\Gamma, \iota_\varepsilon \left(\omega \vee d_A\omega - \frac{1}{3}\omega \vee [\![\omega, \omega]\!] \right) \right\rangle \\ &= \frac{1}{3}\langle -\Gamma, [h_2, h_3] \vee h_1 - [h_1, h_3] \vee h_2 + [h_1, h_2] \vee h_3 \rangle\end{aligned}$$

$$= \frac{1}{3}\langle -\rho(\Gamma), h_2 \wedge h_3 \wedge h_1 - h_1 \wedge h_3 \wedge h_2 + h_1 \wedge h_2 \wedge h_3 \rangle$$
$$= \langle -\varepsilon^\star, h_1 \wedge h_2 \wedge h_3 \rangle = -1. \qquad \square$$

5. Computation of the algebra $H_A(M)$

For an arbitrarily taken element $\Omega \notin \mathbb{R}$ to which we assign the degree n, we form the anticommutative graded algebra $(\bigwedge \Omega, \wedge) \cong H(S^n)$ as the exterior algebra over the 1-dimensional graded vector space $\operatorname{Lin}\Omega$ (homogeneous of degree n) spanned by Ω.

Below, $n = 1$ or $n = 3$, therefore n is odd (see Proposition 2.1).

Theorem 5.1. *Let (A, ε) be any s-Lie algebroid of rank n and let $\Psi \in \Omega^{n+1}(M)$ be a representative of the Euler class $\chi_A = [\Psi]$. According to Remark 3.1, we can take an n-form $\Omega \in \Omega^n_A(M)$ such that $\fint_A \Omega = -1$ and $d_A\Omega = \gamma^\star\Psi$. Consider the skew tensor product $\Omega(M) \bigotimes \bigwedge \Omega$ of anticommutative graded algebras and define the operator $d : \Omega(M) \bigotimes \bigwedge \Omega \to \Omega(M) \bigotimes \bigwedge \Omega$ by the formulae*

$$d(\psi \otimes 1) = d_M\psi \otimes 1$$
$$d(\psi \otimes \Omega) = (-1)^{\deg\psi}(\psi \wedge \Psi) \otimes 1 + d_M\psi \otimes \Omega.$$

Then d is an antiderivation and $(\Omega(M) \bigotimes \bigwedge \Omega, d)$ is a graded differential algebra.

Proof. Clearly, d is a differential of degree $+1$, such that $d(x \wedge y) = dx \wedge y + (-1)^{\deg x} x \wedge dy$ for $x = \psi_1 \otimes 1$, $y = \psi_2 \otimes 1$ or for $x = \psi_1 \otimes 1$, $y = \psi_2 \otimes \Omega$. The remaining third case: $x = \psi_1 \otimes \Omega$, $y = \psi_2 \otimes \Omega$ holds (n is odd), giving zero on both sides of this equality:

$$d(\psi_1 \otimes \Omega \wedge \psi_2 \otimes \Omega) = d\Big((-1)^{n\cdot\deg\psi_2}\psi_1 \wedge \psi_2 \otimes \Omega \wedge \Omega\Big) = d(0) = 0,$$

$$\begin{aligned} d(\psi_1 \otimes \Omega) &\wedge \psi_2 \otimes \Omega + (-1)^{\deg\psi_1 + n}\psi_1 \otimes \Omega \wedge d(\psi_2 \otimes \Omega) \\ &= (-1)^{\deg\psi_1}(\psi_1 \wedge \Psi) \otimes 1 \wedge \psi_2 \otimes \Omega \\ &\quad + (-1)^{\deg\psi_1 + \deg\psi_2 + n}\psi_1 \otimes \Omega \wedge \psi_2 \wedge \Psi \otimes 1 \\ &= (-1)^{\deg\psi_1}\big(1 + (-1)^{n^2}\big)\psi_1 \wedge \Psi \wedge \psi_2 \otimes \Omega = 0. \end{aligned} \qquad \square$$

Theorem 5.2. *The mapping*

$$\mu : \Omega(M) \bigotimes \bigwedge \Omega \longrightarrow \Omega_A(M)$$

defined by $\mu(\psi \otimes 1) = \gamma^\star \psi$, $\mu(\psi \otimes \Omega) = \gamma^\star \psi \wedge \Omega$, *is a degree* 0 *homomorphism of graded algebras.*

Proof. The homogeneity of μ and the commutativity of μ with the differentials d_A and d, as well as the condition $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$ for $x = \psi_1 \otimes 1$, $y = \psi_2 \otimes 1$ or $x = \psi_1 \otimes \Omega$ and $y = \psi_2 \otimes \Omega$ is easy to see. For $x = \psi_1 \otimes \Omega$ and $y = \psi_2 \otimes \Omega$ this condition follows from the observation that $\Omega \wedge \Omega = 0$ in $\bigwedge \Omega$, and $\Omega \wedge \Omega = 0$ in $\Omega(M)$ (n is odd). $\square$

Theorem 5.3. *For an s-Lie algebroid* (A, ε) *over a manifold* M, *the mapping induced by* μ *on the cohomology*

$$\mu_\# : H\Big(\Omega(M) \bigotimes \bigwedge \Omega\Big) \longrightarrow H_A(M)$$

is an isomorphism of graded algebras.

Proof. Consider the short exact sequence

$$0 \longrightarrow \Omega(M) \xrightarrow{i} \Omega(M) \bigotimes \bigwedge \Omega \xrightarrow{\rho} \Omega(M) \longrightarrow 0,$$

$i(\psi) = \psi \otimes 1$, $\rho(\psi_1 \otimes 1 + \psi_2 \otimes \Omega) = -\psi_2$; i and ρ commute with the differentials. The following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega(M) & \xrightarrow{\;i\;} & \Omega(M) \bigotimes \bigwedge \Omega & \xrightarrow{\;\rho\;} & \Omega(M) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \gamma^\star} & & \big\downarrow{\scriptstyle \mu} & & \big\| & & \\
0 & \longrightarrow & \ker \fint_A & \longrightarrow & \Omega_A(M) & \xrightarrow{\;\fint_A\;} & \Omega(M) & \longrightarrow & 0,
\end{array}$$

is commutative; indeed,

$$\begin{aligned}
\fint_A \circ \mu(\psi_1 \otimes 1 + \psi_2 \otimes \Omega) &= \fint_A (\gamma^\star \psi_1 + \gamma^\star \psi_2 \wedge \Omega) \\
&= 0 + \psi_2 \wedge \fint_A \Omega = -\psi_2 = \rho(\psi_1 \otimes 1 + \psi_2 \otimes \Omega), \\
\mu \circ i(\psi) &= \mu(\psi \otimes 1) = \gamma^\star \psi.
\end{aligned}$$

Passing to the long exact sequences, we get from the five-lemma (since $\gamma^{\#}$ is an isomorphism [Theorem 2.1]) that $\mu_{\#}$ is an isomorphism. $\square$

Corollary 5.1. *If $\chi_A = 0$, then $H_A(M) \cong H(M) \bigotimes H(S^n)$ as graded algebras.*

Proof. Let $\chi_A = 0$. Then we can choose $\Psi = 0$ as a representative of the Euler class χ_A, which gives $d(\psi_1 \otimes 1 + \psi_2 \otimes \Omega) = d\psi_1 \otimes 1 + d\psi_2 \otimes \Omega$, i.e. d is equal to the tensor product of the differentials d_M and $0_{\bigwedge \Omega}$ [G]. The Künneth theorem yields the isomorphism of graded algebras

$$H\Big(\Omega(M) \bigotimes \bigwedge \Omega\Big) \cong H(\Omega(M)) \bigotimes H\Big(\bigwedge \Omega\Big) \cong H(M) \bigotimes H(S^n).$$

This result and Theorem 5.3 imply that $H_A(M) \cong H(\Omega(M) \bigotimes \bigwedge \Omega) \cong H(M) \bigotimes H(S^n)$ as graded algebras. $\square$

Theorem 5.4. *If $\dim M = n+1$ and M is compact, connected and oriented, then for arbitrary s-Lie algebroids A and A' over M of rank n with nonzero Euler classes $\chi_A \neq 0 \neq \chi_{A'}$, there exists an isomorphism of cohomology algebras $H_A(M) \cong H_{A'}(M)$.*

Proof. Take the orientation class $[\Delta] \in H^{n+1}(M)$ of M. Then $\chi_A = k \cdot [\Delta] = [k \cdot \Delta]$ and $\chi_{A'} = k' \cdot [\Delta] = [k' \cdot \Delta]$ for some reals $k, k' \in \mathbb{R} \backslash \{0\}$. Therefore $\Psi = k \cdot \Delta$ and $\Psi' = k' \cdot \Delta$ are representatives of the Euler classes. According to Remark 3.1, there exist n-forms $\Omega \in \Omega^n_A(M)$ and $\Omega' \in \Omega^n_{A'}(M)$ such that $\fint_A \Omega = -1$, $\fint_{A'} \Omega' = -1$ and $d_A \Omega = \gamma^\star \Psi$, $d_{A'} \Omega' = \gamma'^\star \Omega'$. Theorem 5.3 says that $H_A(M) \cong H(\Omega(M) \bigotimes \bigwedge \Omega)$ and $H_{A'}(M) \cong H(\Omega(M) \bigotimes \bigwedge \Omega')$ provided that in $\Omega(M) \bigotimes \bigwedge \Omega$ and $\Omega(M) \bigotimes \bigwedge \Omega'$ the following differentials d_k and $d_{k'}$ are defined:

$$\begin{aligned} d_k(\psi \otimes 1) &= d_M \psi \otimes 1 \\ d_k(\psi \otimes \Omega) &= d_M \psi \otimes \Omega + (-1)^{\deg \psi} (\psi \wedge \Psi) \otimes 1 \\ &= \begin{cases} d_M \psi \otimes \Omega, & \deg \psi > 0, \\ (\psi \cdot \Psi) \otimes 1 + d_M \psi \otimes \Omega, & \deg \psi = 0, \end{cases} \end{aligned}$$

and analogously for $d_{k'}$.

Put

$$I : \Omega(M) \bigotimes \bigwedge \Omega \longrightarrow \Omega(M) \bigotimes \bigwedge \Omega',$$
$$\psi \otimes 1 \longmapsto \psi \otimes 1,$$
$$1 \otimes \Omega \longmapsto 1 \otimes \frac{k}{k'}\Omega',$$
$$\psi \otimes \Omega \longmapsto \psi \otimes \Omega', \quad \deg \psi > 0.$$

Of course, I is an isomorphism of graded vector spaces. We check the commutativity of I with differentials:

- $d_{k'} \circ I(\psi \otimes 1) = d_{k'}(\psi \otimes 1) = d\psi \otimes 1 = I \circ d_k(\psi \otimes 1)$,
- $d_{k'} \circ I(1 \otimes \Omega) = d_{k'}(1 \otimes \frac{k}{k'}\Omega') = \frac{k}{k'} \cdot \Psi' \otimes 1 = \Psi \otimes 1 = I \circ d_k(1 \otimes \Omega)$,
- if $\deg \psi > 0$, then $d_{k'} \circ I(\psi \otimes \Omega) = d_{k'}(\psi \otimes \Omega') = d\psi \otimes \Omega' = I \circ d_k(\psi \otimes \Omega)$.

Therefore $I_\# : H(\Omega(M) \bigotimes \bigwedge \Omega) \xrightarrow{\cong} H(\Omega(M) \bigotimes \bigwedge \Omega')$ is an isomorphism of cohomology algebras which implies that $H_A(M) \cong H_{A'}(M)$. $\square$

Remark 5.1. According to the above theorem, the Euler class χ_A is not – in general – an invariant of the cohomology algebra of A and according to [K7, Proposition 7.6.] it has nothing in common with the Euler–Poincaré characteristic of A; the last, when considered for TUIO-Lie algebroids ($\dim M + \operatorname{rank} \boldsymbol{g}$ is odd), is always 0.

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