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Asymptotic analysis of a heterogeneous finite-source communication system operating in random environments

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Dedicated to Professor Lajos Tamássy on his 70th birthday

Abstract. This paper deals with a First-Come, First-Served (FCFS) queueing model to analyse the asymptotic behaviour of a heterogeneous finite-source communication system with a single processor. The sources and the processor are supposed to operate in independent random environments, respectively. Each message is characterized by its own exponentially distributed source and processing time with parameter depending on the state of the corresponding environment. Assuming that the arrival rates of the messages are many times greater than their service rates ("fast" arrival), it is shown that the time to the first system failure converges in distribution, under appropriate norming, to an exponentially distributed random variable. Some simple examples are considered to illustrate the effectiveness of the method proposed by comparing the approximate results to the exact ones.

1. Introduction

Performance evaluation of information system development has become more complex as the size and complexity of the system has increased, (c.f., Mitrani [7], Takagi [14]). Reliability is certainly the most important characteristic for communication networks. The measure of greatest interest is the distribution of the time to the first system failure. It is well-known that the majority of the problems can be treated by the help of Semi-Markov Processes (SMP). Since the failure-free operation time of the system correponds to sojourn time problems we can use the results obtained for SMP. If the exit from a given subset of the state space is a

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"rare" event, that is, it occurs with a small probability it is natural to investigate the asymptotic behaviour of the sojourn time in that subspace, see Anisimov et al. [1], Anisimov and Sztrik [2], Gertsbakh [4–5], Keilson [6].

Realistic consideration of certain stochastic systems, however, often requires the introduction of a random environment where system parameters are subjected to randomly occuring fluctuations. This situation may be attributed to certain changes in the physical environment such as weather, or sudden personal changes and work load alterations. Infinite birth-and-death models in random environments have been the subject of several earlier works (e.g., Neuts [8 ch.6], Purdue [9], Yechiali [15]). Gaver *et al.* [3] proposed an efficient computational approach for the analysis of a generalised structure involving finite state space birth-and-death processes in a Markovian environment. Rosenberg et al. [10] derived necessary and sufficient conditions for the stability of a single server exponential queue with random fluctuations in the intensity of the arrival process. Moreover, they considered applications to the study of a finite resume level queue. Stern and Elwalid [11] has used Markov-modulated processes for analysing some infinite source information systems.

This paper deals with a First-Come, First-Served (FCFS) queueing model to analyse the asymptotic behaviour of a heterogeneous finite-source communication system with a single processor. The sources and the processor are supposed to operate in independent random environments, respectively. Each message is characterized by its own exponentially distributed source and processing time with parameter depending on the state of the corresponding environment. Assuming that the arrival rates of the messages are many times greater than their service rates ("fast" arrival), it is shown that the time to the first system failure converges in distribution, under appropriate norming, to an exponentially distributed random variable. Some simple examples are considered to illustrate the effectiveness of the method proposed comparing the approximate characteristics to the exact ones. This paper generalizes the results of Sztrik and Lukashuk [12] where the sources are homogeneous and the whole system is governed by a single random environment. The technique used here is similar to the one applied in Sztrik [13].

2. Preliminary results

In this section a brief survey is given of the most related theoretical results, mainly due to Anisimov to be applied later on.

Let $(X_\varepsilon(k), k \geq 0)$ be a Markov chain with state space

$$\bigcup_{q=0}^{m+1} X_q, \quad X_i \cap X_j = 0, \quad i \neq j,$$

defined by the transition matrix $\left(p_\varepsilon\left(i^{(q)},\ j^{(z)}\right)\right)$ satisfying the following conditions:

1. $p_\varepsilon(i^{(0)},j^{(0)}) \to p_0(i^{(0)},j^{(0)})$, as $\varepsilon \to 0$, $i^{(0)},j^{(0)} \in X_0$, and matrix $P_0 = \left(p_0(i^{(0)},j^{(0)})\right)$ is irreducible;
2. $p_\varepsilon(i^{(q)},j^{(q+1)}) = \varepsilon\alpha^{(q)}\left(i^{(q)},j^{(q+1)}\right) + o(\varepsilon)$, $i^{(q)} \in X_q, j^{(q+1)} \in X_{q+1}$;
3. $p_\varepsilon(i^{(q)},f^{(q)}) \to 0$, as $\varepsilon \to 0$, $i^{(q)},f^{(q)} \in X_q$, $q \geq 1$;
4. $p_\varepsilon\left(i^{(q)},f^{(z)}\right) \equiv 0$, $i^{(q)} \in X_q$, $f^{(z)} \in X_z$, $z - f \geq 2$.

In the sequel the set of states X_q is called the q-th level of the chain, $q = 1,\ldots,m+1$. Let us single out the subset of states

$$\langle\alpha_m\rangle = \bigcup_{q=0}^{m} X_q.$$

Denote by $\{\pi_\varepsilon\left(i^{(q)}\right),\ i^{(q)} \in X_q\}$, $q = 1,\ldots,m$ the stationary distribution of a chain with transition matrix

$$\left(\frac{p_\varepsilon\left(i^{(q)},j^{(z)}\right)}{1-\sum\limits_{k^{(m+1)}\in X_{m+1}} p_\varepsilon\left(i^{(q)},k^{(m+1)}\right)}\right), \quad i^{(q)} \in X_q,\ j^{(z)} \in X_z,\ q,z \leq m.$$

Furthermore denote by $g_\varepsilon(\langle\alpha_m\rangle)$ the steady state probability of exit from $\langle\alpha_m\rangle$, that is

$$g_\varepsilon(\langle\alpha_m\rangle) = \sum_{i^{(m)}\in X_m} \pi_\varepsilon\left(i^{(m)}\right) \sum_{j^{(m+1)}\in X_{m+1}} p_\varepsilon\left(i^{(m)},j^{(m+1)}\right).$$

Denote by $\left\{\pi_0\left(i^{(0)}\right), i^{(0)} \in X_0\right\}$ the stationary distribution corresponding to P_0 and let

$$\bar{\pi}_0 = \left\{\pi_0\left(i^{(0)}\right),\ i^{(0)} \in X_0\right\}, \quad \bar{\pi}_\varepsilon^{(q)} = \left\{\pi_\varepsilon\left(i^{(q)}\right), i^{(q)} \in X_q\right\},$$

be row vectors. Finally, let the matrix

$$A^{(q)} = \left(\alpha^{(q)}\left(i^{(q)},j^{(q+1)}\right)\right), \quad i^{(q)} \in X_q, \quad j^{(q+1)} \in X_{q+1},\ q = 0,\ldots,m,$$

defined by condition 2.

Conditions 1–4 enable us to compute the main terms of the asymptotic expression for $\bar{\pi}_\varepsilon^{(q)}$ and $g_\varepsilon(\langle\alpha_m\rangle)$. Namely, we obtain

$$\begin{aligned} \bar{\pi}_\varepsilon^{(q)} &= \varepsilon^q \bar{\pi}_0 A^{(0)} A^{(1)} \ldots A^{(q-1)} + o(\varepsilon^q), \quad q = 1,\ldots,m, \\ g_\varepsilon\left(\langle\alpha_m\rangle\right) &= \varepsilon^{m+1}\bar{\pi}_0 A^{(0)} A^{(1)} \ldots A^{(m)}\underline{1} + o\left(\varepsilon^{m+1}\right), \end{aligned} \tag{1}$$

where $\underline{1} = (1, \ldots, 1)^*$ is a column vector, see Anisimov *et al.* [1], pp. 141–153). Let $(\eta_\varepsilon(t),\ t \geq 0)$ be a SMP given by the embedded Markov chain $(X_\varepsilon(k),\ k \geq 0)$ satisfying conditions 1–4. Let the times $\tau_\varepsilon\left(j^{(s)}, k^{(z)}\right)$ – transition times from state $j^{(s)}$ to state $k^{(z)}$– fulfil the condition

$$E \exp\left\{i\theta\beta_\varepsilon\tau_\varepsilon\left(j^{(s)}, k^{(z)}\right)\right\} = 1 + a_{jk}(s, z, \theta)\varepsilon^{m+1} + o\left(\varepsilon^{m+1}\right), \quad (i^2 = -1),$$

where β_ε is some normalizing factor. Denote by $\Omega_\varepsilon(m)$ the instant at which the SMP reaches the $(m+1)$-th level for the first time, exit time from $\langle \alpha_m \rangle$, provided $\eta_\varepsilon(0) \in \langle \alpha_m \rangle$. Then we have:

Theorem 1. *(cf. Anisimov et al. [1] pp. 153) If the above conditions are satisfied then*

$$\lim_{\varepsilon \to 0} E \exp\left\{i\theta\beta_\varepsilon\Omega_\varepsilon(m)\right\} = (1 - A(\theta))^{-1}$$

where

$$A(\theta) = \frac{\sum\limits_{j^{(0)}, k^{(0)} \in X_0} \pi_0\left(j^{(0)}\right) p_0\left(j^{(0)}, k^{(0)}\right) a_{jk}(0, 0, \theta)}{\bar{\pi}_0 A^{(0)} A^{(1)} \ldots A^{(m)} \underline{1}}.$$

Corollary 1. *In particular, if $a_{jk}(s, z, \theta) = i\theta m_{jk}(s, z)$ then the limit is an exponentially distributed random variable with mean*

$$\frac{\sum\limits_{j^{(0)}, k^{(0)} \in X_0} \pi_0\left(j^{(0)}\right) p_0\left(j^{(0)}, k^{(0)}\right) m_{jk}(0, 0)}{\bar{\pi}_0 A^{(0)} A^{(1)} \ldots A^{(m)} \underline{1}}.$$

3. The queueing model

Consider a finite-source communication system with N heterogeneous sources and a single processor. The sources and the receiver operate in independent random environments. The environmental changes are reflected in the values of the acces and service rates that prevail at any point of time. The main objective is to adapt these parameters to respond to random changes effectively and thus maintain derived level of system performance. The sources are assumed to operate in a random environment governed by an ergodic Markov chain $(\xi_1(t), t{\geq}0)$ with state space $(1, \ldots, r_1)$ and with transition density matrix $(a_{i_1 j_1},\ i_1, j_1 = 1, \ldots, r_1,\ a_{i_1 i_1} = \sum\limits_{j \neq i_1} a_{i_1 j})$. Whenever the environmental process is in state i_1 the probability that source p generates a request in the time interval $(t, t+h)$ is $\lambda_p(i_1; \varepsilon)h + o(h)$. Each message is transmitted to a receiver where the service immediately starts if the processor is idle, otherwise a queueing line is formed. The service discipline is First-Come, First-Served (FCFS). The receiver is also supposed to operate in a random environment governed by an ergodic Markov chain $(\xi_2(t), t\geq 0)$ with state space $(1,\ldots,r_2)$ and with transition density matrix $(b_{i_2j_2},\ i_2,j_2 = 1,\ldots,r_2,\ b_{i_2i_2} = \sum_{q\neq i_2} b_{i_2q})$. Similarly, whenever the environmental process is in state i_2 and there are s active sources, $s = 0,\ldots,N-1$ the probability that the service of message p is completed in time interval $(t, t+h)$ is $\mu_p(i_2, s)h + o(h)$. If a given source has sent a message it stays idle and it cannot generate another one. After being serviced each message immediately returns to its source which hence becomes active. All random variables involved here and the random environments are supposed to be independent of each other.

In practical applications it is very important to know the distribution of time until the receiver becomes empty, which is actually the busy period of the processor. For example, if it is a satellite, an aircraft or any flying object which needs orientation.

Let us consider the system under the assumption of "fast" arrivals, i.e., $\lambda_p(i_1,\varepsilon) \to \infty$ as $\varepsilon \to 0$. For simplicity let $\lambda_p(i_1,\varepsilon) = \lambda_p(i_1)/\varepsilon$.

Denote by $Y_\varepsilon(t)$ the number of active sources at time t, and let

$$\Omega_\varepsilon(m) = \inf\{t : t > 0,\ Y_\varepsilon(t) = m + 1/Y_\varepsilon(0)\leq m\},$$

that is, the instant at which the number of active sources reaches the $(m+1)$-th level for the first time, provided that at the beginning their number is not greater than m, $m = 0,\ldots,N-1$. In the following $\Omega_\varepsilon(m)$ is referred to as the time to the first system failure. In particular, for $m = N-1$ we get the case when the processor becomes idle.

Denote by $\pi_0\,(i_1, i_2 : 0; k,\ldots,k_N)$ the steady-state probability that the $\xi_1(t)$ is in state i_1, $\xi_2(t)$ is in i_2, there is no active source and the order of arrival of messages to the processor is $(k_1,\ldots,k_N)$. Similarly, denote by $\pi_1\,(i_1, i_2 : 1; k_2,\ldots,k_N)$ the steady-state probability that the first random environment is in state i_1, the second one is in state i_2, source k_1 is active and the other sources sent their messages in the order $(k_2,\ldots,k_N)$. Clearly, $(k_S,\ldots,k_N)\in V_N^{N-s+1}$, $s = 1,2$ where V_N^{N-s+1} denotes the set of all variations of order $N-s+1$ of elements $1,\ldots,N$. Now we have:

Theorem 2. *For the system in question under the above assumptions, independently of the initial state, the distribution of the normalized random variable $\varepsilon^m\Omega_\varepsilon(m)$ converges weakly to an exponentially distributed random variable with parameter*

$$\Lambda = \sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\sum_{(k_1,\ldots,k_N)\in V_N^N} \pi_0\,(i_1, i_2 : 1; k_2,\ldots,k_N)\,\frac{\mu_{k_2}\,(i_2,1)}{\lambda_{k_1}\,(i_1)}\times$$
$$\times\,\frac{\mu_{k_3}\,(i_2,2)}{\lambda_{k_1}\,(i_1)+\lambda_{k_2}\,(i_1)}\times\cdots\times\frac{\mu_{k_{m+1}}\,(i_2,m)}{\lambda_{k_1}\,(i_1)+\cdots+\lambda_{k_m}\,(i_1)}\,\frac{1}{D},$$

where

$$D = \sum_{j_1=1}^{r_1} \sum_{j_2=1}^{r_2} \sum_{\substack{i_1=1 \\ i_1 \neq j_1}}^{r_1} \sum_{\substack{i_2=1 \\ i_2 \neq j_2}}^{r_2} \sum_{(k_1,\ldots,k_N)\in V_N^N} \pi_0\left(i_1, i_2 : 0; k_1, \ldots, k_N\right) \times$$
$$\times \frac{a_{i_1 j_1} + b_{i_2 j_2}}{\left(a_{i_1 i_1} + b_{i_2 i_2} + \mu_{k_1}\left(i_2, 0\right)\right)^2}.$$

PROOF. Let us introduce the following stochastic process

$$Z_\varepsilon(t) = \left(\xi_1(t), \xi_2(t) : Y_\varepsilon(t); \beta_1(t), \ldots, \beta_{Y_\varepsilon(t)}(t)\right),$$

where $\beta_1(t), \ldots, \beta_{Y_\varepsilon(t)}(t)$ denotes the indices of the messages in order of their arrival at the processor. It is easy to see that $(Z_\varepsilon(t),\, t{\geq}0)$ is a multidimensional Markov chain with state space

$$E = (\ (i_1, i_2 : s; k_1, \ldots, k_{N-s}), \quad i_1 = 1, \ldots, r_1, \quad i_2 = 1, \ldots, r_2,$$
$$(k_1, \ldots, k_{N-s}) \in V_N^{N-s}, \quad s = 0, \ldots, N\ ),$$

where $k_0 = \{0\}$ by definition. Furthermore, let

$$\langle \alpha_m \rangle = (\ (i_1, i_2 : s; k_1, \ldots, k_{N-s}), \quad i_1 = 1, \ldots, r_1, \quad i_2 = 1, \ldots, r_2,$$
$$(k_1, \ldots, k_{N-s}) \in V_N^{N-s}, \quad s = 0, \ldots, N\ ).$$

Hence our aim is to determine the distribution of the first exit time of $Z_\varepsilon(t)$ from $\langle\alpha_m\rangle$, provided that $Z_\varepsilon(o){\in}\langle\alpha_m\rangle$. It can easily be verified that the transition probabilities in any time interval (t,t+h) are the following:

$$(i_1, i_2 : s; k_1, \ldots, k_{N-s}) \xrightarrow{h}$$
$$\xrightarrow{h} \begin{cases} (j_1, i_2 : s; k_1, \ldots, k_{N-s}) & a_{i_1 j_1} h + o(h),\ i_1 \neq j_1, \\ (i_1, j_2 : s; k_1, \ldots, k_{N-s}) & b_{i_2 j_2} h + o(h),\ i_2 \neq j_2, \\ (i_1, i_2 : s+1; k_2, \ldots, k_{N-s}) & \mu_{k_1}\left(i_2, s\right) h + o(h),\ 0 \leq s \leq N-1, \\ (i_1, i_2 : s-1; k_1, \ldots, k_{N-s+1}) & \left(\lambda_{k_{N-s+1}}(i_1)/\varepsilon\right) h + o(h),\ 1 \leq s \leq N. \end{cases}$$

In addition, the sojourn time $\tau_\varepsilon\left(i_1, i_2 : s; k_1, \ldots, k_{N-s}\right)$ of $Z_\varepsilon(t)$ in state $(i_1, i_2 : s; k_1, \ldots, k_{N-s})$ is exponentially distributed with parameter

$$a_{i_1 i_1} + b_{i_2 i_2} \sum_{p \neq k_1, \ldots, k_{N-s}} \lambda_p\left(i_1\right)/\varepsilon + \mu_{k_1}\left(i_2, s\right).$$

Thus, the transition probabilities for the embedded Markov chain are

$$p_\varepsilon\left[(i_1,i_2:s;k_1,\dots,k_{N-s}),(j_1,i_2:s;k_1,\dots,k_{N-s})\right]=$$
$$=\frac{a_{i_1j_1}}{a_{i_1i_1}+b_{i_2i_2}+\sum\limits_{p\neq k_1,\dots,k_{N-s}}\lambda_p(i_1)/\varepsilon+\mu_{k_1}(i_2,s)},\quad s=0,\dots,N,$$

$$p_\varepsilon\left[(i_1,i_2:s;k_1,\dots,k_{N-s}),(i_1,j_2:s;k_1,\dots,k_{N-s})\right]=$$
$$=\frac{b_{i_2j_2}}{a_{i_1i_1}+b_{i_2i_2}+\sum\limits_{p\neq k_1,\dots,k_{N-s}}\lambda_p(i_1)/\varepsilon+\mu_{k_1}(i_2,s)},\quad s=0,\dots,N,$$

$$p_\varepsilon\left[(i_1,i_2:s;k_1,\dots,k_{N-s}),(i_1,i_2:s+1;k_2,\dots,k_{N-s})\right]=$$
$$=\frac{\mu_{k_1}(i_2,s)}{a_{i_1i_1}+b_{i_2i_2}+\sum\limits_{p\neq k_1,\dots,k_{N-s}}\lambda_p(i_1)/\varepsilon+\mu_{k_1}(i_2,s)},\quad s=0,\dots,N-1,$$

$$p_\varepsilon\left[(i_1,i_2:s;k_1,\dots,k_{N-s}),(i_1,i_2:s-1;k_1,\dots,k_{N-s+1})\right]=$$
$$=\frac{\lambda_{k_{N-s+1}}(i_1)/\varepsilon}{a_{i_1i_1}+b_{i_2i_2}+\sum\limits_{p\neq k_1,\dots,k_{N-s}}\lambda_p(i_1)/\varepsilon+\mu_{k_1}(i_2,s)},\quad s=1,\dots,N.$$

As $\varepsilon\to 0$ this implies

$$p_\varepsilon\left[(i_1,i_2:0;k_1,\dots,k_N),(j_1,i_2:0;k_1,\dots,k_N)\right]=\frac{a_{i_1j_1}}{a_{i_1i_1}+b_{i_2i_2}+\mu_{k_1}(i_2,0)},$$

$$p_\varepsilon\left[(i_1,i_2:0;k_1,\dots,k_N),(i_1,j_2:0;k_1,\dots,k_N)\right]=\frac{b_{i_1j_1}}{a_{i_1i_1}+b_{i_2i_2}+\mu_{k_1}(i_2,0)},$$

$$p_\varepsilon\left[(i_1,i_2:s;k_1,\dots,k_{N-s}),(j_1,i_2:s;k_1,\dots,k_{N-s})\right]=o(1),\quad s=1,\dots,N,$$

$$p_\varepsilon\left[(i_1,i_2:s;k_1,\dots,k_{N-s}),(i_1,j_2:s;k_1,\dots,k_{N-s})\right]=o(1),\quad s=1,\dots,N,$$

$$p_\varepsilon\left[(i_1,i_2:0;k_1,\dots,k_N),(i_1,i_2:1;k_2,\dots,k_N)\right]=\frac{\mu_{k_1}(i_2,0)}{a_{i_1i_1}+b_{i_2i_2}+\mu_{k_1}(i_2,s)},$$

$$p_\varepsilon\left[(i_1,i_2:s;k_1,\dots,k_{N-s}),(i_1,i_2:s+1;k_2,\dots,k_{N-s})\right]=$$
$$=\frac{\mu_{k_1}(i_2,s)\,\varepsilon}{\sum\limits_{p\neq k_1,\dots,k_{N-s}}\lambda_p(i_1)}(1+o(1)),\quad s=1,\dots,N-1.$$

This agrees with the conditions 1–4, but here the zero level is the set

$$((i_1, i_2 : s; k_1, \ldots, k_{N-s}),\ i_1 = 1, \ldots, r_1,\ i_2 = 1, \ldots, r_2\ ,\\ \left(k_1\ , \ldots, k_{N-s}\right) \in V_N^{N-s},\ s = 0, 1\right),$$

and the q-th level is the set

$$((i_1, i_2 : q+1; k_1, \ldots, k_{N-q-1}),\ i_1 = 1, \ldots, r_1,\ i_2 = 1, \ldots, r_2\ ,\\ (k_1, \ldots, k_{N-q-1}) \in V_N^{N-q-1}).$$

Since the level 0 in the limit forms an essential class, the probabilities $\pi_0\left(i_1, i_2 : 0; k_1, \ldots, k_N\right)$, $\pi_0\left(i_1, i_2 : 1; k_1, \ldots, k_{N-1}\right)$, $i_1 = 1, \ldots, r_1$, $i_2 = 1, \ldots, r_2$, $(k_1, \ldots, k_{N-s}) \in V_N^{N-s}$, $s = 0, 1$ satisfy the following system of equations

$$\begin{aligned} &\pi_0\left(j_1, j_2 : 0; k_1, \ldots, k_N\right) = \\ &\quad= \sum_{i_1 \neq j_1} \pi_0\left(i_1, j_2 : 0; k_1, \ldots, k_N\right) a_{i_1 j_1} / \left[a_{i_1 i_1} + b_{j_2 j_2} + \mu_{k_1}\left(j_2, 0\right)\right] + \\ &\quad+ \sum_{i_2 \neq j_2} \pi_0\left(j_1, i_2 : 0; k_1, \ldots, k_N\right) b_{i_2 j_2} / \left[a_{j_1 j_1} + b_{i_2 i_2} + \mu_{k_1}\left(i_2, 0\right)\right] + \\ &\quad+ \pi_0\left(j_1, j_2 : 1; k_1, \ldots, k_{N-1}\right), \end{aligned} \tag{2}$$

$$\begin{aligned} &\pi_0\left(j_1, j_2 : 1; k_1, \ldots, k_{N-1}\right) = \pi_0\left(j_1, j_2 : 0; k_N, k_1, \ldots, k_{N-1}\right) \times \\ &\times \mu_{k_N}\left(j_2, 0\right) / \left[a_{j_1 j_1} + b_{j_2 j_2} + \mu_{k_N}\left(j_2, 0\right)\right]. \end{aligned} \tag{3}$$

To apply (1) we need the solution of (2),(3) with normalizing condition

$$\sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \sum_{(k_1, \ldots, k_N)} \{\pi_0\left(i_1, i_2 : 0; k_1, \ldots, k_N\right) + \\ + \pi_0\left(i_1, i_2 : 1; k_1, \ldots, k_{N-1}\right)\} = 1.$$

Suppose that we have this solution. Then by substituting it into (1) we get

$$\begin{aligned} &g\left(\langle \alpha_m \rangle\right) = \\ &= \varepsilon^m \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \sum_{(k_1, \ldots, k_N) \in V_N^N} \pi_0\left(i_1, i_2 : 1; k_2, \ldots, k_N\right) \frac{\mu_{k_2}\left(i_2, 1\right)}{\lambda_{k_1}\left(i_1\right)} \times \\ &\times \frac{\mu_{k_3}\left(i_2, 2\right)}{\lambda_{k_1}\left(i_1\right) + \lambda_{k_2}\left(i_1\right)} \times \cdots \times \frac{\mu_{k_{m+1}}\left(i_2, m\right)}{\lambda_{k_1}\left(i_1\right) + \cdots + \lambda_{k_m}\left(i_1\right)} (1 + o(1)). \end{aligned} \tag{4}$$

Taking into account the exponentiality of $\tau_\varepsilon\left(j_1, j_2 : s; k_1, \ldots, k_{N-s}\right)$ for fixed θ we have

$$
\begin{aligned}
&E \exp\left\{i\varepsilon\theta\tau_\varepsilon\left(j_1, j_2 : 0; k_1, \ldots, k_N\right)\right\} = \\
&\qquad = 1 + \varepsilon^m \frac{i\theta}{a_{j_1 j_1} + b_{j_2 j_2} + \mu_{k_1}\left(j_2, 0\right)}\left(1 + o(1)\right), \\
&E \exp\left\{i\varepsilon^m\theta\tau_\varepsilon\left(j_1, j_2 : s; k_1, \ldots, k_{N-s}\right)\right\} = 1 + o\left(\varepsilon^m\right), \quad s > 0.
\end{aligned}
$$

Notice that $\beta_\varepsilon = \varepsilon^m$ and therefore from Corollary 1 we immediately get that $\varepsilon^m\Omega_\varepsilon(m)$ converges weakly to an exponentially distributed random variable with parameter

$$
\begin{aligned}
\Lambda = &\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\sum_{(k_1,\ldots,k_N)\in V_N^N} \pi_0\left(i_1, i_2 : 1; k_2, \ldots, k_N\right)\frac{\mu_{k_2}\left(i_2, 1\right)}{\lambda_{k_1}\left(i_1\right)} \times \\
&\times \frac{\mu_{k_3}\left(i_2, 2\right)}{\lambda_{k_1}\left(i_1\right) + \lambda_{k_2}\left(i_1\right)} \times \cdots \times \frac{\mu_{k_{m+1}}\left(i_2, m\right)}{\lambda_k\left(i_1\right) + \cdots + \lambda_{k_m}\left(i_1\right)}\frac{1}{D},
\end{aligned}
\tag{5}
$$

which completes the proof.

However, if $\mu_p\left(i_2, s\right) = \mu\left(i_2, s\right)$, $p = 1, \ldots, N$ then by substituting (3) into (2) we get

$$
\begin{aligned}
&\pi_0\left(j_1, j_2 : 0; k_1, \ldots, k_N\right) = \\
&= \sum_{i_1 \neq j_1} \pi_0\left(i_1, j_2 : 0; k_1, \ldots, k_N\right) a_{i_1 j_1} / \left[a_{i_1 i_1} + b_{j_2 j_2} + \mu\left(j_2, 0\right)\right] + \\
&+ \sum_{i_2 \neq j_2} \pi_0\left(j_1, i_2 : 0; k_1, \ldots, k_N\right) b_{i_2 j_2} / \left[a_{j_1 j_1} + b_{i_2 i_2} + \mu\left(i_2, 0\right)\right] + \\
&+ \pi_0\left(j_1, j_2 : 0; k_N, k_1, \ldots, k_{N-1}\right)\mu\left(j_2, 0\right) / \left[a_{j_1 j_1} + b_{j_2 j_2} + \mu\left(j_2, 0\right)\right].
\end{aligned}
\tag{6}
$$

Denote by $(\pi_{i_1}^{(1)},\ i_1 = 1, \ldots, r_1)$, $(\pi_{i_2}^{(2)},\ i_2 = 1, \ldots, r_2)$ the steady-state distribution of the governing Markov chains $(\xi_1(t), t\geq 0)$, $(\xi_2(t), t\geq 0)$, respectively. Clearly,

$$
\pi_{j_1}^{(1)} a_{j_1 j_1} = \sum_{i_1 \neq j_1} \pi_{i_1}^{(1)} a_{i_1 j_1}, \quad \pi_{j_2}^{(2)} b_{j_2 j_2} = \sum_{i_2 \neq j_2} \pi_{i_2}^{(2)} b_{i_2 j_2}.
\tag{7}
$$

It can easily be verified that the solution of (6) together with (7) is

$$
\begin{aligned}
\pi_0\left(i_1, i_2 : 0; k_1, \ldots, k_N\right) &= B\pi_{i_1}^{(1)}\pi_{i_2}^{(2)}\left(a_{i_1 i_1} + b_{i_2 i_2} + \mu\left(i_2, 0\right)\right), \\
\pi_0\left(i_1, i_2 : 1; k_1, \ldots, k_{N-1}\right) &= B\pi_{i_1}^{(1)}\pi_{i_2}^{(2)}\mu\left(i_2, 0\right),
\end{aligned}
$$

where B is the normalizing constant, i.e.

$$1/B = N! \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \pi_{i_1}^{(1)} \pi_{i_2}^{(2)} \left(a_{i_1 i_1} + b_{i_2 i_2} + 2\mu\left(i_2, 0\right) \right).$$

Thus from (5) it follows that $\varepsilon^m \Omega_\varepsilon(m)$ converges weakly to an exponentially distributed random variable with parameter

$$\Lambda = \frac{1}{N!} \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \sum_{(k_1,\ldots,k_N)\in V_N^N} \pi_{i_1}^{(1)} \pi_{i_2}^{(2)} \mu\left(i_2, 0\right) \frac{\mu\left(i_2, 1\right)}{\lambda_{k_1}\left(i_1\right)} \times \\ \times \frac{\mu\left(i_2, 2\right)}{\lambda_{k_1}\left(i_1\right) + \lambda_{k_2}\left(i_1\right)} \times \cdots \times \frac{\mu\left(i_2, m\right)}{\lambda_{k_1}\left(i_1\right) + \cdots + \lambda_{k_m}\left(i_1\right)}.$$

Consequently, the distribution of the time while the number of active sources reaches the $(m+1)$-th level for the first time is approximated by

$$P\left(\Omega_\varepsilon(m) > t\right) = P(\varepsilon^m \Omega_\varepsilon(m) > \varepsilon^m t) \approx \exp\left(-\varepsilon^m \Lambda t\right).$$

In particular, when $m = N-1$, we get that the busy period length of the processor is asymptotically an exponentially distributed random variable with parameter

$$\varepsilon^{N-1}\Lambda = \varepsilon^{N-1} \frac{1}{N!} \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \sum_{(k_1,\ldots,k_N)\in V_N^N} \pi_{i_1}^{(1)} \pi_{i_2}^{(2)} \mu\left(i_2, 0\right) \frac{\mu\left(i_2, 1\right)}{\lambda_{k_1}\left(i_1\right)} \times \\ \times \frac{\mu\left(i_2, 2\right)}{\lambda_{k_1(i_1)} + \lambda_{k_2}\left(i_1\right)} \times \cdots \times \frac{\mu\left(i_2, N-1\right)}{\lambda_{k_1}\left(i_1\right) + \cdots + \lambda_{k_{N-1}}\left(i_1\right)}. \tag{8}$$

In the case when there are no random environments, that is $\mu\left(i_2, s\right) = \mu(s)$, $s = 0, \ldots, N-1$, $\lambda_p\left(i_1\right) = \lambda_p$, $i_1 = 1, \ldots, r_1$, $i_2 = 1, \ldots, r_2$, $p = 1, \ldots, N$ from (8) we get

$$\varepsilon^{N-1}\Lambda = \\ \varepsilon^{N-1} \frac{1}{N!} \sum_{(k_1,\ldots,k_N)\in V_N^N} \mu(0) \frac{\mu(1)}{\lambda_{k_1}} \frac{\mu(2)}{\lambda_{k_1} + \lambda_{k_2}} \times \cdots \times \frac{\mu(N-1)}{\lambda_{k_1} + \cdots + \lambda_{k_{N-1}}}. \tag{9}$$

Finally, for the totally homogeneous case and constant service rate from (9) we obtain

$$\varepsilon^{N-1}\Lambda = \frac{1}{(N-1)!} \frac{\mu^N}{(\lambda/\varepsilon)^{N-1}}. \tag{10}$$

It is clear that the mean idle period length I of the processor is

$$I = \sum_{i_1=1}^{r_1} \pi_{i_1}^{(1)} \frac{1}{\sum\limits_{p=1}^{N} \lambda_p (i_1) / \varepsilon}. \tag{11}$$

The utilization U_p of the processor, which is the long-run fraction of time during which it is busy, can be obtained by

$$U_p = \frac{1}{\varepsilon^{N-1}\Lambda} \left(\frac{1}{\varepsilon^{N-1}\Lambda} + I \right)^{-1}. \tag{12}$$

4. Numerical results

This section presents a number of validation experiments (c.f., Tables 1–8) examining the credibility of the proposed approximation against exact results for the performance measure of processor utilization at equilibrium. Note that, to best of our knowledge, an exact formula for the utilization is known only when the system is not effected by random environment and it is given (via Palm-formula) by

$$U_p^* = \frac{1}{N} \frac{\sum\limits_{k=1}^{N} \binom{N}{k} k! \rho^k}{1 + \sum\limits_{k=1}^{N} \binom{N}{k} k! \rho^k},$$

where $\rho = \frac{\lambda/\varepsilon}{\mu}$. This is the reason why there are validation experiments for this case only. Now relations (10–12) reduce to the following approximation

$$U_p = \frac{1}{N} \frac{N!}{N! + \left(\frac{\mu}{\lambda/\varepsilon} \right)^N}.$$

The numerical results can be seen in Tables 1–8. It can be observed that the approximate values for $\{U_p\}$ are very much comperable in accuracy to those provided by the exact results for $\left\{U_p^*\right\}$. However, the computational complexity, due to the proposed approximation, has been considerably reduced. As λ/ε becomes greater than μ, the $\{U_p\}$ approximations, as expected, approach the exact values of $\left\{U_p^*\right\}$. Clearly, the greater the number of processors the less number of steps are needed to reach the exact results.

Table 1

$N = 3$

ρ	U_p^*	U_p
1	0.3125	0.285714286
2	0.329113924	0.326530612
2^2	0.332657201	0.332467532
2^3	0.333237575	0.333224862
2^4	0.333320592	0.333319771
2^5	0.333333169	0.333331638
2^6	0.333333125	0.333333121
2^7	0.333333307	0.333333307
2^8	0.333333333	0.333333333

Table 2

$N = 4$

ρ	U_p^*	U_p
1	0.246153846	0.24
2	0.249605055	0.249350649
2^2	0.249968310	0.249959317
2^3	0.249997756	0.249997457
2^4	0.249999999	0.249999999
2^5	0.25	0.25

Table 3

$N = 5$

ρ	U_p^*	U_p
1	0.199386503	0.198347107
2	0.199968409	0.199947930
2^2	0.199998732	0.199998372
2^3	0.199999955	0.199999949
2^4	0.199999998	0.199999998
2^5	0.2	0.2

Table 4

$N = 6$

ρ	U_p^*	U_p
1	0.166581502	0.166435506
2	0.166664473	0.166666305
2^2	0.166666623	0.166666661
2^3	0.166666666	0.166666666

Table 5

$N = 7$

ρ	U_p^*	U_p
1	0.142846715	0.142828804
2	0.142857009	0.142856921
2^2	0.142857142	0.142857141
2^3	0.142857143	0.142857143

Table 6

$N = 8$

ρ	U_p^*	U_p
1	0.124998860	0.1249969
2	0.124999993	0.124999988
2^2	0.125	0.125

Table 7

$N = 9$

ρ	U_p^*	U_p
1	0.111110998	0.111110805
2	0.111111111	0.111111111

Table 8

$N = 10$

ρ	U_p^*	U_p
1	0.099999999	0.099999999
2	0.1	0.1

5. Concluding remarks

In this paper we have presented a queueing model to analyse the behaviour of a FCFS heterogeneous finite-source communication system with a single processor. The system operates in Markovian environments and the messages arrive fast compared to their service. An asymptotic approach has been provided to obtain the distribution of the time to the first system failure. The credibility of the method is illustrated with some simple validation experiments and favourable comparisons against exact results are made.

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References

[1] V.V. Anisimov, O.K. Zakusilo and V.S. Donchenko, Elements of queueing theory and asymptotic analysis of systems, *Visha Skola*, *Kiev*, 1987 (in Russian).

[2] V.V. Anisimov and J. Sztrik, Asymptotic analysis of some controlled finite-source queueing systems, *Acta Cybernetica* **9** (1989), 27–38.

[3] D.P. Gaver, P.A. Jacobs and G. Latouche, Finite birth-and-death models in randomly changing environments, *Advances in Applied Probability* **16** (1984), 715–731.

[4] I.B. Gertsbakh, Asymptotic methods in reliability theory; A review, *Advances in Applied Probability* **16** (1984), 147–175.

[5] I.B. Gertsbakh, Statistical Reliability Theory, *Marcel Dekker, Inc.*, *New York*, 1989.

[6] J. Keilson, Markov chain models – Rarity and exponentiality, *Sringer–Verlag*, *Berlin*, 1979.

[7] I. Mitrani, Modelling of Computer and Communication Systems, *Cambridge University Press*, *Cambridge*, 1987.

[8] M. Neuts, Matrix-Geometric Solutions in Stochastic Models — An Algorithmic Approach, *The John Hopkins Univ. Press. Baltimore, MD*, 1981.

[9] P. Purdue, The M/M/1 Queue in a Markovian Environment, *Operations Research* **22** (1974), 562–569.

[10] C. Rosenberg, R. Mazumdar and L. Kleinrock, On the analysis of Exponential Queueing Systems with Randomly Changing Arrival Rates: Stability Conditions and Finite Buffer Scheme with a Resume Level, *Performance Evaluation* **11** (1990), 283–293.

[11] T.E. Stern and A.I. Elwalid, Analysis of Separable Markov-modulated Rate models for Information-handling Systems, *Advances in Applied Probability* **23** (1991), 105–139.

[12] J. Sztrik and L. Lukashuk, Modelling of a Communication System Evolving in a Random Environment, *Acta Cybernetica* **10** (1991), 85–91.

[13] J. Sztrik, Modelling of heterogeneous multiprocessor systems with randomly changing parameters, *Acta Cybernetica* **10** (1991), 71–84.

[14] H. Takagi, Bibliography on Performance Evaluation, Stochastic Analysis of Computer and Communication Systems, H. Takagi (Editor), IFIP 1990, *North-Holland*, *Amsterdam*, 1990.

[15] U. Yechiali, A Queueing-Type Birth-and-Death Process Defined on a Continuous-Time Markov Chain, *Operations Research* **21** (1973), 604–609.

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