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A locally symmetric pseudo-Riemannian structure on the tangent bundle

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Dedicated to Professor V. Oproiu on the occasion of his 60-th birthday

Abstract. We consider a certain pseudo-Riemannian metric G on the tangent bundle TM of a Riemannian manifold (M, g) and obtain necessary and sufficient conditions for the pseudo-Riemannian manifold (TM, G) to be a locally symmetric space.

1. Introduction

The tangent bundle TM of a Riemannian manifold (M, g) can be organized as a Riemannian or a pseudo-Riemannian manifold in many ways. The most known such structures are given by the Sasaki metric (see [14], [1], [7]) and the complete lift type pseudo-Riemannian metrics (see [15], [16], [8]), both defined on TM with the help of g. It is also known that the tangent bundle TM of a Riemannian manifold (M, g) has a structure of almost Kaehlerian manifold with the Sasaki metric and an almost complex structure defined by the splitting of the tangent bundle to TM into the vertical and horizontal distributions VTM, HTM (the last one being determined by the Levi Civita connection on M). However, this strucure is Kaehler only in the case where the base manifold is locally Euclidean ([14], [1]). Note also that the Sasaki metric on TM is locally symmetric if and only if the base manifold is locally Euclidean. The Sasaki metric

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is rather rigid and it should be interesting to get another Riemannian or pseudo-Riemannian metrics on TM, having some better properties. One possibility is to consider some (pseudo-)Riemannian metrics involving the natural lifts of the Riemannian metric g on M (for the definition and the expression of the natural 1-st order lifts of the Riemannian metric g to TM see [4], [5], [3]).

A particular case of such a Riemannian metric which can be considered is a slight generalization of the Sasaki metric and it has been studied by V. Oproiu in [9]. The Riemannian metric G on TM considered in [9] has been defined by using the Levi Civita connection of the Riemannian metric g on M and two smooth real valued functions $u(t)$, $v(t)$ depending on the energy density only and such that $u(t) > 0$ and $u(t) + 2tv(t) > 0$ for all $t \in [0, \infty)$. Remark that the Sasaki metric can be obtained from G in the case where $u(t) = 1$, $v(t) = 0$ for all $t \in [0, \infty)$. Next, V. Oproiu has considered an almost complex structure J on TM, related to the metric G and has studied the conditions under which (TM, G, J) is a Kaehler–Einstein manifold. Note that in [9], the author excludes some important cases which appeared, in a certain sense, as singular cases. Note also that one of the cases excluded in [9] is when the Riemannian metric G on TM is defined by using a certain Lagrangian L on the base manifold (M, g) depending on the energy density only (i.e. the case when $v(t) = u'(t)$). This important case has been studied by V. Oproiu and the present author in [11]. In [12], considering on TM the Riemannian metric G defined in [9], V. Oproiu and the present author have studied the conditions under which the Riemannian manifold (TM, G) or a tube around the zero section in the Riemannian manifold (TM, G) is a locally symmetric manifold, without using the almost complex structure J. Roughly speaking, if (TM, G) or a tube around the zero section in TM is a locally symmetric space, then (M, g) must have constant sectional curvature c and TM must carry a Kaehler–Einstein structure. Note that in the singular case studied in [11] when $v(t) = u'(t)$, the Riemannian manifold (TM, G) cannot be a locally symmetric space. On the other hand, in [8], V. Oproiu has defined and studied a pseudo-Riemannian structure on the tangent bundle of a Lagrange manifold M, considering the pseudo-Riemannian metric G on TM as being the complete lift of a quadratic form defined by the considered Lagrangian L. Such a pseudo-Riemannian metric defines a pairing between the vertical and horizontal distributions VTM and HTM. Remark that this metric has signature (n, n) and the distributions VTM, HTM are isotropic. Inspired from [9] and [8], in [13], we have considered another special natural 1-st order lift G of g which defines a pseudo-Riemannian metric on TM (so that, in general, G is no longer obtained as the complete lift by using a Lagrangian on M). This new metric G has been defined by using also the Levi Civita connection of the Riemannian metric g and two smooth real valued functions $u(t)$, $v(t)$ such that $u(t) > 0$ and $u(t) + 2tv(t) > 0$ for all $t \in [0, \infty)$. Next we have studied necessary and sufficient conditions for the pseudo-Riemannian manifold (TM, G) to be Ricci flat.

In the present note we study other geometric properties of the pseudo-Riemannian metric G defined in [13]. In fact we study necessary and sufficient conditions for the pseudo-Riemannian manifold (TM, G) to be a locally symmetric space. The obtained main result is: The pseudo-Riemannian manifold (TM, G) is a locally symmetric space if and only if the base manifold (M, g) is a locally symmetric space and the functions u and v which appear in the expression of G are related by $v = u'$ (Theorem 2). By using some known results from the Lagrange geometry (see [2], [8], [11]), it is shown that the condition $v = u'$ is equivalent to the fact that the considered pseudo-Riemannian metric G on TM is the complete lift of a quadratic form defined by the special Lagrangian L on M considered in [11] (Corollary 3).

The manifolds, tensor fields and geometric objects we consider in this paper, are assumed to be differentiable of class C^∞ (i.e. smooth). The well known summation convention is used throughout this paper, the range for the indices i, j, k, l, h, s, t, a, b being always$\{1, \dots, n\}$. We shall denote by $\Gamma(TM)$ the module of smooth vector fields on TM.

2. The pseudo-Riemannian metric G on TM

Let (M, g) be a smooth n-dimensional Riemannian manifold, $n > 1$, and denote its tangent bundle by $\tau : TM \longrightarrow M$. Recall that TM has a structure of $2n$-dimensional smooth manifold induced from the smooth manifold structure of M. A local chart $(U, \varphi) = (U, x^1, \dots, x^n)$ of M induces a local chart $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^n,\ y^1, \dots, y^n)$ of TM where the local coordinates x^i, y^i; $i = 1, \dots, n$ are defined as follows. The first n local coordinates $x^i = x^i \circ \tau$; $i = 1, \dots, n$ of $y \in \tau^{-1}(U)$ are the local coordinates in the local chart (U, φ) of the base point $\tau(y) \in U$ and the last n local coordinates y^i; $i = 1, \ldots, n$ are the vector space coordinates of the same tangent vector y with respect to the natural local basis $\frac{\partial}{\partial x^i}$; $i = 1, \ldots, n$ defined by (U, φ).

This special structure of TM allows us to introduce the notion of M-tensor field on it (see [6]). An M-tensor field of type (p, q) on TM is defined by sets of functions

$$T^{i_1 \ldots i_p}_{j_1 \ldots j_q}(x, y);\ i_1, \ldots, i_p, j_1, \ldots, j_q = 1, \ldots, n$$

assigned to any induced local chart $(\tau^{-1}(U), \Phi)$ on TM, such that the change rule is that of the components of a tensor field of type (p, q) on the base manifold, when a change of local charts on the base manifold is performed. We shall use currently M-tensor fields defined on TM. Remark also that any ordinary tensor field on the base manifold may be thought of as an M-tensor field on TM, having the same type and with the components in the induced local chart on TM, equal to the local coordinate components of the given tensor field in the chosen local chart on the base manifold.

Recall that the Levi Civita connection $\dot{\nabla}$ of g defines a direct sum decomposition

$$TTM = VTM \oplus HTM$$

of the tangent bundle to TM into the vertical distribution $VTM = \operatorname{Ker} \tau_*$ and the horizontal distribution HTM. The vector fields $(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n})$ define a local frame field for VTM, and for HTM we have the local frame field $(\frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n})$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{i0} \frac{\partial}{\partial y^h}; \qquad \Gamma^h_{i0} = \Gamma^h_{ik} y^k$$

and $\Gamma^h_{ik}(x)$ are the Christoffel symbols of g.

The distributions VTM and HTM are isomorphic to each other and it is possible to derive an almost complex structure on TM which, together with the Sasaki metric, determines a structure of almost Kaehlerian manifold on TM (see [1], [16]).

Consider now the energy density (kinetic energy)

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2} g_{\tau(y)}(y, y) = \frac{1}{2} g_{ik}(x) y^i y^k, \quad y \in \tau^{-1}(U)$$

of the tangent vector y, where g_{ik} are the components of g in the local chart (U,φ). Let $u,v:[0,\infty)\longrightarrow\mathbb{R}$ be two real smooth functions such that $u(t)>0$ and $u(t)+2tv(t)>0$ for all $t\in[0,\infty)$. Then we may consider the following symmetric M-tensor field of type (0,2) on TM, defined by the components (see [9])

$$G_{ij}=u(t)g_{ij}+v(t)g_{0i}g_{0j},$$

where $g_{0i}=g_{hi}y^h$. The matrix (G_{ij}) is symmetric and positive definite and has the inverse with the entries

$$H^{kl}=\frac{1}{u}g^{kl}-\frac{v}{u(u+2tv)}y^ky^l,$$

where g^{kl} are the components of the inverse of the matrix (g_{ij}). The components $H^{kl}(x,y)$ define a symmetric M-tensor field of type $(2,0)$ on TM. We shall use also the components $H_{ij}(x,y)$ of a symmetric M-tensor field of type $(0,2)$ obtained from H^{kl} by "lowering" the indices, i.e.

$$H_{ij}=g_{ik}H^{kl}g_{lj}=\frac{1}{u}g_{ij}-\frac{v}{u(u+2tv)}g_{0i}g_{0j}.$$

In the same vein we shall use the M-tensor fields on TM defined by the components

$$G^{kl}=g^{ki}G_{ij}g^{jl},\quad G^i_k=G^{ih}g_{hk},\quad H^i_k=H^{ih}g_{hk}$$

and remark that the matrix (H^i_k) is the inverse of the matrix (G^i_k) (see [9]).

The following pseudo-Riemannian metric of natural 1-st order lift type may be considered on TM (see [13])

$$G=2G_{ij}\dot{\nabla}y^idx^j=2(ug_{ij}+vg_{0i}g_{0j})\dot{\nabla}y^idx^j, \tag{1}$$

where $\dot{\nabla}y^i=dy^i+\Gamma^i_{j0}dx^j$ is the absolute differential of y^i with respect to the Levi Civita connection $\dot{\nabla}$ of g. Equivalently, we have

$$G\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)=0,\quad G\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)=0,$$

$$G\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right)=G\left(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^i}\right)=G_{ij}.$$

We note that, due to the above conditions satisfied by the functions $u(t)$, $v(t)$, the pseudo-Riemannian metric G defined by (1) is balanced, i.e. has the signature (n,n), and both distributions VTM, HTM are isotropic. Remark also that, in the particular case when the components G_{ij} from the expression (1) of G are obtained as in usual Lagrange geometry from a Lagrangian L on M, it follows that G coincides with the pseudo-Riemannian metric studied by V. Oproiu in [8]. In the case where $u(t)=1$, $v(t)=0$ for all $t\in[0,\infty)$ we have $G=g^c$, where g^c denotes the complete lift of the Riemannian metric g on M.

Observe that the system of 1-forms $(dx^1,\dots,dx^n,\dot\nabla y^1,\dots,\dot\nabla y^n)$ defines a local frame of T^*TM, dual to the local frame $(\frac{\delta}{\delta x^1},\dots,\frac{\delta}{\delta x^n},\frac{\partial}{\partial y^1},\dots$ $\dots,\frac{\partial}{\partial y^n})$.

In the following we determine the Levi Civita connection ∇ of the pseudo-Riemannian metric G on TM, where G is defined by (1). To do this we need the following well known formulas for the brackets of the vector fields $\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^i}$; $i=1,\dots,n$:

$$\left[\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right]=0,\quad \left[\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right]=-\Gamma^h_{ij}\frac{\partial}{\partial y^h},\quad \left[\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right]=-R^h_{0ij}\frac{\partial}{\partial y^h},$$

where $R^h_{0ij}=R^h_{kij}y^k$ and R^h_{kij} are the local coordinate components of the curvature tensor field of $\dot\nabla$ on M.

Recall that the Levi Civita connection ∇ on the pseudo-Riemannian manifold (TM,G) is obtained from the formula

$$\begin{aligned}2G(\nabla_XY,Z)&=X(G(Y,Z))+Y(G(X,Z))-Z(G(X,Y))+G([X,Y],Z)\\&\quad-G([X,Z],Y)-G([Y,Z],X);\quad \forall X,Y,Z\in\Gamma(TM).\end{aligned}$$

Theorem 1. *Let (M,g) be a Riemannian manifold. Then the Levi Civita connection ∇ of the pseudo-Riemannian manifold (TM,G) has the following expression in the local adapted frame $\big(\frac{\partial}{\partial y^1},\dots,\frac{\partial}{\partial y^n},\frac{\delta}{\delta x^1},\dots,\frac{\delta}{\delta x^n}\big)$:*

$$\nabla_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}=Q^h_{ij}\frac{\partial}{\partial y^h};\quad \nabla_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j}=\Gamma^h_{ij}\frac{\partial}{\partial y^h}+P^h_{ji}\frac{\delta}{\delta x^h};$$

$$\nabla_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j}=P^h_{ij}\frac{\delta}{\delta x^h};\quad \nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j}=\Gamma^h_{ij}\frac{\delta}{\delta x^h}+S^h_{ij}\frac{\partial}{\partial y^h},$$

where the M-tensor fields $P^h_{ij}, Q^h_{ij}, S^h_{ij}$ are given by:

$$P^h_{ij} = \frac{u'-v}{2u}\left(g_{0i}\delta^h_j - \frac{u}{u+2tv}g_{ij}y^h - \frac{v}{u+2tv}g_{0i}g_{0j}y^h\right),$$

$$Q^h_{ij} = \frac{u'+v}{2u}(g_{0i}\delta^h_j + g_{0j}\delta^h_i) + \frac{v}{u+2tv}g_{ij}y^h + \frac{v'u-u'v-v^2}{u(u+2tv)}g_{0i}g_{0j}y^h,$$

$$S^h_{ij} = g^{hk}R_{0ikj} + \frac{v}{u+2tv}R_{0ij0}y^h,$$

R_{likj} denoting the local coordinate components of the Riemann–Christoffel tensor of $\dot\nabla$ on M and $R_{0ikj} = R_{likj}y^l$, $R_{0ij0} = R_{lijk}y^l y^k$.

3. A locally symmetric structure on (TM, G)

In this section we shall study the necessary and sufficient conditions under which the pseudo-Riemannian manifold (TM, G) is a locally symmetric space. The curvature tensor field K of the Levi Civita connection ∇ of the pseudo-Riemannian metric G on TM is defined by well known formula

$$K(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \quad X,Y,Z \in \Gamma(TM).$$

The formal expression of K in the local frame $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$, $i = 1,\dots,n$, is obtained by a straightforward computation

$$(2)\qquad \begin{aligned} K\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} &= XXX^h_{kij}\frac{\delta}{\delta x^h} \\ &\quad + y^l\left(\dot\nabla_l R^h_{kij} + \frac{v}{u+2tv}\dot\nabla_l R_{k0ij}y^h\right)\frac{\partial}{\partial y^h}, \\ K\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k} &= XXY^h_{kij}\frac{\partial}{\partial y^h}, \\ K\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)\frac{\delta}{\delta x^k} &= YYX^h_{kij}\frac{\delta}{\delta x^h}, \\ K\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^k} &= YYY^h_{kij}\frac{\partial}{\partial y^h}, \end{aligned}$$

$$K\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} = YXX^h_{kij}\frac{\partial}{\partial y^h},$$

(2)

$$K\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k} = YXY^h_{kij}\frac{\delta}{\delta x^h},$$

where the components $XXX^h_{kij}, \ldots$ define M-tensor fields on TM and have the following expressions

$$XXX^h_{kij} = R^h_{kij} + \frac{u'-v}{2(u+2tv)}(R_{0j0k}\delta^h_i - R_{0i0k}\delta^h_j),$$

$$\begin{aligned} XXY^h_{kij} &= R^h_{kij} + \frac{v}{u}g_{0k}R^h_{0ij} - \frac{v}{u+2tv}R_{0kij}y^h \\ &- \frac{u'-v}{2(u+2tv)}(g_{kj}R^h_{00i} - g_{ki}R^h_{00j}) \\ &- \frac{v(u'-v)}{2u(u+2tv)}(g_{0j}g_{0k}R^h_{00i} - g_{0i}g_{0k}R^h_{00j}), \end{aligned}$$

(3)

$$\begin{aligned} YYX^h_{kij} &= \frac{\alpha - 2uv(u'-v)}{4u(u+2tv)^2}(g_{0j}g_{ik} - g_{0i}g_{jk})y^h \\ &+ \frac{u'-v}{2(u+2tv)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i) \\ &+ \frac{v(u'-v)}{2u(u+2tv)}(g_{0i}g_{0k}\delta^h_j - g_{0j}g_{0k}\delta^h_i), \end{aligned}$$

$$\begin{aligned} YYY^h_{kij} &= \frac{\alpha}{4u^2(u+2tv)}(g_{0i}g_{0k}\delta^h_j - g_{0j}g_{0k}\delta^h_i) \\ &+ \frac{u'-v}{2(u+2tv)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i), \end{aligned}$$

$$\begin{aligned} YXX^h_{kij} &= R^h_{kij} + \frac{v}{u}g_{0i}R^h_{k0j} + \frac{v^2}{u(u+2tv)}g_{0i}R_{0jk0}y^h \\ &- \frac{v}{u+2tv}R_{0kij}y^h - \frac{u'-v}{2(u+2tv)}R_{0jk0}\delta^h_i \\ &+ \frac{u'-v}{2(u+2tv)}g_{ik}R^h_{00j} + \frac{v(u'-v)}{2u(u+2tv)}g_{0i}g_{0k}R^h_{00j}, \end{aligned}$$

$$
\begin{aligned}
YXY^h_{kij} = {} & \frac{u'-v}{2(u+2tv)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i) \\
& + \frac{v(u'-v)}{2u(u+2tv)}(g_{0i}g_{0k}\delta^h_j - g_{0j}g_{0k}\delta^h_i) + \frac{\alpha - 2uv(u'-v)}{4u(u+2tv)} \\
& \times \left[\frac{1}{u}g_{0i}g_{0k}\delta^h_j - \frac{1}{u+2tv}g_{0i}g_{jk}y^h - \frac{v}{u(u+2tv)}g_{0i}g_{0j}g_{0k}y^h\right]
\end{aligned} \tag{3}
$$

and where we have denoted

$$\alpha = 2u(u+2tv)u'' - 3u(u')^2 - 2u^2v' + 3uv^2 - 4tuu'v' - 2t(u')^2v + 2tv^3. \tag{4}$$

In the following we shall study the conditions under which the pseudo-Riemannian manifold (TM, G) is a locally symmetric space i.e.

$$\nabla K = 0.$$

First of all, by using the expressions of the components of K given by (2) and (3), by computing $(\nabla_{\frac{\delta}{\delta x^l}} K)\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^k}$ we obtain

$$\left(\nabla_{\frac{\delta}{\delta x^l}} K\right)\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^k} = XYYY^h_{lkij}\frac{\delta}{\delta x^h},$$

where $XYYY^h_{lkij}$ are the components of an M-tensor field of type (1,4) on TM, given by

$$XYYY^h_{lkij} = \frac{(u'-v)^2}{4u(u+2tv)}(g_{0j}\delta^h_i - g_{0i}\delta^h_j)(g_{lk} - g_{0l}g_{0k}). \tag{5}$$

It can be checked that the last two factors from the expresion (5) of $XYYY^h_{lkij}$ cannot vanish for all tangent vector $y \in \tau^{-1}(U)$ (see [10]). Thus we may conclude that the relation $(\nabla_{\frac{\delta}{\delta x^l}} K)(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k} = 0$ is equivalent to the condition

$$v = u'. \tag{6}$$

From now on we shall assume that the functions u and v satisfy the condition (6).

Remark. It is obvious that the condition (6) implies $\alpha = 0$, where α is given by (4). Then, from the condition (6) we obtain that the expression (2) of the curvature tensor field K becomes simpler

$$\begin{cases} K\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} = R^h_{kij}\frac{\delta}{\delta x^h} + y^l\left(\dot{\nabla}_l R^h_{kij} + \frac{u'}{u+2tu'}\dot{\nabla}_l R_{k0ij}y^h\right)\frac{\partial}{\partial y^h}, \\ K\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k} = \left[R^h_{kij} + \frac{u'}{u}g_{0k}R^h_{0ij} - \frac{u'}{u+2tu'}R_{0kij}y^h\right]\frac{\partial}{\partial y^h}, \\ K\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)\frac{\delta}{\delta x^k} = 0, \\ K\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^k} = 0, \\ K\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} = \left[R^h_{kij} + \frac{u'}{u}g_{0i}R^h_{k0j} + \frac{(u')^2}{u(u+2tu')}g_{0i}R_{0jk0}y^h \right. \\ \qquad \left. - \frac{u'}{u+2tu'}R_{0kij}y^h\right]\frac{\partial}{\partial y^h}, \\ K\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k} = 0. \end{cases}$$

In order to obtain the other covariant derivatives of K, we shall denote, for convenience,

$$B^h_{kij} = y^l\left(\dot{\nabla}_l R^h_{kij} + \frac{u'}{u+2tu'}\dot{\nabla}_l R_{k0ij}y^h\right)$$

and

$$\overline{\nabla}_l B^h_{kij} = \frac{\delta B^h_{kij}}{\delta x^l} - \Gamma^s_{lk}B^h_{sij} - \Gamma^s_{li}B^h_{ksj} - \Gamma^s_{lj}B^h_{kis} + \Gamma^h_{ls}B^s_{kij}.$$

By using these notations, we get by a straightforward computation that only the following four covariant derivatives of K are not identical zero:

$$\begin{aligned} \left(\nabla_{\frac{\delta}{\delta x^l}}K\right)\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} &= \dot{\nabla}_l R^h_{kij}\frac{\delta}{\delta x^h} + \Big[\overline{\nabla}_l B^h_{kij} + y^t\left(R^a_{stl}R^s_{kij}\right. \\ &\left. - R^a_{ksj}R^s_{itl} + R^a_{ksi}R^s_{jtl} - R^a_{sij}R^s_{ktl}\right)\left(\delta^h_a - \frac{u'}{u+2tu'}g_{0a}y^h\right)\Big]\frac{\partial}{\partial y^h}, \end{aligned} \tag{7}$$

$$(7)\qquad \begin{aligned}
\left(\nabla_{\frac{\partial}{\partial y^l}}K\right)\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} &= \Big[\dot{\nabla}_l R^h_{kij}+\frac{u'}{u}g_{0l}B^h_{kij}\\
&\qquad +\frac{u'}{u+2tu'}\dot{\nabla}_l R_{k0ij}y^h\Big]\frac{\partial}{\partial y^h},\\
\left(\nabla_{\frac{\delta}{\delta x^l}}K\right)\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k} &= \Big[\dot{\nabla}_l R^h_{kij}+\frac{u'}{u}g_{0k}\dot{\nabla}_l R^h_{0ij}\\
&\qquad -\frac{u'}{u+2tu'}\dot{\nabla}_l R_{0kij}y^h\Big]\frac{\partial}{\partial y^h},\\
\left(\nabla_{\frac{\delta}{\delta x^l}}K\right)\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} &= \Big[\dot{\nabla}_l R^h_{kij}+\frac{u'}{u}g_{0i}\dot{\nabla}_l R^h_{k0j}\\
&\quad -\frac{u'}{u+2tu'}\dot{\nabla}_l R_{0kij}y^h+\frac{(u')^2}{u(u+2tu')}g_{0i}\dot{\nabla}_l R_{0jk0}y^h\Big]\frac{\partial}{\partial y^h}.
\end{aligned}$$

From a careful examination of the formulas (7), it follows that the pseudo-Riemannian manifold (TM,G) is locally symmetric if and only if the following conditions are satisfied

$$(8)\qquad \begin{cases} (i) & \dot{\nabla}_l R^h_{kij}=0,\\ (ii) & R^a_{stl}R^s_{kij}-R^a_{ksj}R^s_{itl}+R^a_{ksi}R^s_{jtl}-R^a_{sij}R^s_{ktl}=0. \end{cases}$$

Due to the Ricci identity for the curvature tensor field R^h_{kij} we get that the condition (8)(ii) is satisfied whenever the condition (8)(i) is satisfied.

Thus, we obtain the main result of this paper:

Theorem 2. *Let (M,g) be a Riemannian manifold and consider the pseudo-Riemannian manifold (TM,G), where G is given by (1). Then (TM,G) is a locally symmetric space if and only if the functions $u(t)$, $v(t)$ satisfy the condition $v=u'$ and the base manifold (M,g) is a locally symmetric space.*

Finally, we describe a geometric interpretation of the condition $v=u'$ in the expression (1) of the pseudo-Riemannian metric G by using a Lagrangian function $L:TM\longrightarrow\mathbb{R}$ defined by

$$(9)\qquad L=\int u(t)dt,$$

where $u:\mathbb{R}_+=[0,\infty)\longrightarrow\mathbb{R}$ is a smooth function such that $u(t)>0$, $\forall\, t\geq 0$ (see [11]). As in usual Lagrange geometry (see [2], [7], [8]), we obtain the symmetric M-tensor field of type $(0,2)$ on TM, defined by the components

$$G_{ij}=\frac{\partial^2 L}{\partial y^i \partial y^j}=ug_{ij}+u' g_{0i}g_{0j}. \tag{10}$$

Hence it follows that the condition (6) is fulfilled if and only if G_{ij} are obtained from the Lagrangian L defined by (9).

Remarks. (i) In [11], V. Oproiu and the present author have proved that the usual nonlinear connection determined by the Euler–Lagrange equations asociated to the Lagrangian L defined by (9) coincides with the nonlinear connection defined by the Levi Civita connection $\dot\nabla$ of g (see Proposition 1 from [11]).

(ii) Taking into account remark *(i)*, it follows that the condition $v=u'$ in the expression (1) of G is equivalent to the fact that the pseudo-Riemannian metric G defined on TM coincides with the pseudo-Riemannian metric h^c, where h^c is the complete lift of the quadratic form $h=G_{ij}(x,y)dx^i dx^j$ and where G_{ij} are defined by (10) (see [8]). In the particular case when $u(t)=1$ for all $t\in[0,\infty)$ we have $G=g^c$, where g^c is the complete lift of the Riemannian metric g on M.

(iii) By using the results obtained by V. Oproiu in [8] and the above remarks, the pseudo-Riemannian metric $G=2G_{ij}\dot\nabla y^i dx^j$, where G_{ij} are defined by (10) (i.e. G is the complete lift of the quadratic form $h=G_{ij}dx^i dx^j$), we see that (TM,G) is a locally symmetric space if and only if the base manifold (M,g) is a locally symmetric space. In particular, the pseudo-Riemannian manifold (TM,g^c) is a locally symmetric space if and only if (M,g) is a locally symmetric space.

From the above remarks it follows that Theorem 2 can be formulated by

Corollary 3. *Let (M,g) be a Riemannian manifold and consider the pseudo-Riemannian manifold (TM,G), where G is defined by* (1)*. Then (TM,G) is a locally symmetric space if and only if the base manifold (M,g) is a locally symmetric space and the pseudo-Riemannian metric G is the complete lift of the quadratic form $h=G_{ij}(x,y)dx^i dx^j$, where G_{ij} are obtained as in usual Lagrange geometry by* (10)*, considering on the base manifold the Lagrangian L defined by* (9)*.*

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