Publ. Math. Debrecen
59 / 3-4 (2001), 317–326

Filling space with cubes of two sizes

By ATTILA BÖLCSKEI (Budapest)

Abstract. The problem to classify the unilateral and equitransitive tilings of the plane by squares of different sizes has been revived in the last few years ([6], [1], [8]). The analogous problem in three-dimensional space seems to be more difficult and has not been investigated so far systematically ([2], [3]). In this paper we prove that there is only one unilateral tiling of $\mathbb{R}^3$ by cubes of two sizes and that is necessarily equitransitive. Finally we describe the maximal crystallographic group the tiling is equipped with.

1. Introduction

The investigation of unilateral and equitransitive tilings of $\mathbb{R}^2$ by squares of three different sizes has been revived in the last few years. After the constructions of D. Schattschneider [4, p. 76], Martini, Makai and Soltan [6] and B. Grünbaum [6], the classification problem was finally solved in [1] and in [8], by different methods.

A similar question in three-dimensional space has not been investigated yet. The papers [2] and [3] contain some results on whether the space can be filled with cubes so that no neighbouring cubes have the same edge-length and they shortly cite the construction of Rogers filling the space using cubes of two sizes only.

Our purpose is to describe *all* the possible unilateral, and then necessarily equitransitive, tilings of the three-dimensional space with cubes of *just two different sizes* using combinatorial and crystallographic methods.

For the purpose of classification, two tilings (T, Γ) and (T', Γ'), with corresponding symmetry groups Γ and Γ', respectively, will be considered

Mathematics Subject Classification: 52C22.
Key words and phrases: tiling of the space by cubes, unilaterality, equitransitivity, crystallography.

equivalent if they are topologically equivariant (homeomeric). It means that there exists a homeomorphism ψ that maps T onto T' (preserving tiles, faces, edges and vertices) such that $\psi\Gamma\psi^{-1} = \Gamma'$.

The main statement of the the paper will be

Theorem 1. *Using only cubes of two sizes and having no two cubes with a face in common, there is only one way to tile space. It is known as Rogers filling.*

Remark 1. In the literature this type of tiling is known as *unilateral* ([4], [6], [1]).

Remark 2. We say that a tiling is *equitransitive* if for any two congruent tiles S and S' there is a congruence transform of the space which maps S onto S' in such a manner, that the whole tiling is mapped onto itself. The filling of Theorem 1 has this property.

Based on this result and Theorem 2 (see below) we formulate a

Conjecture. In every dimension d there exists precisely one equitransitive unilateral tiling by $d-$dimensional cubes of two sizes.

Throughout the paper we use the following notation for squares and cubes. The sizes of the objects we distinguish denoting the smaller ones by λ_1 and the bigger ones by λ_2. Individual objects will be denoted by capital letters: $A, B, C \dots$ Furthermore, we use the method of Dawson for identifying the faces, edges and vertices of a cube. Namely, we characterize the faces by indices

- u-meaning the upper
- d-meaning the downmost
- l-meaning the left
- r-meaning the right
- b-meaning the back
- f-meaning the front

face of the cube as e.g. $A_u, A_d \dots.$ The edges can be represented by double, the vertices by triple indices in a unique way.

2. Non-edge-to-edge planar arrangements by squares

Before the spatial case we deal with arrangements in the plane by squares of two sizes. The following assertion just repeats the well-known fact (see [4] and Figure 1) that

Theorem 2. *In the plane there exists only one unilateral and equitransitive tiling by squares of two different sizes.*

Figure 1

Figure 2

From Figure 1 we can easily read off the unique neighbourhood of a λ_1- and of a λ_2-type square.

We can observe that if we do not require equitransitivity the λ_1-environment has to be the one above. Namely, if X has a λ_1-neighbour, then the other neighbours are uniquely determined as shown in Figure 2. But the last neighbour has again to be a λ_1-square, which contradicts the unilaterality condition.

In this way, if we require non-edge-to-edge configuration just at three sides, there are three possible arrangements, shown in Figures 2 and 3.

Figure 3 *Figure 4*

Moreover, if unilaterality holds just at two sides, the arrangements are those in Figure 4.

3. Environments of λ_i-cubes in a unilateral tiling

In this section we focus our attention on three-dimensional unilateral tilings not necessarily equitransitive. First we prove an important lemma in three parts.

Lemma 1. *For the mutual position of neighboring λ_1- and λ_2-cubes there is only one possibility in the unilateral tiling of cubes of two sizes shown in Figure 5.*

Proof. We conduct the proof by excluding the other possibilities as follows.

1. There does not exist such a unilateral construction, where a smaller cube, X, would stand in the interior of a face of a bigger one (on A_u).

   Namely, suppose on the contrary that such an arrangement exists. In this case the common neighbours of X and A would form a non-edge-to-edge tiling of squares of two sizes locally on A_u. Then the environment of the λ_1-square consists of λ_2-squares only. Thus the faces of X are adjacent to λ_2-cubes except for X_u. But the only possible neighbour at this face would be a λ_1-cube, which contradicts the unilaterality.

Figure 5 *Figure 6*

2. There does not exist such a unilateral tiling, where the mutual position of the cubes is that shown in Figure 6.

   If there existed such a tiling, the neighbours of X would form a non-edge-to-edge environment of X_d just at three sides on A_u. But again, each of the three allowed neighbourhoods would cause that at X_u we are able to border only with another λ_1-cube, a contradiction.

3. There does not exist such a unilateral construction, where the mutual position of the cubes would be that in Figure 7 by projecting them parallelly e.g. with edge A_{rf}.

Figure 7 *Figure 8*

Let us suppose the contrary. Consider first Figure 8. Here we depicted the arrangement arising when the neighbourhood of X_d is of the first type from among the three ones in Figure 4. The other neighbours of X are then strictly determined: now we have to put a λ_2-cube to X_u, next we are forced to put another λ_2-cube to X_r. But in this way we necessarily get a gap between the last cube and A, shown in Figure 8, right. Therefore, this construction is not permitted, neither the other two possible neighbourhoods of Figure 4 because of similar argumentation. □

Figure 8b *Figure 9*

Now we can formulate our observations about the environments of the cubes of different types. First let us consider a λ_1-cube.

Lemma 2. *The environment of a cube of λ_1-type is uniquely determined up to an isometry.*

Proof. In Figure 5 we have depicted a λ_1-cube X and its λ_2-neighbour called A. Now we consider the common neighbours of X and A. We have a planar arrangement of squares where unilaterality at two sides is needed. The only possibility is just the first case in Figure 4 up to an isometry. This is true because we have to set smaller squares to a corner which is obviously impossible for the other cases. In this way we are forced to build two λ_2 cubes to X_l and X_b respectively, and the arrangement is straightforward (see Figure 9). There is another possible solution with starting configuration which is just the reflected image of the former one that refers to the tiling on A_u, changing the orientation. □

For the purpose of algebraic description we introduce a coordinate system with O as origo, OI, OJ and OK as axis and with the lengths $a =$

OI, $b = OE$ (Figure 5). The coordinates of the centers of the central λ_1-cube are $(\frac{a}{2}, \frac{a}{2}, \frac{a}{2})$, the centers of the surrounding λ_2-cubes are $(\frac{b}{2}, \frac{b}{2}, -\frac{b}{2})$, $(a+\frac{b}{2}, a-\frac{b}{2}, \frac{b}{2})$, $(\frac{b}{2}, a+\frac{b}{2}, \frac{b}{2})$, $(-\frac{b}{2}, \frac{b}{2}, a-\frac{b}{2})$, $(a-\frac{b}{2}, -\frac{b}{2}, a-\frac{b}{2})$, $(a-\frac{b}{2}, a-\frac{b}{2}, a+\frac{b}{2})$.

Surprisingly, a similar statement is true for λ_2-cubes:

Lemma 3. *The environment of a cube of λ_2-type is uniquely determined up to an isometry.*

Proof. In Figure 10 we can see the previously discussed arrangement of a λ_1-cube but from the opposite direction. (To simplify the figure we omitted the cube D.) Our aim is to form the neighbours of A. We assert that either A_l or A_b borders a λ_1-square. This is necessary because we cannot cover these faces with λ_2-cubes only by avoiding common faces. The smaller cube has to join either the corner $A_{lb} - A_{ul}$ or $A_{rb} - A_{ub}$.

Figure 10

Indeed, first consider the face A_l. The two corners above are not proper because the gap between them and B_d allows at most a λ_1-square in contradiction with the previous lemma. By a similar reasoning, the corner A_{ufl} is not good either.

The only remaining possibility is A_{lbu}. Indeed, we can build the neighbourhood of A beginning at this corner with a cube of λ_1-type. After forming its environment, we can easily see that we have to continue on A_b at A_{rbu}. Building up its environment we are forced to set the next smaller cube and the other ones step by step in a similar way. The neighbours of A are uniquely determined.

If we start at A_b, our only chance is indeed the corner A_{rbu}. Namely, if we put λ_1-cubes to the other corners, we cannot form their environments just by overlapping. But if we put a cube to A_{rbu} we can build the environment in a unique way as above. Our solution will be the same as the former one.

Of course another congruent solution comes into consideration as with the previous lemma. □

The analytic description of the environment is the following: the center of the base λ_2-cube is located in $(\frac{b}{2}, \frac{b}{2}, -\frac{b}{2})$.

The centers of the λ_1-cubes are: $(\frac{a}{2}, \frac{a}{2}, \frac{a}{2})$, $(b - \frac{a}{2}, b + \frac{a}{2}, -\frac{a}{2})$, $(b + \frac{a}{2}, \frac{a}{2}, -\frac{a}{2})$, $(\frac{a}{2}, -\frac{a}{2}, -b + \frac{a}{2})$, $(-\frac{a}{2}, b - \frac{a}{2}, -b + \frac{a}{2})$, $(b - \frac{a}{2}, b - \frac{a}{2}, -b - \frac{a}{2})$.

The centers of the λ_2-cubes are: $(a + \frac{b}{2}, a - \frac{b}{2}, \frac{b}{2})$, $(\frac{b}{2}, a + \frac{b}{2}, \frac{b}{2})$, $(-\frac{b}{2}, \frac{b}{2}, a - \frac{b}{2})$, $(a - \frac{b}{2}, -\frac{b}{2}, a - \frac{b}{2})$, $(-a + \frac{b}{2}, -a + \frac{3b}{2}, -\frac{3b}{2})$, $(\frac{b}{2}, -a + \frac{b}{2}, -\frac{3b}{2})$, $(\frac{3b}{2}, \frac{b}{2}, -a - \frac{b}{2})$, $(-a + \frac{3b}{2}, \frac{3b}{2}, -a - \frac{b}{2})$, $(\frac{3b}{2}, a + \frac{b}{2}, -a + \frac{b}{2})$, $(-\frac{b}{2}, -a + \frac{b}{2}, a - \frac{3b}{2})$, $(a + \frac{b}{2}, -\frac{b}{2}, -\frac{b}{2})$, $(-a + \frac{b}{2}, \frac{3b}{2}, -\frac{b}{2})$.

Having proved the lemmas our Theorem 1 simply follows.

Proof of Theorem 1. The statement is just a simple consequence of the uniqueness of the environments above. If we start either with a cube of λ_1 or of λ_2-type the neighbours of it are determined as the further neighbourhoods. In addition, the isometry, which may occur at the local environments, preserves topological equivariance. This tiling is known as Rogers filling. □

4. The maximal symmetry group

If we consider the configuration of cubes in the filling we see that there are many symmetries of the space which map the tiling onto itself. E.g. the translations in three directions that move the centers of small cubes to the centers of any other small cubes form a space group 1. (**P1** of [5].) In order to determine the corresponding **maximal** symmetry group we use the analytic description.

Firstly, we take a fundamental domain and give the face identifications due to the Poincaré algorithm (for more information see [7]). In Figure 11

Figure 11

a fundamental domain with our base configuration is drawn. The generators are the following:

$$z_1 : LHJ \to JKL$$

$$z_2 : EKD \to DGE$$

$$s_1 : GJHE \to DTSK$$

$$s_2 : HLK \to GJT$$

We can give the analytic forms of the transformations as usual in crystallography [5] by homogeneous coordinates according to Figure 5 as $OI = \boldsymbol{i}$, $OJ = \boldsymbol{j}$, $OK = \boldsymbol{k}$:

$$\boldsymbol{Z}_1 = \begin{bmatrix} 0 & 0 & -1 & a \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{Z}_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -b \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{S}_1 = \begin{bmatrix} 0 & 0 & 1 & \frac{2b}{3} \\ -1 & 0 & 0 & \frac{4b}{3} \\ 0 & -1 & 0 & -\frac{b}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{S}_2 = \begin{bmatrix} 0 & -1 & 0 & a+b \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first two generators are rotoinversions ($\overline{3}$), the last ones are screw motions of order three (3_1). The axes are parallel. These transformations generate the corresponding crystallographic group which is 148. ($R\overline{3}$ in [5].) This is the maximal symmetry group of the tiling.

References

[1] A. Bölcskei, Classification of unilateral and equitransitive tilings by squares of three sizes, Beiträge zur Algebra und Geometrie (Contributions to Algebra and Geometry) **41** no. 1 (2000), 267–277.

[2] R. J. M. Dawson, On filling space with different integer cubes, *J. Combin. Theory* A **36** (1984), 221–229.

[3] D. E. Daykin, Space filling with unequal cubes, *Bull. Amer. Math. Soc.* **70** (1964), 340–341.

[4] B. Grünbaum and G. C. Shepard, Tilings and Patterns, *W. H. Freeman* (1987), 72–81.

[5] N. F. M. Henry and K. Lonsdale, International Tables for *X*-ray Crystallography, Symmetry Groups, Vol. 1, *Kynoch Press*, *Birmingham*, 1969; New edition by Theo Hahn, Vol. A, *Reidel Co*, *Dordrecht,* 1983.

[6] H. Martini, E. Makai and V. Soltan, Unilateral Tilings of the Plane with Squares of Three Sizes, *Beiträge zur Algebra und Geometrie (Contributions to Algebra and Geometry)* **39** (1998), 481–495.

[7] E. Molnár, Polyhedron complexes with simply transitive group actions and their realizations, *Acta Math. Hung* **59** (1–2) (1992), 175–216.

[8] D. Schattschneider, Unilateral and Equitransitive Tilings by Squares, *Discrete Comput Geom* **24** (2000), 519–525.

ATTILA BÖLCSKEI
DEPARTMENT OF GEOMETRY
TU BUDAPEST
H–1521 BUDAPEST
HUNGARY

E-mail: bolcskei@math.bme.hu

(Received September 11, 2000; revised January 26, 2001)