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Generalizations of jets and reflections in the theory of connections

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Abstract. Classical holonomic jets are generalized to nonholonomic jets and furthermore, to quasijets, which are in fact certain vector bundle morphisms. Higher order connections are viewed as sections of r-th order jet prolongation of a fibered manifold. This approach is prolonged to quasijets and obtained quasiconnections are studied.

1. Introduction

In this paper, we present the influence of a development of the jet theory upon the modern theory of connections. We recall that the general connection is defined as a section of the first jet prolongation of an arbitrary fibered manifold. Analogously, the r-th order connection is defined as a section of the r-th jet prolongation, preferably the nonholonomic one. We recommend the monograph of Kolář, Michor and Slovák, [8], for a self-contained introduction to jet theory. [3]–[7] and [12]–[14] are the fundamental papers dealing with nonholonomic jets and higher order connections.

The generalization of jets to quasijets is very natural. The concept of a quasijet was introduced in [9]. Quasijets are investigated in papers of Dekrét, first of all in [2]. The recent paper [11] uses modern language of Weil bundles in this investigation. Surely, quasijets may play an important role in the study of several geometrical objects. In the paper, we introduce

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the investigation of sections of the r-th order quasijet prolongation of a fibered manifold. These sections are not studied up to now. We show that they generalize the higher order connections by a natural way and we shall call them quasiconnections. Especially, we take the properties of the product of higher order connections into account of this generalization.

As to the product of higher order connections, the following result is more or less known (see e.g. [1]): if the product is a holonomic connection, then the factors must be holonomic and, moreover, satisfying a certain conditions expressible in the first order case of them as the vanishing of curvatures. We introduce the product of quasiconnections. It represents a nonholonomic connection if and only if the factors are also nonholonomic connections.

The paper has the following structure. In Section 2 we recall the basic facts from the theory of jets. Section 3 is devoted to quasijets: the starting point is the paper [2] and we contribute to it by slight generalizations and simplifications. Section 4 deals with connections, we evaluate the explicit formulae of the product of connections in Lemma 1 and we precise the conditions under which the product represents a holonomic connection in Proposition 4. Section 5 contains quite original results: we discuss the basic properties of sections of higher order quasijet prolongations by analogy with the Section 4. In addition, we find one-to-one correspondence between second order quasiconnections and couples of nonholonomic connections.

We remind that we do not study the important subbundle of semi-holonomic jets in this paper.

2. Jets

Holonomic jets

Roughly speaking, (holonomic) jets are certain equivalence classes of smooth maps between manifolds, which are represented by Taylor polynomials. We make this idea precise. Two maps $f, g : M \to N$ are said to *determine the same r-jet* at $x \in M$, if for every curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = x$ the curves $f \circ \gamma$ and $g \circ \gamma$ have the r-th order contact at zero. In such a case we write $j^r_x f = j^r_x g$ and an equivalence class of this relation is called an *r-jet* of M into N. The set of all r-jets of M into N is denote by $J^r(M, N)$. If the *source* of a r-jet is $x \in M$ and the *target* of this jet is $\bar{x} = f(x) \in N$, then $\alpha := j^r_x f \mapsto x$ and $\beta := j^r_x f \mapsto \bar{x}$ are projections of fibered manifolds $\alpha : J^r(M,N) \to M$, $\beta : J^r(M,N) \to N$. Further, by $J^r_x(M,N)$ or $J^r(M,N)_{\bar{x}}$ we mean the set of all r-jets of M into N with source $x \in M$ or target $\bar{x} \in N$, respectively, and we write $J^r_x(M,N)_{\bar{x}} = J^r_x(M,N) \cap J^r(M,N)_{\bar{x}}$. As r-th order contact of maps is preserved under composition, we define the composition of r-jets as the r-jet of the composed map.

Let $p : Y \to M$ be a fibered manifold. The set J^rY of all r-jets of the local sections of Y is called the *r-th jet prolongation of Y* and $J^rY \subset J^r(M,Y)$ is a closed submanifold. (If $Y \to M$ is a vector bundle, then $J^rY \to M$ is also a vector bundle.) Let $q : Z \to N$ be another fibered manifold and $f : Y \to Z$ a fibered bundle morphism with the property that the base map $f_0 : M \to N$ is a local diffeomorphism. There is an induced map $J^r(f_0, f)(X) := j^r_{\beta(X)} f \circ X \circ j^r_{f_0(\alpha(X))} f_0^{-1}$ for $X \in J^r(M,Y)$. If we restrict it to local sections, we obtain a map denoted by $J^rf : J^rY \to J^rZ$ which is called the *r-th jet prolongation of f*.

Nonholonomic jets

For $r = 1$, the set of nonholonomic 1-jets $\tilde{J}^1(M,N) := J^1(M,N)$. By induction, let $\alpha : \tilde{J}^{r-1}(M,N) \to M$ denote the source projection and $\beta : \tilde{J}^{r-1}(M,N) \to N$ the target projection of $(r-1)$-th nonholonomic jets. Then X is said to be a *nonholonomic r-jet* with the source $x \in M$ and the target $\bar{x} \in N$, if there is a local section $\sigma : M \to \tilde{J}^{r-1}(M,N)$ such that $X = j^1_x \sigma$ and $\beta(\sigma(x)) = \bar{x}$. There is a natural embedding $J^r(M,N) \subset \tilde{J}^r(M,N)$.

Every $X \in \tilde{J}^r(M,N)$ induces a map $\mu X : (\underbrace{T \dots T}_{r\text{-times}} M)_x \to (\underbrace{T \dots T}_{r\text{-times}} N)_{\bar{x}}$ in the following way. For $r = 1$ and $X = j^1_x f$ is μX defined as $T_x f$. By induction, let $X = j^1_x \sigma$ for a local α-section $\sigma : M \to \tilde{J}^{r-1}(M,N)$. Then $\sigma(u) \in J^{r-1}_u(M,N)$, $\mu(\sigma(u)) : (\underbrace{T \dots T}_{r-1\text{-times}} M)_u \to (\underbrace{T \dots T}_{r-1\text{-times}} N)_{\beta(\sigma(u))}$ and we put $\mu X = T_x \mu(\sigma(u))$. The constructed map $\mu X : (\underbrace{T \dots T}_{r\text{-times}} M)_x \to (\underbrace{T \dots T}_{r\text{-times}} N)_{\bar{x}}$ is a vector bundle morphism with respect to all vector bundle structures $\underbrace{T \dots T}_{r\text{-times}} \to \underbrace{T \dots T}_{r-1\text{-times}}$. However, μX is not an entirely general vector bundle morphism of this type.

Let $p : Y \to M$ be a fibered manifold. We construct the *r–th jet nonholonomic prolongation of* Y denoted by $\tilde{J}^r Y$ as the set of all nonholonomic r-jets of the local sections of Y. The construction of the *r-th jet nonholonomic prolongation of* f for a fibered bundle morphism $f : Y \to Z$ with the property that the base map $f_0 : M \to N$ is a local diffeomorphism is analogous to the holonomic case.

Local expressions

Let x^i are local coordinates on M, $i = 1, \ldots, m = \dim M$, $\bar{x}^p$ local coordinates on N, $p = 1, \ldots, n = \dim N$. On $J^r(M, N)$ we have local coordinates x^i, $\bar{x}^p$ and the induced coordinates $a^p_{i_1 \ldots i_q}$, $q = 1, \ldots, r$, $i_1, \ldots, i_q = 1, \ldots, m$. The induced coordinates are symmetric in all subscripts. On $\tilde{J}^r(M, N)$ we have local coordinates x^i, $\bar{x}^p$ and the induced coordinates $b^p_{i_1 \ldots i_r}$, $i_1, \ldots, i_r = 0, 1, \ldots, m$, which are not symmetric in the subscripts. Let us realize that we obtain the same coordinate descriptions also in the cases of $J^r Y$ or $\tilde{J}^r Y$, respectively.

If the induced coordinates of a nonholonomic jet are invariant with respect to any transposition of subscripts, we have in fact a holonomic jet. We realize the transfer from $b^p_{i_1 \ldots i_r}$ to $a^p_{i_1 \ldots i_q}$ by deleting of all zeros in subscripts.

3. Quasijets

The iterated tangent bundle

We introduce the following denotation of projections in the iterated tangent bundle $\underbrace{T \ldots T}_{r\text{-times}} M$. For every s, $0 < s \leq r$, we denote by $\pi^s : \underbrace{T \ldots T}_{s\text{-times}} M \to M$ the canonical projection to the base. Further, we denote $\pi^s_b := \pi^s_{\underbrace{T \ldots\ T}_{b\text{–times}} M} : \tilde{T}^s(\underbrace{T \ldots T}_{b\text{-times}} M) \to \underbrace{T \ldots T}_{b\text{-times}} M$ projection with $\underbrace{T \ldots T}_{b\text{-times}} M$ as the base space, ${}_a\pi^s := \underbrace{T \ldots T}_{a\text{-times}} \pi^s : \underbrace{T \ldots T}_{a\text{-times}}(\underbrace{T \ldots T}_{s\text{-times}} M) \to \underbrace{T \ldots T}_{a\text{-times}} M$ induced projection originating by the posterior application of the functor $\underbrace{T \ldots T}_{a\text{-times}}$, ${}_a\pi^s_b := \underbrace{T \ldots T}_{a\text{-times}} \pi^s_{\underbrace{T \ldots\ T}_{b\text{–times}} M}$ the general case containing both previous cases. If a or b equal zero, we do not write them. We recall the concept of the kernel injection. For a vector bundle $q : E \to M$ we have two vector bundle structures on TE, namely $p : TE \to E$ and $Tq : TE \to TM$. The *heart* HE of the vector bundle $E \to M$ is defined as the vector bundle $Vp \cap VTq \to M$. We can identify E with HE and there is the canonical *kernel injection* $\iota : E \approx HE \to TE$.

Now we take ${}_a\pi^1_b : \underbrace{T \dots T}_{s\text{-times}} M \to \underbrace{T \dots T}_{s-1\text{-times}} M$ in the role of the mentioned vector bundle and we denote the kernel injection by ${}_{a+1}\iota_b : \underbrace{T \dots T}_{s\text{-times}} M \to \underbrace{T \dots T}_{s+1\text{-times}} M$.

The definition and the basic properties of quasijets

Let $x \in M$, $\bar{x} \in N$. A map $\phi : (\underbrace{T \dots T}_{r\text{-times}} M)_x \to (\underbrace{T \dots T}_{r\text{-times}} N)_{\bar{x}}$ is said to be a *quasijet* of order r with the source x and the target $\bar{x}$, if it is a vector bundle morphism with respect to all vector bundle structures ${}_a\pi^1_b : (\underbrace{T \dots T}_{r\text{-times}} M)_x \to (\underbrace{T \dots T}_{r-1\text{-times}} M)_x$ and ${}_a\pi^1_b : (\underbrace{T \dots T}_{r\text{-times}} N)_{\bar{x}} \to (\underbrace{T \dots T}_{r-1\text{-times}} N)_{\bar{x}}$, $a + b = r - 1$. The set of all such quasijets is denoted by $QJ^r_x(M, N)_{\bar{x}}$ and $QJ^r(M, N)$ means the set of all quasijets from M to N.

There is a bundle structure $QJ^r(M, N) \to M \times N$ and, analogously to J^r, the set QJ^rY of all r-jets of the local sections of a fibered manifold $Y \to M$ Y is called the *r-th quasijet prolongation of* Y. We compose quasijets as maps. Further, let $q : Z \to N$ be another fibered manifold and $f : Y \to Z$ a fibered bundle morphism with the property that the base map $f_0 : M \to N$ is a local diffeomorphism. There is an induced map $QJ^r(f_0, f)(X) := j^r_{\beta(X)} f \circ X \circ j^r_{f_0(\alpha(X))} f_0^{-1}$ for $X \in QJ^r(M, Y)$. The composition denoted the composition of quasijets, where the holonomic jets $j^r_{\beta(X)} f$, $j^r_{f_0(\alpha(X))} f_0^{-1}$ are considered as quasijets by the use of the map μ from Section 2. If we confine ourselves to local sections, we obtain a map denoted by $QJ^r f : QJ^rY \to QJ^rZ$ which is called the *r-th quasijet prolongation of* f.

For every s, $0 < s < r$ and for every $a, b \geq 0$ such that $a + b + s = r$, we denote by ${}_a\phi^s_b$ the underlying map with respect to the projection ${}_a\pi^s_b : \underbrace{T \dots T}_{r\text{-times}} \to \underbrace{T \dots T}_{r-s\text{-times}}$, i.e. ${}_a\phi^s_b \circ {}_a\pi^s_b = {}_a\pi^s_b \circ \phi$.

Proposition 1. ${}_a\phi^s_b \in QJ^{r-s}_x(M, N)_{\bar{x}}$.

Proof. The assertion is only a slight generalization of Proposition 1 in [2], where it is formulated and proved for the case $s = 1$. Certainly, it is sufficient to decompose the projection ${}_a\pi^s_b$ to projections $\underbrace{T\dots T}_{r\text{-times}} \to \underbrace{T\dots T}_{r-1\text{-times}}, \dots, \underbrace{T\dots T}_{r-s+1\text{-times}} \to \underbrace{T\dots T}_{r-s\text{-times}}$ and applicate the mentioned proposition s-times. $\square$

For every s, $0 < s < r$ and for every $a, b \geq 0$ such that $a + b + s = r$, we define ${}^{r-s-1}_{a}\psi_b := (\underbrace{T\dots T}_{r-s-1\text{-times}}\ {}_a\iota_b)^{-1} \circ \phi \circ \underbrace{T\dots T}_{r-s-1\text{-times}}\ {}_a\iota_b$. The following proposition was proved in [2].

Proposition 2. ${}^{r-s-1}_{a}\psi_b \in QJ^{r-1}_x(M,N)_{\bar{x}}$.

We have the induced projections of quasijet spaces now. We define ${}_a\Pi^s_b : QJ^r(M,N) \to QJ^{r-s}(M,N)$ by ${}_a\Pi^s_b(\phi) = {}_a\phi^s_b$ and ${}^{r-s-1}_{a}\Pi_b : QJ^r(M,N)QJ^{r-1}(M,N)$ by ${}^{r-s-1}_{a}\Pi_b(\phi) = {}^{r-s-1}_{a}\psi_b$. Now, we are able to describe a wide class of subbundles of $QJ^r(M,N)$ by identifications of some of these projections. We intend to determine the most significant subbundle now. We know, that nonholonomic jets can be viewed as special quasijets by use μ, cf. Section 2. The following proposition was also proved in [2].

Proposition 3. *ϕ represents a nonholonomic jet if and only if*

$$ {}_a\Pi^1_{r-a-1} = {}^{a}_{h}\Pi_{r-a-h-1} \tag{1} $$

for all $a = 0, \dots, r-2$ and all $h = 1, \dots, r-a-1$.

Remark 1. The canonical involution in the second order case $\kappa : TTM \to TTM$ is well known. We remark that we obtain holonomic jets simply by generalized involutions.

Local expressions

If x^i are local coordinates on M, $i = 1, \dots, m = \dim M$, $\bar{x}^p$ local coordinates on N, $p = 1, \dots, n = \dim N$, then x^i, $\bar{x}^p$ and the induced coordinates $c^{p\gamma_1\dots\gamma_q}_{i_1\dots i_q}$, $q = 1, \dots, r$, $i_1, \dots, i_q = 1, \dots, m$, where $\gamma_1, \dots, \gamma_q$ are multiindexes of the length r with elements from $\{0,1\}$ representing an ordered decomposition of every multiindex of this form. The ordering of them respects the rule of the increasing number of left zeros in $\gamma_1, \dots, \gamma_q$.

If the condition (1) from Proposition 3 comes true, we obtain the identification of coordinates, in which it turns out that only first units in multiindexes $\gamma_1, \dots, \gamma_q$ (in each of them) of them are essential. So, we remove all units except the first ones in multiindexes and we obtain only one unit in each multiindex in this way. We replace the remaining units by related $i_1, \dots, i_q$ and take the sum of these multiindexes. This is the way of the transfer from quasijet local coordinates to the nonholonomic ones.

Example. For example, we take $r = 3$. If we have local coordinates on $TTTM$ as x^i, $y^i_{100} = y^i$, $y^i_{010} = X^i$, $y^i_{110} = Y^i$, $y^i_{001} = \xi^i$, $y^i_{101} = \eta^i$, $y^i_{011} = \Xi^i$, $y^i_{111} = H^i$ and on $TTTN$ as $\bar{x}^p$, $\bar{y}^p_{100} = \bar{y}^p$, $\bar{y}^p_{010} = \bar{X}^p$, $\bar{y}^p_{110} = \bar{Y}^p$, $\bar{y}^p_{001} = \bar{\xi}^p$, $\bar{y}^p_{101} = \bar{\eta}^p$, $\bar{y}^p_{011} = \bar{\Xi}^p$, $\bar{y}^p_{111} = \bar{H}^p$, a quasijet $\phi : TTTM \to TTTN$ has the coordinate expression $\bar{x}^p = f^p(x^i)$ and, in fiber coordinates,

$$
\begin{aligned}
\bar{y}^p &= c_i^{p100} y^i \\
\bar{X}^p &= c_i^{p010} X^i \\
\bar{Y}^p &= c_{ij}^{p100\,010} y^i X^j + c_i^{p110} Y^i \\
\bar{\xi}^p &= c_i^{p001} \xi^i \\
\bar{\eta}^p &= c_{ij}^{p100\,001} y^i \xi^j + c_i^{p101} \eta^i \\
\bar{\Xi}^p &= c_{ij}^{p010\,001} X^i \xi^j + c_i^{p011} \Xi^i \\
\bar{H}^p &= c_{ijk}^{p100\,010\,001} y^i X^j \xi^k + c_{ij}^{p101\,010} \eta^i X^j \\
&\quad + c_{ij}^{p100\,011} y^i \Xi^j + c_{ij}^{p110\,001} Y^i \xi^j + c_i^{p111} H^i .
\end{aligned}
$$

In view of the Proposition 3 ϕ represents a nonholonomic jet if and only if

$$
\begin{aligned}
\Pi_2^1 &= {}_1\Pi_1 = {}_2\Pi \\
{}_1\Pi_1^1 &= {}_1^1\Pi \,.
\end{aligned}
$$

It means the identification of the quasijet coordinates

$$
\begin{aligned}
c_i^{p100} &= c_i^{p110} = c_i^{p101} = c_i^{p111} \\
c_i^{p010} &= c_i^{p011}
\end{aligned}
$$

$$c_{ij}^{p100\,010} = c_{ij}^{p101\,010} = c_{ij}^{p100\,011}$$
$$c_{ij}^{p100\,001} = c_{ij}^{p110\,001},$$

and we see that we make do only with coordinates c_i^{p100}, c_i^{p010}, $c_{ij}^{p100\,010}$, c_i^{p001}, $c_{ij}^{p100\,001}$, $c_{ij}^{p010\,001}$, $c_{ijk}^{p100\,010\,001}$. They represent nonholonomic local coordinates b_{i00}^p, b_{0i0}^p, b_{ij0}^p, b_{00i}^p, b_{i0j}^p, b_{0ij}^p, b_{ijk}^p, respectively.

Moreover, ϕ represents a holonomic jet if and only if the induced coordinates of a nonholonomic jet are invariant with respect to any transposition of subscripts. It means the identification of the nonholonomic coordinates

$$b_{i00}^p = b_{0i0}^p = b_{00i}^p$$
$$b_{ij0}^p = b_{i0j}^p = b_{0ij}^p.$$

By deleting of zeros in subscripts, we obtain the holonomic coordinates a_i^p, a_{ij}^p, a_{ijk}^p (symmetric in subscripts).

Remark 2. We remark that further generalization of the concept of quasijets does not carry an enrichment to the theory. If we consider a map $\phi : (\underbrace{T_k^1 \dots T_k^1}_{r\text{-times}} M)_x \to (\underbrace{T_k^1 \dots T_k^1}_{r\text{-times}} N)_{\bar{x}}$ in the definition of the generalized quasijet (T_k^1 is the functor of k-dimensional 1-velocities, $T_1^1 = T$), we obtain the same coordinate expression of it only with matrix multiindices. It holds even for a map $\phi : (\underbrace{T_{k_1}^1 \dots T_{k_r}^1}_{r\text{-times}} M)_x \to (\underbrace{T_{k_1}^1 \dots T_{k_r}^1}_{r\text{-times}} N)_{\bar{x}}$.

4. Connections

A *nonholonomic r-th order connection* in a fibered manifold $p : Y \to M$ is a section $\Gamma : Y \to \tilde{J}^r Y$. Holonomic r-th order connections can be viewed as special nonholonomic ones. If Γ is a nonholonomic r-th order connection and Δ is a nonholonomic s-th order connection, we define $\Gamma * \Delta := \tilde{J}^s \Gamma \circ \Delta$. $\Gamma * \Delta$ is a nonholonomic $(r+s)$-th order connection called the *product* of Γ and Δ.

Now, we provide the coordinate expression of the product of Γ and Δ, which is necessary for practical computations and which was known only for $r = 1$ and $s = 1$ up to now. Let $j_1 \dots j_s$ is a sequence of integers, $j_h = 0, 1, \dots, m$ (i.e. we understand non-negative integers whenever write integers).

Firstly, we define the *underlying sequence of integers* $k_1 \dots k_s$ as a sequence of integers such that $k_h = j_h$ or $k_h = 0$ for $h = 0, \dots, m$ excluding $k_1 = \dots k_s = 0$. Further, we denote $l_h = j_h - k_h$ and we take from the integers $l_1, \dots, l_s$ the non-zero ones and denote them by $L_1, \dots, L_\mu$, $\mu \le s$.

We denote the choice of an underlying sequence of integers from the given sequence of integers $j_1 \dots j_s$ by ζ. Secondly, we decompose the sequence of integers $k_1 \dots k_s$ to ν sequences of integers $k_1^1 \dots k_s^1, \dots, k_1^\nu \dots k_s^\nu$ in this way:

(i) $k_h^1 + \dots + k_h^\nu = k_h$

(ii) if $k_h > 0$, then for a unique ν_0, $1 \le \nu_0 \le \nu$ is $k_h^{\nu_0} = k_h$

(iii) the ordering of sequences $k_1^1 \dots k_s^1, \dots, k_1^\nu \dots k_s^\nu$ submits to the positions of their first non-zero elements

We denote such a decomposition of the sequence of integers $k_1 \dots k_s$ by ρ. Moreover, we write shortly $\Gamma^p_{i_1 \dots i_r, j_1 \dots j_s}$ instead of $\frac{\partial^s \Gamma^p_{i_1 \dots i_r}}{\partial x^{j_1} \dots \partial x^{j_s}}$. With the use of this denotation, we obtain the following lemma by a direct evaluation.

Lemma 1. *The product $\Gamma * \Delta$ of a nonholonomic r-th order connection Γ with a nonholonomic s-th order connection Δ has the coordinate expression*

$$
(2) \quad b^p_{i_1 \dots i_r j_1 \dots j_s} = \begin{cases} \Gamma^p_{i_1 \dots i_r} & \text{for } j_1 = \dots = j_s = 0 \\ \Delta^p_{j_1 \dots j_s} & \text{for } i_1 = \dots = i_r = 0 \\ \Gamma^p_{i_1 \dots i_r, j_1 \dots j_s} + \sum\limits_\zeta \sum\limits_\rho \Gamma^p_{i_1 \dots i_r, q_1 \dots q_\nu L_1 \dots L_\mu} \Delta^{q_1}_{k_1^1 \dots k_s^1} \dots & \\ \quad \dots \Delta^{q_\nu}_{k_1^\nu \dots k_s^\nu} \text{ in another case.} & \end{cases}
$$

The product is associative and evidently non-commutative. By the derived formula, we deduce conditions under which $\Gamma * \Delta$ represents a holonomic $(r+s)$-th order connection.

Proposition 4. *If $\Gamma * \Delta$ is a holonomic connection, then Γ and Δ are holonomic and one of them is the projection of the other. For $r = s = 1$, $\Gamma * \Delta$ is a holonomic connection, if and only if $\Gamma = \Delta$ and this is a holonomic curvature-free connection.*

Proof. If $\Gamma * \Delta$ is invariant with respect to any transposition of subscripts, then $\Gamma^p_{i_1 \dots i_r}$ and $\Delta^p_{j_1 \dots j_s}$ must be invariant with respect to any transposition of subscripts (see (2)). It means Γ and Δ are holonomic. If $r \leq s$, then $a^p_{i_1\dots i_r\underbrace{0\dots0}_{s\text{-times}}} = \Gamma^p_{i_1\dots i_r} = a^p_{\underbrace{0\dots0}_{s\text{-times}} j_{s-r+1}\dots j_s} = \Delta^p_{j_{s-r+1}\dots j_s} = \Delta^p_{j_1\dots j_r}$ and Γ equals $\pi(\Delta)$, where π is the jet projection $\pi : \tilde{J}^s Y \to \tilde{J}^r Y$. The case $r \geq s$ is analogous. For $r = s = 1$, $a^p_{ij} = a^p_{ji}$ from (2) reads as

$$\Gamma^p_{i,j} + \Gamma^p_{i,q}\Gamma^q_j = \Gamma^p_{j,i} + \Gamma^p_{j,q}\Gamma^q_i.$$

The curvature C of a connection $\Gamma : Y \to J^1 Y$ is defined as $C := \frac{1}{2}[\Gamma, \Gamma]$ (Frölicher–Nijenhuis bracket) and the vanishing of C means just the evaluated equation. $\square$

5. Quasiconnections

A *r-th order quasiconnection* in a fibered manifold $p : Y \to M$ is a section $\Gamma : Y \to QJ^r Y$. As the nonholonomic jets can be viewed as special quasijets, nonholonomic r-th order connections are special quasiconnections. We revise the procedure known from Section 4. If Γ is a r-th order quasiconnection and Δ is a s-th order quasiconnection, we define $\Gamma * \Delta := QJ^s\Gamma \circ \Delta$.

Proposition 5. *$\Gamma * \Delta$ is a $(r+s)$-th order quasiconnection.*

Proof. It is necessary to realize the fact that the elements of $QJ^r Y$ can be represented as sections $M \to QJ^r Y$ equally as vector bundle morphisms (with respect to all vector bundle structures) $(\underbrace{T\dots T}_{r\text{-times}} M)_x \to (\underbrace{T\dots T}_{r\text{-times}} Y)_{\bar{x}}$. In other words, there is one-to-one correspondence between such sections and such vector bundle morphisms. That is why there is one-to-one correspondence between induced iterated tangent maps $\underbrace{T\dots T}_{s\text{-times}} M \to \underbrace{T\dots T}_{s\text{-times}} QJ^r Y$ and $(\underbrace{T\dots T}_{s\text{-times}}\underbrace{T\dots T}_{r\text{-times}} M)_x \to (\underbrace{T\dots T}_{s\text{-times}}\underbrace{T\dots T}_{r\text{-times}} Y)_{\bar{x}}$, which enables to see the elements of $QJ^s(QJ^r Y)$ as the elements of $QJ^{r+s} Y$. Of course, this proves the claim. $\square$

We call $\Gamma * \Delta$ the *product* of Γ and Δ. The definition of the product of quasiconnections extends naturally the concept of product introduced in the Section 4. We intend to express the product of Γ and Δ in local coordinates. For multiindices γ_i, $i = 1, \dots, q$, we denote by $\underleftarrow{\gamma_i}$ or $\underrightarrow{\gamma_i}$ the multiindices consisted of r first elements or s last elements of γ_i, respectively.

Such coordinates $c^{p\gamma_1\dots\gamma_q}_{i_1\dots i_q}$ of $\Gamma * \Delta$, in which

$$\underleftarrow{\gamma_i} \neq \underbrace{0\dots0}_{r\text{-times}} \tag{3}$$

for all $i = 1, \dots, q$, fulfil evidently $c^{p\gamma_1\dots\gamma_q}_{i_1\dots i_q} = \Gamma^{p\underleftarrow{\gamma_1}\dots\underleftarrow{\gamma_q}}_{i_1\dots i_q}$. Similarly, such coordinates $c^{p\gamma_1\dots\gamma_q}_{i_1\dots i_q}$ of $\Gamma * \Delta$, in which

$$\underleftarrow{\gamma_i} = \underbrace{0\dots0}_{r\text{-times}} \tag{4}$$

for all $i = 1, \dots, q$, fulfil $c^{p\gamma_1\dots\gamma_q}_{i_1\dots i_q} = \Delta^{p\underrightarrow{\gamma_1}\dots\underrightarrow{\gamma_q}}_{i_1\dots i_q}$. It remains to derive a general case, i.e. to express coordinates $c^{p\gamma_1\dots\gamma_q}_{i_1\dots i_q}$ of $\Gamma * \Delta$, in which (3) holds for all $i = 1, \dots, u$, $u < q$, and (4) holds for all $i = u+1, \dots, q$. We choose an equally ordered nonempty sequence of multiindices $\delta_1 \dots \delta_w$ from the sequence of multiindices $\underrightarrow{\gamma_{u+1}} \dots \underrightarrow{\gamma_q}$, $1 \leq w \leq q$. We denote the remaining sequence of multiindices by $\epsilon_1 \dots \epsilon_{\bar{w}}$, $w + \bar{w} = q$. We denote $k_1 \dots k_w$ the lower indexes corresponding with $\delta_1 \dots \delta_w$ and $l_1 \dots l_{\bar{w}}$ the lower indices corresponding with $\epsilon_1 \dots \epsilon_{\bar{w}}$. The choice is denoted by ζ. Further, we decompose the sequence of multiindices $\delta_1 \dots \delta_w$ to ν sequences of multiindices: $\delta^1_1 \dots \delta^1_{w_1}, \dots\ \delta^\nu_1 \dots \delta^\nu_{w_\nu}$. We denote such a decomposition by ρ. With this denotation, we derive directly the following lemma.

Lemma 2. *The product $\Gamma * \Delta$ of a r-th order quasiconnection Γ with a s-th order quasiconnection Δ has the coordinate expression*

$$c^{p\gamma_1\dots\gamma_q}_{i_1\dots i_q} = \begin{cases} \Gamma^{p\underleftarrow{\gamma_1}\dots\underleftarrow{\gamma_q}}_{i_1\dots i_q} \quad \text{for } \underleftarrow{\gamma_i} \neq \underbrace{0\dots0}_{r\text{-times}},\ i = 1, \dots, q \\ \Delta^{p\underrightarrow{\gamma_1}\dots\underrightarrow{\gamma_q}}_{i_1\dots i_q} \quad \text{for } \underleftarrow{\gamma_i} = \underbrace{0\dots0}_{r\text{-times}},\ i = 1, \dots, q \\ \Gamma^{p\underleftarrow{\gamma_1}\dots\underleftarrow{\gamma_u}}_{i_1\dots i_u, i_{u+1}\dots i_q} + \sum_\zeta \sum_\rho \Gamma^{p\underleftarrow{\gamma_1}\dots\underleftarrow{\gamma_u}}_{i_1\dots i_u, q_1\dots q_\nu l_1\dots l_{\bar{w}}} \Delta^{q_1\delta^1_1\dots\delta^1_{w_1}}_{k^1_1\dots k^1_{w_1}} \dots \\ \quad \dots \Delta^{q_\nu\delta^\nu_1\dots\delta^\nu_{w_\nu}}_{k^\nu_1\dots k^\nu_{w_\nu}} \quad \text{for } \underleftarrow{\gamma_i} \neq \underbrace{0\dots0}_{r\text{-times}},\ i = 1, \dots, u \\ \quad \text{and } \underleftarrow{\gamma_i} = \underbrace{0\dots0}_{r\text{-times}},\ i = u+1, \dots, q. \end{cases} \tag{5}$$

Remark 3. The associativity and non-commutativity of the product of quasiconnections follows directly from this coordinate expression.

The following assertion completes properties of the product $\Gamma * \Delta$ of r-th order quasiconnection Γ and s-th order quasiconnection Δ and it proves that the introduced product generalizes naturally of the product of higher order connections.

Proposition 6. *$\Gamma * \Delta$ is a nonholonomic connection if and only if both Γ and Δ are nonholonomic connections.*

Proof. We suppose that $\Gamma * \Delta$ is a nonholonomic connection. It means (cf. Section 3) that we have the identification of coordinates of $\Gamma * \Delta$, in which it turns out that only first units in multiindices $\gamma_1, \dots, \gamma_q$ of them are essential. It follows that we have the identification of this type for coordinates with multiindices $\underleftarrow{\gamma_1}, \dots, \underleftarrow{\gamma_q}$ or $\underrightarrow{\gamma_1}, \dots, \underrightarrow{\gamma_q}$, in other words for coordinates of Γ or Δ, respectively and that is why both Γ and Δ are nonholonomic connections.

Conversely, we suppose that Γ and Δ are nonholonomic connections. For coordinates of $\Gamma * \Delta$, we have evidently the described identification of coordinates with respect to r first elements or s last elements of multiindices (separately). This implication is clear, if we recall that the product of nonholonomic connections was defined correctly and we have the extended definition. □

Characterization of second order quasiconnections

Second order nonholonomic connections play an important role in the study of functional bundles of all smooth maps between the fibers over the same base point of two fibered manifold over the same base and they have something to do with the Schrödinger connection in a double fibered manifold. In [1] was deduced a useful identification of second order nonholonomic connection with two first order connection and a special section. We contribute to it by the following identification.

Proposition 7. *Second order quasiconnections in Y are in bijection with couples (Θ_1, Θ_2), where Θ_1 is a first order connection in Y and Θ_2 is a second order nonholonomic connection in Y.*

Proof. As the nonholonomic jets in QJ^2Y are characterized by the condition $\mathit{\Pi}_1^1 = {}_1\mathit{\Pi}$, there is the canonical projection $P : QJ^2Y \to \tilde{J}^2Y$ satisfying

$$\mathit{\Pi}_1^1 \circ P = \mathit{\Pi}_1^1. \tag{6}$$

Let Γ be a second order quasiconnection in Y. We set $\Theta_1 = {}_1\Pi(\Gamma)$, $\Theta_2 = P(\Gamma)$. Locally, coordinates $(\Gamma_i^{p10}, \Gamma_i^{p01}, \Gamma_{ij}^{p10\,01}, \Gamma_i^{p11})$ of Γ are projected to coordinates $(\Gamma_i^{p10}, \Gamma_i^{p01}, \Gamma_{ij}^{p10\,01})$ of Θ_2 and (Γ_i^{p11}) of Θ_1. It follows directly from this expression that Γ is in bijection with the couple (Θ_1, Θ_2). (Alternatively, (6) can be replaced by ${}_1\Pi \circ \bar{P} = {}_1\Pi$ and then $\Theta_1 = \Pi_1^1(\Gamma)$, $\Theta_2 = \bar{P}(\Gamma)$.) $\square$

Now, it follows from [1]:

Corollary. *Second order quasiconnections in Y are in bijection with quadruples $(\Theta, \bar{\Theta}, \bar{\bar{\Theta}}, \Sigma)$, where $\Theta, \bar{\Theta}, \bar{\bar{\Theta}}$ are first order connections in Y and $\Sigma : Y \to VY \otimes \bigotimes^2 T^*M$ is a section, VY being the vertical tangent bundle of Y.*

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