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The natural operators transforming affinors to tensor fields of type (4,4)

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Abstract. We give a complete classification of natural operators transforming affinors to tensor fields of type (4,4).

The *affinors* on a smooth manifold M are, by definition, the tensor fields of type (1,1) on M. All natural operators transforming affinors to tensor fields of type (r,r) for $r=0$ and $r=1$ are classified in [2], for $r=2$ in [3], for $r=3$ in [4]. Unfortunately, the methods used in these cases are inadequate to investigate the cases $r\geq 4$. In this paper we give a modification of the method presented in [4]. It enables us to receive a full characterization of natural operators transforming affinors to tensor fields of type (4,4). Since they form a module over the ring consisting of the known natural operators transforming affinors to functions, we prove that this module is free and finite-dimensional, and we find a basis of it.

Let p, q be non-negative integers. We will use the symbol X^p_qM to denote the vector space of all smooth tensor fields of type (p,q) on a smooth manifold M. If V is a vector space then we will write T^p_qV for

$$\underbrace{V\otimes\cdots\otimes V}_{p\text{ times}}\otimes\underbrace{V^*\otimes\cdots\otimes V^*}_{q\text{ times}}$$

and if $f:V\longrightarrow W$ is an isomorphism between two vector spaces we will write T^p_qf for

$$\underbrace{f\otimes\cdots\otimes f}_{p\text{ times}}\otimes\underbrace{f^{-1*}\otimes\cdots\otimes f^{-1*}}_{q\text{ times}}:T^p_qV\longrightarrow T^p_qW.$$

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If $\varphi : M \longrightarrow N$ is an immersion between two smooth manifolds of the same dimension then two tensor fields $t \in X^p_q M$ and $u \in X^p_q N$ are said to be φ-related, if $u(\varphi(x)) = T^p_q(T_x\varphi)t(x)$ for every $x \in M$.

Let n, p, q, r, s be non-negative integers. A family of maps $A_M : X^p_q M \longrightarrow X^r_s M$, where M is an arbitrary n-dimensional smooth manifold, is called the *natural operator* transforming tensor fields of type (p, q) to tensor fields of type (r, s) if for every injective immersion $\varphi : M \longrightarrow N$ between two n-dimensional smooth manifolds, for every $t \in X^p_q M$ and every $u \in X^p_q N$ the tensor fields $A_M(t)$ and $A_N(u)$ are φ-related whenever t and u are φ-related. (This is a special case of a general definition of natural operators, see [5].)

Let k be a non-negative integer. A natural operator A transforming tensor fields of type (p, q) to tensor fields of type (r, s) is said to be of order k if for any n-dimensional smooth manifold M, any $x \in M$ and all $t, u \in X^p_q M$ the following implication

$$j^k_x t = j^k_x u \Longrightarrow A_M(t)(x) = A_M(u)(x)$$

holds (here $j^k_x t$ denotes the k-jet of t at x). It is known (see [2]) that if $p = q$ and $r = s$ then every natural operator transforming tensor fields of type (p, q) to tensor fields of type (r, s) has order zero. This reduces the problem of finding natural operators to determining equivariant maps (see [5], [6]).

We racall that the group $GL(n, K)$, where K is a field, acts on $T^p_q K^n$ in the following way: if $t \in T^p_q K^n$ and $A \in GL(n, K)$ then

$$(t \cdot A)^{i_1 \ldots i_p}_{j_1 \ldots j_q} = (A^{-1})^{i_1}_{k_1} \ldots (A^{-1})^{i_p}_{k_p} t^{k_1 \ldots k_p}_{l_1 \ldots l_q} A^{l_1}_{j_1} \ldots A^{l_q}_{j_q}$$

for all $i_1, \ldots, i_p$, $j_1, \ldots, j_q \in \{1, \ldots, n\}$.

Definition. A map $a : T^p_q \mathbb{R}^n \longrightarrow T^r_s \mathbb{R}^n$ is called to be *equivariant* if $a(t \cdot A) = a(t) \cdot A$ for all $t \in T^p_q \mathbb{R}^n$, $A \in GL(n, \mathbb{R})$, and if $a \circ b$ is smooth for every smooth map $b : \mathbb{R}^n \longrightarrow T^p_q \mathbb{R}^n$. (The latter condition forces the smoothness of a, but a proof of this is not simple, see [1].)

The set of all such equivariant maps will be denoted by $E_{(p,q),(r,s),n}$. Using standard methods (see [5], [6], [2]) we can show that there is a one-to-one correspondence between natural operators of order zero transforming tensor fields of type (p, q) to tensor fields of type (r, s) and equivariant maps from $E_{(p,q),(r,s),n}$. Namely, if A is a natural operator then the corresponding equivariant map a is defied by

$$a(t(0)) = A_{\mathbb{R}^n}(t)(0)$$

for any $t \in X^p_q\mathbb{R}^n$ (since A has order zero, the definition is independent of a choice of t). Conversely, if $a \in E_{(p,q),(r,s),n}$ then the corresponding natural operator is for every n-dimensional smooth manifold M, every $t \in X^p_q M$ and every $x \in M$ defined by

$$A_M(t)(x) = T^r_s(T_x\varphi)^{-1}a(T^p_q(T_x\varphi)t(x)),$$

where φ is a chart on M.

Since we have established the relation between all natural operators and equivariant maps for $p = q$ and $r = s$, from now on we will study equivariant maps instead of natural operators.

We first observe that $E_{(p,q),(0,0),n}$ is a ring and $E_{(p,q),(r,s),n}$ is a module over $E_{(p,q),(0,0),n}$. In the paper [2] it is given a classification of equivariant maps transforming tensors of type (1,1) to tensors of type (0,0). Namely, for every $a \in E_{(1,1),(0,0),n}$ there is a uniquely determined smooth function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$a(t) = f(c_1(t), \dots, c_n(t)) \tag{1}$$

for every $t \in T^1_1\mathbb{R}^n$, where $c_i : T^1_1\mathbb{R}^n \longrightarrow \mathbb{R}$ for $i \in \{1, \dots, n\}$ are the coefficients of the characteristic polynomial of a linear endomorphism i.e.

$$\det(\lambda \operatorname{id}_{\mathbb{R}^n} - t) = \lambda^n + \sum_{i=1}^{n} c_i(t)\lambda^{n-i} \tag{2}$$

for every $\lambda \in \mathbb{R}$ and $t \in T^1_1\mathbb{R}^n$. Of course, the converse statement also is true: for every smooth map $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ formula (1) defines an equivariant map $a \in E_{(1,1),(0,0),n}$.

We can also construct examples of equivariant maps from $E_{(1,1),(r,r),n}$ for each non-negative integer r.

Example. Suppose that $\psi : \{1, \dots, r\} \longrightarrow \mathbb{N}$ and $\sigma \in S_r$, where $\mathbb{N}$ is the set of all non-negative integers and S_r denotes the set of all permutations of the set $\{1, \dots, r\}$. Put

$$e_{\psi,\sigma}(t)^{i_1\dots i_r}_{j_1\dots j_r} = (t^{\psi(1)})^{i_{\sigma(1)}}_{j_1} \dots (t^{\psi(r)})^{i_{\sigma(r)}}_{j_r} \tag{3}$$

for every $t \in T^1_1\mathbb{R}^n$ and all $i_1, \dots, i_r,\ j_1, \dots, j_r \in \{1, \dots, n\}$. Here t^k, where k is a non-negative integer, stands for

$$\underbrace{t \circ \cdots \circ t}_{k \text{ times}}.$$

It is immediate that $e_{\psi,\sigma} \in E_{(1,1),(r,r),n}$.

We are now in a position to formulate our main result.

Theorem. *The equivariant maps* $e_{\psi,\sigma}$ *for*

$$\psi : \{1, 2, 3, 4\} \longrightarrow \{0, \dots, n-1\}$$

and $\sigma \in S_4$ *satisfying one out of the following ten conditions:*

1. $\psi(1) = n-1$ *and*
$$\sigma = \left(\begin{smallmatrix} 1&2&3&4 \\ 1&2&3&4 \end{smallmatrix}\right),$$
2. $\psi(1) \le n-2$, $\psi(2) = n-1$ *and* σ *is an element of the following set*
$$\left\{\left(\begin{smallmatrix} 1&2&3&4 \\ 1&2&3&4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3&4 \\ 2&1&3&4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3&4 \\ 3&1&2&4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3&4 \\ 4&1&2&3 \end{smallmatrix}\right)\right\},$$
3. $\psi(1) = n-2$, $\psi(2) \le n-2$, $\psi(3) = n-1$ *and* σ *is an element of one out of the following two sets*
$$\left\{\left(\begin{smallmatrix} 1&2&3&4 \\ 1&2&3&4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3&4 \\ 2&1&3&4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3&4 \\ 3&1&2&4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3&4 \\ 4&1&2&3 \end{smallmatrix}\right)\right\},$$
$$\left\{\left(\begin{smallmatrix} 1&2&3&4 \\ 1&3&2&4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3&4 \\ 1&4&2&3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3&4 \\ 2&3&1&4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3&4 \\ 2&4&1&3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3&4 \\ 3&4&1&2 \end{smallmatrix}\right)\right\},$$
4. $\psi(1) \le n-3$, $\psi(2) \le n-2$, $\psi(3) = n-1$ *and* σ *is an element of one out of the following three sets*
$$\left\{\left(\begin{smallmatrix} 1&2&3&4 \\ 1&2&3&4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3&4 \\ 2&1&3&4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3&4 \\ 3&1&2&4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3&4 \\ 4&1&2&3 \end{smallmatrix}\right)\right\},$$
$$\left\{\left(\begin{smallmatrix} 1&2&3&4 \\ 1&3&2&4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3&4 \\ 1&4&2&3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3&4 \\ 2&3&1&4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3&4 \\ 2&4&1&3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3&4 \\ 3&4&1&2 \end{smallmatrix}\right)\right\},$$
$$\left\{\left(\begin{smallmatrix} 1&2&3&4 \\ 3&2&1&4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3&4 \\ 4&2&1&3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3&4 \\ 4&3&1&2 \end{smallmatrix}\right)\right\},$$

5. $\psi(1) = n-2$, $\psi(2) \le n-2$, $\psi(3) \le n-2$ *and* σ *is an element of one out of the following three sets*

$$\left\{\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&1&2&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&3&2&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&4&2&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&4&1&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&2&4&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\right\},$$

6. $\psi(1) \le n-3$, $\psi(2) = n-2$, $\psi(3) \le n-2$ *and* σ *is an element of one out of the following five sets*

$$\left\{\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&1&2&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&3&2&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&4&2&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&4&1&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&2&4&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&3&1&2\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&1&3&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&2&3&1\end{pmatrix}\right\},$$

7. $\psi(1) = n-3$, $\psi(2) \le n-3$, $\psi(3) \le n-2$ *and* σ *is an element of one out of the following six sets*

$$\left\{\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&1&2&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&3&2&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&4&2&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&4&1&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&2&4&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&3&1&2\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&1&3&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&2&3&1\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} 1&2&3&4\\ 1&4&3&2\end{pmatrix},\ \begin{pmatrix} 1&2&3&4\\ 2&4&3&1\end{pmatrix},\ \begin{pmatrix} 1&2&3&4\\ 3&4&2&1\end{pmatrix}\right\},$$

8. $\psi(1) \leq n-4$, $\psi(2) \leq n-3$, $\psi(3) \leq n-2$,
9. $\psi(1) = n-2$, $\psi(2) = 0$, $\psi(3) \leq n-2$ *and*
$$\sigma = \begin{pmatrix} 1&2&3&4\\ 3&1&4&2\end{pmatrix},$$
10. $\psi(1) = n-2$, $\psi(2) = 0$, $\psi(3) \leq n-2$ *and*
$$\sigma = \begin{pmatrix} 1&2&3&4\\ 2&1&4&3\end{pmatrix}$$

form a basis of the module $E_{(1,1),(4,4),n}$.

We must give a lemma before we start to prove our theorem. Let us denote by $M_{r,n}$ the set of all pairs (α, β) of maps $\alpha : \{1, \ldots, r\} \longrightarrow \{1, \ldots, n\}$ and $\beta : \{1, \ldots, r\} \longrightarrow \{1, \ldots, n\}$ such that for every $i \in \{1, \ldots, n\}$ the numbers of elements of the sets $\alpha^{-1}(\{i\})$ and $\beta^{-1}(\{i\})$ are equal. The number of elements of $M_{r,n}$ will be denoted by $m(r, n)$.

Lemma. *Let $t \in T^1_1\mathbb{R}^n$ be a linear endomorphism of $\mathbb{R}^n$ with n different complex eingenvalues. Then there is a vector subspace $V \subset T^r_r\mathbb{R}^n$ such that $\dim V \leq m(r, n)$ and that for every $a \in E_{(1,1),(r,r),n}$ we have $a(t) \in V$.*

A proof of the lemma can be found in [4].

Proof of the theorem. We will assume that the set

$$K_{r,n} = \{0, \ldots, n-1\}^{\{1,\ldots,r\}} \times S_r$$

is equiped with the following order: for $\psi, \omega : \{1, \ldots, r\} \longrightarrow \{0, \ldots, n-1\}$ and $\sigma, \tau \in S_r$ we have $(\psi, \sigma) < (\omega, \tau)$ if and only if

$$(\psi(1), \sigma(1), \ldots, \psi(r), \sigma(r)) < (\omega(1), \tau(1), \ldots, \omega(r), \tau(r))$$

with respect to the lexicographic order i.e.

$$(\psi, \sigma) < (\omega, \tau) \Longleftrightarrow \exists_{k\in\{1,\ldots,r\}} \left(\forall_{l\in\{1,\ldots,k-1\}}\ \psi(l) = \omega(l) \wedge \sigma(l) = \tau(l)\right)$$
$$\wedge \left(\psi(k) < \omega(k) \vee \left(\psi(k) = \omega(k) \wedge \sigma(k) < \tau(k)\right)\right).$$

For $z \in \mathbb{R}^n$ the linear endomorphism $t_z : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is defined by imposing the following conditions: $t_z(e_i) = e_{i+1}$ for $i \in \{1, \dots, n-1\}$ and $t_z(e_n) = -z^n e_1 - \dots - z^1 e_n$, where $e_1, \dots, e_n$ denotes the canonical basis of $\mathbb{R}^n$, i.e.

$$t_z = \begin{bmatrix} 0 & & & -z^n \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & -z^2 \\ & & 1 & -z^1 \end{bmatrix} \tag{4}$$

in the canonical basis. Finally, let us denote by $L_{r,n}$ the set consisting of all pairs $(\psi, \sigma) \in K_{r,n}$ with the property that there exists any standard coordinate on $T^r_r \mathbb{R}^n$ such that this coordinate of $e_{\psi,\sigma}(t_z)$ equals 1 for every $z \in \mathbb{R}^n$ and that (ψ, σ) is the minimal element of $K_{r,n}$ for which this coordinate of $e_{\psi,\sigma}(t_z)$ does not vanish for some $z \in \mathbb{R}^n$ i.e.

$$L_{r,n} = \{(\psi,\sigma) \in K_{r,n} : \exists_{i_1,\dots,i_r,j_1,\dots,j_r \in \{1,\dots,n\}} (\forall_{z \in \mathbb{R}^n}\ e_{\psi,\sigma}(t_z)^{i_1 \dots, i_r}_{j_1 \dots j_r} = 1)$$
$$\wedge (\forall_{(\omega,\tau) \in K_{r,n}}\ (\omega,\tau) < (\psi,\sigma) \Longrightarrow (\forall_{z \in \mathbb{R}^n}\ e_{\omega,\tau}(t_z)^{i_1 \dots, i_r}_{j_1 \dots j_r} = 0))\}.$$

By (4),

$$(k \leq n - j \wedge k \neq i - j) \Longrightarrow (t^k_z)^i_j = 0,$$
$$k = i - j \Longrightarrow (t^k_z)^i_j = 1$$

for every $z \in \mathbb{R}^n$, every non-negative integer k and all $i, j \in \{1, \dots, n\}$. Therefore from (3) we see that

$$(\exists_{k \in \{1,\dots,r\}} \psi(k) \leq n - j_k \wedge \psi(k) \neq i_{\sigma(k)} - j_k) \Longrightarrow e_{\psi,\sigma}(t_z)^{i_1,\dots,i_r}_{j_1,\dots,j_r} = 0, \tag{5}$$

$$(\forall_{k \in \{1,\dots,r\}}\ \psi(k) = i_{\sigma(k)} - j_k) \Longrightarrow e_{\psi,\sigma}(t_z)^{i_1,\dots,i_r}_{j_1,\dots,j_r} = 1 \tag{6}$$

for every $(\psi, \sigma) \in K_{r,n}$, every $z \in \mathbb{R}^n$ and all $i_1, \dots, i_r, j_1, \dots, j_r \in \{1, \dots, n\}$.

Fix $i_1, \dots, i_r, j_1, \dots, j_r \in \{1, \dots, n\}$. We now describe an algorithm for finding the minimal pair $(\psi, \sigma) \in K_{r,n}$ such that $e_{\psi,\sigma}(t_z)^{i_1,\dots,i_r}_{j_1,\dots,j_r} \neq 0$ for some $z \in \mathbb{R}^n$. The construction of (ψ, σ) is by induction. Our algorithm does not work for arbitrary $i_1, \dots, i_r, j_1, \dots, j_r \in \{1, \dots, n\}$. Necessary conditions will be formulated in the course of the construction.

Algorithm. *Suppose that* $\psi(1), \sigma(1), \dots, \psi(k-1), \sigma(k-1)$ *are defined, where* $k \in \{1, \dots, r\}$*. We will define* $\psi(k)$ *and* $\sigma(k)$*. Let*

$$G_k = \{i \in \{1, \dots, n\} : \exists_{u \in \{1,\dots,r\}} (\forall_{v \in \{1,\dots,k-1\}} u \neq \sigma(v)) \wedge i = i_u \wedge i \geq j_k\}.$$

If the set G_k *is empty, then our algorithm breaks down. If not, we define* $g_k = \min G_k$ *and put* $\psi(k) = g_k - j_k$*. Let*

$$H_k = \{u \in \{1, \dots, r\} : (\forall_{v \in \{1,\dots,k-1\}} u \neq \sigma(v)) \wedge i_u = g_k\}.$$

We define $h_k = \min H_k$ *and put* $\sigma(k) = h_k$*.*

Of course, since $g_k \in G_k$ whenever g_k is defined, $H_k \neq \emptyset$ whenever H_k is defined. It is also seen at once that if it is possible to continue the construction to the very end (i.e. if $G_k \neq \emptyset$ for every $k \in \{1, \dots, r\}$), then we obtain $(\psi, \sigma) \in K_{r,n}$. We now show that such (ψ, σ) is the minimal pair from $K_{r,n}$ with the property that $e_{\psi,\sigma}(t_z)^{i_1,\dots,i_r}_{j_1,\dots,j_r} \neq 0$ for some $z \in \mathbb{R}^n$. From (6) we have $e_{\psi,\sigma}(t_z)^{i_1,\dots,i_r}_{j_1,\dots,j_r} = 1$ for every $z \in \mathbb{R}^n$, as $h_k \in H_k$ for every $k \in \{1, \dots, r\}$. Suppose that $(\omega, \tau) \in K_{r,n}$ and that $(\omega, \tau) < (\psi, \sigma)$. Thus there is $k \in \{1, \dots, r\}$ such that $\omega(l) = \psi(l)$ and $\tau(l) = \sigma(l)$ for every $l \in \{1, \dots, k-1\}$ and $\omega(k) < \psi(k)$ or $\omega(k) = \psi(k)$ and $\tau(k) < \sigma(k)$. We have to prove that $e_{\omega,\tau}(t_z)^{i_1,\dots,i_r}_{j_1,\dots,j_r} = 0$ for every $z \in \mathbb{R}^n$. If $\omega(k) < \psi(k)$ then $\omega(k) < g_k - j_k$. Since $\tau(l) = \sigma(l)$ for every $l \in \{1, \dots, k-1\}$, it must be either $i_{\tau(k)} < j_k$ or $i_{\tau(k)} \in G_k$. Since $g_k = \min G_k$, the last condition implies $g_k \leq i_{\tau(k)}$. Therefore $e_{\omega,\tau}(t_z)^{i_1,\dots,i_r}_{j_1,\dots,j_r} = 0$ for every $z \in \mathbb{R}^n$ as follows from (5). If $\omega(k) = \psi(k)$ and $\tau(k) < \sigma(k)$ then $\omega(k) = g_k - j_k$ and $\tau(k) < h_k$. Since $h_k = \min H_k$, the last inequality implies that $\tau(k) \notin H_k$, and since $\tau(l) = \sigma(l)$ for every $l \in \{1, \dots, k-1\}$, it must be $i_{\tau(k)} \neq g_k$. Therefore $e_{\omega,\tau}(t_z)^{i_1,\dots,i_r}_{j_1,\dots,j_r} = 0$ for every $z \in \mathbb{R}^n$ as follows from (5). This is the desired conclusion. Actually, we have proved that $(\psi, \sigma) \in L_{r,n}$.

Our next goal is to show that all pairs $(\psi, \sigma) \in K_{4,n}$ specified in the theorem can be obtain as a result of applying our algorithm.

We first observe that if $(\psi, \sigma) \in K_{r,n}$, if $m_1, \dots, m_r \in \{1, \dots, n\}$ are such that $m_1 \leq \dots \leq m_r$ and if we set $i_1 = m_{\sigma^{-1}(1)}, \dots, i_r = m_{\sigma^{-1}(r)}$, $j_1 = m_1 - \psi(1), \dots, j_r = m_r - \psi(r)$, then our algorithm applying to $i_1, \dots, i_r, j_1 \dots, j_r$ yields (ψ, σ), whenever

$$\forall_{k,l \in \{1,\dots,r\}} (k < l \wedge m_k = m_l) \Longrightarrow \sigma(k) < \sigma(l)$$

and whenever $j_1, \dots, j_r \in \{1, \dots, n\}$. This remark facilitate us to produce (ψ, σ) from items 1–8 of the theorem.

1. In order to obtain by our algorithm any (ψ, σ) from item 1 of the theorem it suffices to take $m_1 = n$, $m_2 = n$, $m_3 = n$, $m_4 = n$.
2. Suppose ψ is as in item 2 of the theorem. If σ is an element of the set from item 2 of the theorem then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-1$, $m_2 = n$, $m_3 = n$, $m_4 = n$.
3. Suppose ψ is as in item 3 of the theorem.
   If σ is an element of the first set from item 3 of the theorem then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-1$, $m_2 = n$, $m_3 = n$, $m_4 = n$.
   If σ is an element of the second set from item 3 of the theorem then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-1$, $m_2 = n-1$, $m_3 = n$, $m_4 = n$.
4. Suppose ψ is as in item 4 of the theorem.
   If σ is an element of the first set from item 4 of the theorem then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-1$, $m_2 = n$, $m_3 = n$, $m_4 = n$.
   If σ is an element of the second set from item 4 of the theorem then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-1$, $m_2 = n-1$, $m_3 = n$, $m_4 = n$.
   If σ is an element of the third set from item 4 of the theorem then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-2$, $m_2 = n-1$, $m_3 = n$, $m_4 = n$.
5. Suppose ψ is as in item 5 of the theorem.
   If σ is an element of the first set from item 5 of the theorem then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-1$, $m_2 = n$, $m_3 = n$, $m_4 = n$.
   If σ is an element of the second set from item 5 of the theorem then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-1$, $m_2 = n-1$, $m_3 = n$, $m_4 = n$.
   If σ is an element of the third set from item 5 of the theorem then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-1$, $m_2 = n-1$, $m_3 = n-1$, $m_4 = n$.
6. Suppose ψ is as in item 6 of the theorem.
   If σ is an element of the first set from item 6 of the theorem then in

order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-1$, $m_2 = n$, $m_3 = n$, $m_4 = n$.

If σ is an element of the second set from item 6 of the theorem then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-1$, $m_2 = n-1$, $m_3 = n$, $m_4 = n$.

If σ is an element of the third set from item 6 of the theorem then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-1$, $m_2 = n-1$, $m_3 = n-1$, $m_4 = n$.

If σ is an element of the fourth set from item 6 of the theorem then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-2$, $m_2 = n-1$, $m_3 = n$, $m_4 = n$.

If σ is an element of the fifth set from item 6 of the theorem then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-2$, $m_2 = n-1$, $m_3 = n-1$, $m_4 = n$.

7. Suppose ψ is as in item 7 of the theorem.

   If σ is an element of the first set from item 7 of the theorem then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-1$, $m_2 = n$, $m_3 = n$, $m_4 = n$.

   If σ is an element of the second set from item 7 of the theorem then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-1$, $m_2 = n-1$, $m_3 = n$, $m_4 = n$.

   If σ is an element of the third set from item 7 of the theorem then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-1$, $m_2 = n-1$, $m_3 = n-1$, $m_4 = n$.

   If σ is an element of the fourth set from item 7 of the theorem then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-2$, $m_2 = n-1$, $m_3 = n$, $m_4 = n$.

   If σ is an element of the fifth set from item 7 of the theorem then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-2$, $m_2 = n-1$, $m_3 = n-1$, $m_4 = n$.

   If σ is an element of the sixth set from item 7 of the theorem then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-2$, $m_2 = n-2$, $m_3 = n-1$, $m_4 = n$.

8. Suppose ψ is as in item 8 of the theorem. If σ is an arbitrary permutation from S_4 then in order to obtain by our algorithm (ψ, σ) it suffices to take $m_1 = n-3$, $m_2 = n-2$, $m_3 = n-1$, $m_4 = n$.

9. In order to obtain any (ψ,σ) from item 9 of the theorem it suffices to apply our algorithm to $i_1=n$, $i_2=n$, $i_3=n-1$, $i_4=n-1$, $j_1=1$, $j_2=n$, $j_3=n-1-\psi(3)$, $j_4=n-\psi(4)$.
10. In order to obtain any (ψ,σ) from item 10 of the theorem it suffices to apply our algorithm to $i_1=n$, $i_2=n-1$, $i_3=n$, $i_4=n-1$, $j_1=1$, $j_2=n$, $j_3=n-1-\psi(3)$, $j_4=n-\psi(4)$.

It is easy to check that ten conditions formulated in the theorem exclude each other and that the sets specified in each item are pairwise disjoint.

Let P_n denote the set consisting of all pairs $(\psi,\sigma)\in K_{4,n}$ specified in the theorem. The set of the sequences of integers $(i_1,i_2,i_3,i_4,j_1,j_2,j_3,j_4)$, which we used above to obtain by our algorithm the elements of P_n, will be denoted by Q_n. We will write $(i_1,i_2,i_3,i_4,j_1,j_2,j_3,j_4)=\pi(\psi,\sigma)$ if $(\psi,\sigma)\in P_n$ is the result of our algorithm applied to
$(i_1,i_2,i_3,i_4,j_1,j_2,j_3,j_4)\in Q_n$. Therefore $\pi : P_n \longrightarrow Q_n$ is a bijection. We have proved that $P_n\subset L_{4,n}$. The definition of $L_{4,n}$ makes it obvious that there is an injection $\rho : L_{4,n}\longrightarrow\{1,\dots,n\}^8$ such that $\rho|P_n=\pi$ and that if $(i_1,i_2,i_3,i_4,j_1,j_2,j_3,j_4)=\rho(\psi,\sigma)$ for any $(\psi,\sigma)\in L_{4,n}$ then $e_{\psi,\sigma}(t_z)^{i_1i_2i_3i_4}_{j_1j_2j_3j_4}=1$ for every $z\in\mathbb{R}^n$ and $e_{\omega,\tau}(t_z)^{i_1i_2i_3i_4}_{j_1j_2j_3j_4}=0$ for every $(\omega,\tau)\in K_{4,n}$ such that $(\omega,\tau)<(\psi,\sigma)$ and for every $z\in\mathbb{R}^n$.

Let $z\in\mathbb{R}^n$. We now show that the vectors $e_{\psi,\sigma}(t_z)\in T^4_4\mathbb{R}^n$ for $(\sigma,\psi)\in L_{4,n}$ are linearly independent. Suppose that $\lambda_{\psi,\sigma}\in\mathbb{R}$ for $(\psi,\sigma)\in L_{4,n}$ are such that

$$\sum_{(\psi,\sigma)\in L_{4,n}}\lambda_{\psi,\sigma}e_{\psi,\sigma}(t_z)=0.$$

We have to prove that $\lambda_{\psi,\sigma}=0$ for $(\psi,\sigma)\in L_{4,n}$. The proof is by induction on (ψ,σ). Fix $(\psi,\sigma)\in L_{4,n}$ and assume $\lambda_{\omega,\tau}=0$ for $(\omega,\tau)\in L_{4,n}$ such that $(\omega,\tau)>(\psi,\sigma)$. Taking $(i_1,i_2,i_3,i_4,j_1,j_2,j_3,j_4)=\rho(\psi,\sigma)$ we get

$$\lambda_{\psi,\sigma}=\sum_{(\psi,\sigma)\in L_{4,n}}\lambda_{\psi,\sigma}e_{\psi,\sigma}(t_z)^{i_1i_2i_3i_4}_{j_1j_2j_3j_4}=0,$$

which is our claim.

If $z\in\mathbb{R}^n$ is such that t_z has n different complex eingenvalues then, by our lemma, there is a subspace $V\subset T^4_4\mathbb{R}^n$ such that $\dim V\le m(4,n)$ and $a(t_z)\in V$ for every $a\in E_{(1,1),(4,4),n}$. Consequently $e_{\psi,\sigma}(t_z)\in V$

for every $(\psi,\sigma) \in L_{4,n}$. A trivial computation shows that $m(4,n) = 24n^4 - 72n^3 + 82n^2 - 33n$. Since ten conditions formulated in the theorem exclude each other and the sets specified in each item are pairwise disjoint, we see that the number of elements of P_n equals $n^3+4n^2(n-1)+9n(n-1)+12n(n-1)(n-2)+12n(n-1)^2+20n(n-1)(n-2)+23n(n-1)(n-2)+24n(n-1)(n-2)(n-3)+n(n-1)+n(n-1) = 24n^4-72n^3+82n^2-33n$, which is equal to $m(4,n)$. Since $P_n \subset L_{4,n}$ and the vectors $e_{\psi,\sigma}(t_z)$ for $(\psi,\sigma) \in L_{4,n}$ are linearly independent, we deduce that $e_{\psi,\sigma}(t_z)$ for $(\psi,\sigma) \in P_n$ form a basis of V. Furthermore, we see that $P_n = L_{4,n}$, which is worth pointing out. We now prove that if $x \in V$ is such that $x^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4} = 0$ for every $(i_1,i_2,i_3,i_4 j_1,j_2,j_3,j_4) \in Q_n$, then $x = 0$. The vector x is a linear combination of the vectors of our basis of V, i.e.

$$x = \sum_{(\psi,\sigma)\in P_n} x_{\psi,\sigma} e_{\psi,\sigma}(t_z),$$

where $x_{\psi,\sigma} \in \mathbb{R}$ for $(\psi,\sigma) \in P_n$. Thus it is sufficient to show that $x_{\psi,\sigma} = 0$ for every $(\psi,\sigma) \in P_n$. The proof is by induction on (ψ,σ). Fix $(\psi,\sigma) \in P_n$ and assume $x_{\omega,\tau} = 0$ for $(\omega,\tau) \in P_n$ such that $(\omega,\tau) > (\psi,\sigma)$. Taking $(i_1,i_2,i_3,i_4,j_1,j_2,j_3,j_4) = \pi(\psi,\sigma)$ we get

$$x_{\psi,\sigma} = \sum_{(\psi,\sigma)\in P_n} x_{\psi,\sigma} e_{\psi,\sigma}(t_z)^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4} = 0,$$

which is our claim.

We next prove that if $a, b \in E_{(1,1),(4,4),n}$ are such that $a(t_z)^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4} = b(t_z)^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4}$ for every $(i_1,i_2,i_3,i_4,j_1,j_2,j_3,j_4) \in Q_n$ and every $z \in \mathbb{R}^n$, then $a = b$. Clearly, it suffices to show that if $a \in E_{(1,1),(4,4),n}$ is such that $a(t_z)^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4} = 0$ for every $(i_1,i_2,i_3,i_4,j_1,j_2,j_3,j_4) \in Q_n$ and every $z \in \mathbb{R}^n$, then $a = 0$. Let $u \in T^1_1\mathbb{R}^n$. We have to prove that $a(u) = 0$. We first consider the case that u has n different complex eingenvalues. An easy computation shows that for every $z \in \mathbb{R}^n$ the coefficients of the characteristic polynomial of t_z coincide with the coordinates of z, i.e.

$$\det \begin{bmatrix} \lambda & & & z^n \\ -1 & \ddots & & \vdots \\ & \ddots & \lambda & z^2 \\ & & -1 & \lambda + z^1 \end{bmatrix} = \lambda^n + \sum_{i=1}^{n} z^i \lambda^{n-i} \tag{7}$$

for every $\lambda \in \mathbb{R}$, where $(z^1, \dots, z^n) = z$. Thus, writing $c(u)$ for the vector $(c_1(u), \dots, c_n(u)) \in \mathbb{R}^n$, where $c_1(u), \dots, c_n(u)$ are the coefficients of the characteristic polynomial of u, we see that the characteristic polynomial of u is the same as that of $t_{c(u)}$. Combining this with the fact that both u and $t_{c(u)}$ have n different complex eingenvalues we conclude, by Jordan's theorem, that there is $A \in GL(n, \mathbb{R})$ such that $u = t_{c(u)} \cdot A$. Since $a(t_{c(u)}) = 0$, which is due to the fact proved in the previous paragraph, we have $a(u) = a(t_{c(u)} \cdot A) = a(t_{c(u)}) \cdot A = 0 \cdot A = 0$ as desired. We now turn to the case of an arbitrary u. Let $v \in T^1_1\mathbb{R}^n$ be an arbitrary matrix with n different complex eingenvalues and let R be an n-dimensional affine subspace in $T^1_1\mathbb{R}^n$ such that $u \in R$ and $v \in R$. Suppose that $D(Z)$ denotes the discriminant of the characteristic polynomial of a matrix $Z \in T^1_1\mathbb{R}^n$. Then $D : T^1_1\mathbb{R}^n \longrightarrow \mathbb{R}$ is a polynomial and $D(Z) \neq 0$ if and only if Z has n different complex eingenvalues. Of course, $D|R \neq 0$, because $D(v) \neq 0$. Therefore $S = \{Z \in R : D(Z) \neq 0\}$ is a dense subset of R. We known that $a|S = 0$. Suppose that $P : \mathbb{R}^n \longrightarrow T^1_1\mathbb{R}^n$ is an affine parametrization of R. By the definition of equivariant maps, the composition $a \circ P$ is smooth and so is $a|R = (a \circ P) \circ P^{-1}$. Since each continous map vanishing on a dense subset vanishes everywhere, we have $a|R = 0$. In particular $a(u) = 0$ as required.

Fix $a \in E_{(1,1),(4,4),n}$. Our next goal is to determine smoth functions $f_{\psi,\sigma} : \mathbb{R}^n \longrightarrow \mathbb{R}$ for $(\psi, \sigma) \in P_n$ such that

$$a(t_z)^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4} = \sum_{(\psi,\sigma)\in P_n} f_{\psi,\sigma}(z) e_{\psi,\sigma}(t_z)^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4} \tag{8}$$

for every $(i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4) \in Q_n$ and every $z \in \mathbb{R}^n$. The definition is by induction on $(\psi, \sigma) \in P_n$. Suppose that $(\psi, \sigma) \in P_n$ and that $f_{\omega,\tau}$ for $(\omega, \tau) \in P_n$ such that $(\omega, \tau) > (\psi, \sigma)$ are defined. We take $(i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4) = \pi(\psi, \sigma)$ and put

$$f_{\psi,\sigma}(z) = a(t_z)^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4} - \sum_{\substack{(\omega,\tau)\in P_n \\ (\omega,\tau)>(\psi,\sigma)}} f_{\omega,\tau}(z) e_{\omega,\tau}(t_z)^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4} \tag{9}$$

for every $z \in \mathbb{R}^n$. It is easily seen that, by the smoothness of the map $\mathbb{R}^n \ni z \longrightarrow a(t_z) \in T^4_4\mathbb{R}^n$, we obtain smooth functions which satisfy the claimed conditions (8). Write

$$\widetilde{a} : T^1_1\mathbb{R}^n \ni t \longrightarrow \sum_{(\psi,\sigma)\in P_n} f_{\psi,\sigma}(c_1(t), \dots, c_n(t)) e_{\psi,\sigma}(t) \in T^4_4\mathbb{R}^n,$$

where $c_1, \dots, c_n$ are given by (2). Clearly, $\widetilde{a} \in E_{(1,1),(4,4),n}$. By (7) and (8), we have $\widetilde{a}(t_z)^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4} = a(t_z)^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4}$ for every $(i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4) \in Q_n$ and every $z \in \mathbb{R}^n$. Hence $\widetilde{a} = a$, which is due to the fact proved in the previous paragraph. Therefore $e_{\psi,\sigma}$ for $(\psi, \sigma) \in P_n$ are generators of $E_{(1,1),(4,4),n}$, because for every smooth map $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ formula (1) defines an equivariant map from $E_{(1,1),(0,0),n}$.

It remains to prove that they are linearly independent. Assume that

$$\sum_{(\psi,\sigma)\in P_n} g_{\psi,\sigma}(c_1(t), \dots, c_n(t)) e_{\psi,\sigma}(t) = 0$$

for every $t \in T^1_1\mathbb{R}^n$, where $g_{\psi,\sigma} : \mathbb{R}^n \longrightarrow \mathbb{R}$ for $(\psi, \sigma) \in P_n$ are smooth functions and $c_1, \dots, c_n$ are given by (2). Hence, according to (7),

$$\sum_{(\psi,\sigma)\in P_n} g_{\psi,\sigma}(z) e_{\psi,\sigma}(t_z) = 0$$

for every $z \in \mathbb{R}^n$. We have to prove that $g_{\psi,\sigma} = 0$ for $(\psi, \sigma) \in P_n$. The proof will be by induction on (ψ, σ). Suppose that $(\psi, \sigma) \in P_n$ and that $g_{\omega,\tau} = 0$ for $(\omega, \tau) \in P_n$ such that $(\omega, \tau) > (\psi, \sigma)$. We take $(i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4) = \pi(\psi, \sigma)$. Then

$$0 = \sum_{(\omega,\tau)\in P_n} g_{\omega,\tau}(z) e_{\omega,\tau}(t_z)^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4} = g_{\psi,\sigma}(z)$$

for every $z \in \mathbb{R}^n$, and so $g_{\psi,\sigma} = 0$. This proves the theorem. □

It is worth pointing out that the final part of the proof (formula (9)) yields a method of calculating the coordinates of an arbitrary equivariant map (for instance $e_{\psi,\sigma}$ with an arbitrary $\psi : \{1, 2, 3, 4\} \longrightarrow \mathbb{N}$ and an arbitrary $\sigma \in S_4$) in our basis.

Recall that we have proved the equality $P_n = L_{4,n}$. It enables us to write our theorem in the following equivalent form.

Theorem. *The equivariant maps $e_{\psi,\sigma}$ for $(\psi, \sigma) \in L_{4,n}$ form a basis of the module $E_{(1,1),(4,4),n}$.*

Moreover, the proof of our theorem leads to the following corollary.

Corollary. *$E_{(1,1),(4,4),n}$ is a free module of dimension $24n^4 - 72n^3 + 82n^2 - 33n$.*

Remark. Using the same arguments we can obtain the classification of equivariant maps from $E_{(0,0),(r,r),n}$ for $r = 1, 2, 3$, as it is described in [4]. On the other hand the method presented here is essentialy stronger than that from [4], because applying the algorithm from [4] in the case $r = 4$ we can obtain only the pairs $(\psi, \sigma) \in K_{4,n}$ specified in items 1–8 of our theorem, omitting those from items 9–10. Unfortunately, for $r \geq 5$ also the new method brakes down. For instance, applying our algorithm in the case $r = 5$ and $n = 3$ we can obtain only 4644 equivariant maps, while $m(5, 3) = 4653$.

We are ending off the paper with some remarks about possibile applications of our result. Generally, it seems that classifications of the natural operators transforming affinors to tensor fields of type (r, r), where r is a non-negative integer, can be applied to investigate other type natural operators transforming affinors. For instance, in [3] a classification of the natural operators transforming affinors to tensor fields of type $(2, 2)$ enabled us to find a classification of the natural operators transforming affinors to tensor fields of type $(0, 1)$. If we try to use the same methods for the natural operators transforming affinors to tensor fields of type $(r-2, r-1)$, where r is a non-negative integer and $r \geq 2$, there will appear just natural operators transforming affinors to tensor fields of type (r, r).

As a more complicated example we consider natural operators lifting affinors to the cotangent bundle. Such a natural operator is, by definition, a family of maps $A_M : X^1_1 M \longrightarrow X^1_1(T^*M)$, where M is an arbitrary n-dimensional smooth manifold and T^* denotes the functor of the cotangent bundle, such that for every injective imersion $\varphi : M \longrightarrow N$ between two n-dimensional smooth manifolds, for every $t \in X^1_1 M$ and every $u \in X^1_1 N$ the affinors $A_M(t)$ and $A_N(u)$ are $T^*\varphi$-related whenever t and u are φ-related. (This is a special case of a general definition of natural operators, see [5].) For such A and all $t \in X^1_1\mathbb{R}^n$, $p \in T^*_0\mathbb{R}^n$ put $a(j^\infty_0 t, p) = A_{\mathbb{R}^n}(t)(0, p)$. It can be proved that a is well defined. Suppose $a(j^\infty_0 t, p)$ depends on a finite jet only and a is smooth. Then, by the homogeneous function theorem (see [5]), we have

$$a(j^\infty_0 t, p) = \begin{bmatrix} b(t(0)) & 0 \\ c(j^2_0 t, p) & d(t(0)) \end{bmatrix},$$

where

$$c_{j_1j_2}(j^2_0t,p)=e^{i_1i_2}_{j_1j_2}(t(0))p_{i_1}p_{i_2}+f^{i_1i_2i_3}_{j_1j_2j_3}(t(0))\frac{\partial t^{j_3}_{i_1}}{\partial x^{i_2}}(0)p_{i_3}$$
$$+g^{i_1i_2i_3}_{j_1j_2j_3}(t(0))\frac{\partial^2 t^{j_3}_{i_1}}{\partial x^{i_2}\partial x^{i_3}}(0)+h^{i_1i_2i_3i_4}_{j_1j_2j_3j_4}(t(0))\frac{\partial t^{j_3}_{i_1}}{\partial x^{i_2}}(0)\frac{\partial t^{j_4}_{i_3}}{\partial x^{i_4}}(0)$$

for all $j_1,j_2\in\{1,\dots,n\}$, $t\in X^1_1\mathbb{R}^n$, $p\in T^*_0\mathbb{R}^n$. Of course, we may assume that $e^{i_2i_1}_{j_1j_2}=e^{i_1i_2}_{j_1j_2}$ for $i_1,i_2,j_1,j_2\in\{1,\dots,n\}$, $g^{i_1i_3i_2}_{j_1j_2j_3}=g^{i_1i_2i_3}_{j_1j_2j_3}$ for $i_1,i_2,i_3,j_1,j_2,j_3\in\{1,\dots,n\}$, $h^{i_3i_4i_1i_2}_{j_1j_2j_4j_3}=h^{i_1i_2i_3i_4}_{j_1j_2j_3j_4}$ for $i_1,i_2,i_3,i_4,j_1,j_2,j_3,$ $j_4\in\{1,\dots,n\}$. Now a standard computation shows that $b,d\in E_{(1,1),(1,1),n}$, $e\in E_{(1,1),(2,2),n}$, $f,g\in E_{(1,1),(3,3),n}$, $h\in E_{(1,1),(4,4),n}$. Therefore b, d, e, f, g, h are elements of the modules we have described in this paper. This may be helpful in further studying the natural operators lifting affinors to the cotangent bundle.

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