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On ϕ-skew symmetric conformal vector fields

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Abstract. The notion of the J-skew symmetric vector field was introduced in [MNR]. In the present paper, we deal with ϕ-skew symmetric conformal vector fields on a Kenmotsu manifold $M(\phi,\Omega,\eta,\xi,g)$. A necessary and sufficient condition for M to admit such a vector field C is given. In this case, C defines an infinitesimal relative conformal transformation of Ω and ϕC is a relatively integral invariant of Ω.

0. Introduction

Let $M(\phi,\Omega,\eta,\xi,g)$ be a $(2m+1)$-dimensional almost contact Riemannian manifold, where the structure tensors ϕ, η and ξ are a $(1,1)$-tensor field, a closed 1-form and the Reeb vector field, respectively, satisfying

$$\phi^2 = -Id + \eta\otimes\xi, \quad \eta(\xi) = 1.$$

M is said to be a *Kenmotsu manifold* if the following conditions

$$(\nabla_Z\phi)Z' = -\eta(Z')\phi Z - g(Z,\phi Z')\xi, \tag{0.1}$$

$$\nabla_Z\xi = Z - \eta(Z)\xi, \quad Z, Z' \in \Gamma TM \tag{0.2}$$

hold good.

In the present paper, we assume that M carries a ϕ-skew symmetric conformal (abbr. SSC) vector field C (in the sense of [R1]), that is

$$\nabla C = f dp + C \wedge \phi C, \tag{0.3}$$

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where dp denotes the canonical vector valued 1-form and $\wedge$ the wedge product of vector fields on M.

It is known that (0.3) implies

$$\mathcal{L}_C g = \rho g, \quad \rho = 2f \in \Lambda^0 M.$$

We prove that the dual 1-form $C^\flat$ of C with respect to g is an exterior recurrent 1-form having $(\phi C)^\flat$ as the recurrent form and that $C^\flat$, $(\phi C)^\flat$ and η belong to the same ideal I. In consequence, M is foliated by 3 surfaces tangent to the distributions spanned by $\{\xi, C\}$, $\{\xi, \phi C\}$ and $\{C, \phi C\}$ respectively, and if Z is any vector field orthogonal to $\{\xi, \phi C\}$, then $\mathcal{L}_C \mathcal{R}(C, Z)$ defines an infinitesimal conformal transformation for $g(C, Z)$, where $\mathcal{R}$ denotes the Ricci tensor field of M.

Finally, by using the Lie algebra induced by $C^\flat$ and $(\phi C)^\flat$, it is shown that C defines an infinitesimal relative conformal transformation of Ω, i.e.

$$d(\mathcal{L}_C \Omega) = (d\rho + 2\rho\eta) \wedge \Omega$$

and that Ω is a relatively integral invariant [AM] of Ω, i.e.

$$d(\mathcal{L}_{\phi C} \Omega) = 0.$$

1. Preliminaries

Let (M, g) be an n-dimensional connected manifold and let ∇ be the covariant differential operator defined by the metric tensor g (we assume that M is oriented and ∇ is the Levi–Civita connection).

Let ΓTM be the set of sections of the tangent bundle and $\flat : TM \to T^*M$ and $\sharp : T^*M \to TM$ the classical isomorphisms defined by g (i.e., $\flat$ is the index lowering operator and $\sharp$ is the index raising operator).

We denote by

$$A^q(M, TM) = \Gamma \operatorname{Hom}(\Lambda^q TM, TM)$$

the set of vector valued q-forms ($q \leq \dim M$) and following [P] we write for the covariant derivative with respect to ∇

$$d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM)$$

(it should be noticed that in general $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$).

If $p \in M$, then $dp \in A^1(M, TM)$ is the canonical vector valued 1-form and is called *the soldering form* of M. Since ∇ is symmetric, one has $d^\nabla(dp) = 0$.

The cohomology operator [GL] is defined by

$$d^\omega = d + e(\omega) \tag{1.1}$$

and is acting on ΛM, where $e(\omega)$ denotes the exterior product by the closed 1-form ω. One has $d^\omega \circ d^\omega = 0$ and a form $\omega \in \Lambda M$ with $d^\omega u = 0$ is said to be d^ω-closed.

Let $\mathcal{O} = \{e_A \mid A = 1, \ldots, n\}$ be a local field of orthonormal frames over M and let $\mathcal{O}^* = \{\omega^A\}$ be its associated coframe. The E. Cartan's structure equation [C] written in the index-less manner are

$$\nabla e = \theta \otimes e, \tag{1.2}$$

$$d\omega = -\theta \wedge \omega, \tag{1.3}$$

$$d\theta = -\theta \wedge \theta + \Theta. \tag{1.4}$$

In the above equations θ (resp. Θ) are the local connections forms in the tangent bundle TM (resp. the curvature forms on M).

2. ϕ-skew symmetric conformal vector fields

Let $M(\phi, \Omega, \eta, \xi, g)$ be a $(2m+1)$-dimensional Kenmotsu manifold [K], [MRV].

As is known, the quintuple of the structure tensor fields $(\phi, \Omega, \eta, \xi, g)$ satisfies the following equations:

$$\begin{cases} \phi^2 = -Id + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \\ g(Z, Z') = g(\phi Z, \phi Z') + \eta(Z)\eta(Z'), \quad \eta(Z) = g(\xi, Z), \\ (\nabla \phi) Z = -\eta(Z)\phi dp - (\phi Z)^\flat \otimes \xi, \\ \nabla \xi = dp - \eta \otimes \xi, \\ \Omega(Z, Z') = g(\phi Z, Z'), \end{cases} \tag{2.1}$$

for any vector fields $Z, Z' \in \Gamma TM$, and moreover we have

$$d\eta = 0, \quad d\Omega = 2\eta \wedge \Omega. \tag{2.2}$$

It should be noted that the equations (2.2) show that the pairing (η, Ω) defines a *conformal cosymplectic structure* $1 \times CS(m, \boldsymbol{R})$ [R1], [BR]. We also recall [R2] that the structure vector field ξ is (as in the case of a Sasakian manifold) always exterior concurrent (abbr. EC), that is

$$d^{\nabla}(\nabla\xi) = \nabla^2\xi = \xi \wedge dp. \tag{2.3}$$

In the present paper, we assume that M carries a vector field C such that its covariant differential satisfies

$$\nabla C = f dp + C \wedge \phi C, \quad f \in \Lambda^0 M. \tag{2.4}$$

As an extension of the concept of J-skew symmetric vector field [MNR], we agree to define C as a *ϕ-skew symmetric conformal vector field.*

Let Z be any vector field on M. If we denote by Z^A $(A \in \{0, \ldots, 2m\})$ its components with respect to an orthonormal frame $\mathcal{O} = \{e_0 = \xi, e_1, \ldots, e_m, e_{m+1} = \phi e_1, \ldots, e_{2m} = \phi e_m\}$, then, on behalf of the 4-th equation of (2.1), its covariant derivative is expressed by

$$\nabla Z = (dZ^A + Z^B \theta^A_B + Z^0 \omega^A) \otimes e_A + (dZ^0 + Z^\flat) \otimes \xi, \; Z^0 = \eta(Z). \tag{2.5}$$

With respect to $\mathcal{O}^*$, we have

$$\Omega = \sum_{a=1}^{m} \omega^a \wedge \omega^{a^*}, \; a^* = a + m. \tag{2.6}$$

We come back to the case under discussion. Since (2.4) is expressed as

$$\nabla C = f dp + (\phi C)^\flat \otimes C - C^\flat \otimes \phi C, \tag{2.7}$$

one quickly finds

$$g(\nabla_Z C, Z') + g(\nabla_{Z'} C, Z) = 2fg(Z, Z'),$$

which is equivalent to $\mathcal{L}_C g = 2fg$.

This, as is known, shows that C is a conformal vector field having $\rho = 2f$ as the conformal scalar.

Further by (2.5) and (2.7) one may write

$$\left\{\begin{array}{l} dC^a + C^A\theta_A^a = (f - C^a)\omega^a + C^a(\phi C)^\flat + C^{a^*}C^\flat, \\ dC^{a^*} + C^A\theta_A^{a^*} = \theta - C^0\omega^{a^*} + C^{a^*}(\phi C)^\flat - C^A C^\flat, \\ dC^0 = (f-1)\eta + C^0(\phi C)^\flat + C^\flat. \end{array}\right. \tag{2.8}$$

Since $C^\flat = C^0\eta + \sum_{A=1}^{2m} C^A\omega^A$, then by E. Cartan's structure equations one infers from (2.8)

$$dC^\flat = 2(\phi C)^\flat \wedge C^\flat. \tag{2.9}$$

This proves that $C^\flat$ is a recurrent form [D] having $2(\phi C)^\flat$ as the recurrence form, and so one refinds Rosca's lemma induced by the concept of skew symmetric vector fields [R1], [R2].

Next, since

$$(\phi C)^\flat = \sum_{a=1}^{m}(C^a\omega^{a^*} - C^{a^*}\omega^a), \tag{2.10}$$

one infers by (2.8) and E. Cartan's structure equations

$$d(\phi C)^\flat = 2(f - C^0)\Omega + \eta \wedge (C^\flat + (\phi C)^\flat). \tag{2.11}$$

Now by the exterior differentiation of (2.11), one derives on behalf of (2.2)

$$f = C^0 \tag{2.12}$$

and

$$\eta \wedge C^\flat \wedge (\phi C)^\flat = 0. \tag{2.13}$$

Hence by (2.13) one may say that the forms η, $C^\flat$ and $(\phi C)^\flat$ belong to the same ideal I.

Conversely, by straightforward computations, one may prove that if a vector field C on M satisfies (2.9) and (2.13), then it implies (2.4).

By (2.12) one has

$$d(\phi C)^\flat = \eta \wedge (C^\flat + (\phi C)^\flat) = 0. \tag{2.14}$$

Then by (2.9) and (2.13) it is seen that the three 2-forms $(\phi C)^\flat \wedge C^\flat$, $\eta \wedge C^\flat$ and $\eta \wedge (\phi C)^\flat$ are closed. Therefore, if the Kenmotsu manifold M

under consideration carries a ϕ-SSC vector field C, then it is foliated by 3 surfaces tangent to the distributions spanned by $\{\xi, C\}$, $\{\xi, \phi C\}$ and $\{C, \phi C\}$, respectively.

Next by (2.12) and the third equation of (2.8) one may write

$$\operatorname{grad}\rho = (\rho - 2)\xi + \rho\phi C + 2C. \tag{2.15}$$

On the other hand, by the third equation of (2.1) one derives

$$\nabla\phi C = (\phi C)^\flat \otimes \phi C + C^\flat C - \left((\phi C)^\flat + \frac{\rho}{2}C^\flat\right) \otimes \xi \tag{2.16}$$

and by a standard calculation one gets

$$\operatorname{div}\phi C = \|C\|^2 - \frac{\rho^2}{4}. \tag{2.17}$$

Since C is a conformal vector field on M, one has as is known $\operatorname{div} C = \frac{2m+1}{2}\rho$ and also finds

$$\langle dp, \phi C\rangle = \rho\left(2f - \frac{\rho^2}{4}\right), \quad \langle dp, \xi\rangle = 2(\rho - 1). \tag{2.18}$$

Hence by (2.15) and (2.16) one gets

$$\Delta\rho = -\operatorname{div}(\operatorname{grad}\rho) = 2(1+2m) - (3+4m+4l)\rho + \frac{\rho}{2},$$

where $2l = \|C\|^2$.

Now by Yano's formula [B], that is

$$\mathcal{L}_C K = 2m\Delta\rho - K\rho,$$

one may write

$$\mathcal{L}_C K = 2(1+2m)2m - [2m(3+4m+2l) + K]\rho - \frac{\rho^3}{2},$$

where K denotes the scalar curvature of M.

Next, by the general formula for conformal vector fields (see [B]), since we know that

$$2\mathcal{L}_C\mathcal{R}(Z, Z') = (\Delta\rho)g(Z, Z') - (2m-1)\operatorname{Hess}_\rho(Z, Z'),$$

where $\mathcal{R}$ means the Ricci tensor of M, one finds, after some calculation, that

$$2\mathcal{L}_C\mathcal{R}(C,Z) = [\Delta\rho - 2\rho(1+l)(2m-1) + (2m-1)]g(C,Z) - (2m-1)\frac{\rho}{2}(\rho-2)g(\phi C,Z) + 4l\left(1-\frac{\rho^2}{4}\right)\eta(Z).$$

Hence for any Z orthogonal to the surface S tangent to the distribution spanned by $\{\xi, \phi C\}$, $\mathcal{L}_C\mathcal{R}(C,Z)$ defines an infinitesimal transformation for $g(C,Z)$.

On the other hand, by (2.1) one has

$$\mathcal{L}_C\Omega = \rho\Omega + \eta\wedge(C^\flat - (\phi C)^\flat). \tag{2.19}$$

From (2.19) and (2.13) one derives

$$d(\mathcal{L}_C\Omega) = (d\rho + 2\rho\eta)\wedge\Omega \tag{2.20}$$

and so, according to the definition, it follows that C is an *infinitesimal relative conformal transformation* of Ω.

It should be noticed that since η is closed, then making use of the cohomological transformation operator d^η (see Section 1), one may also write

$$d^\eta(\mathcal{L}_C\Omega) = \eta\wedge\mathcal{L}_C\Omega \tag{2.21}$$

and say that C defines an *infinitesimal conformal cohomological transformation* of Ω.

Next, by a short calculation, one gets from (2.2)

$$\mathcal{L}_{\phi C}\Omega = \frac{1}{2}d\rho\wedge\eta - d(\phi C)^\flat,$$

and consequently

$$d(\mathcal{L}_{\phi C}\Omega) = 0.$$

Hence, following the definition (see also [AM]), the above equation says that ϕC is a *relatively integral invariant* of Ω.

Summarizing, up these computations, we have the following

Theorem. *Let $M(\phi, \Omega, \eta, \xi, g)$ be a $(2m+1)$-dimensional Kenmotsu manifold. Then the necessary and sufficient condition in order that M carries a ϕ-skew symmetric conformal vector field C, that is*

$$\nabla_Z C = fZ + g(Z, \phi C)C - g(Z, C)\phi C, \quad Z \in \Gamma TM,$$

is that

$$dC^\flat = 2(\phi C)^\flat \wedge C^\flat, \quad C^\flat \wedge (\phi C)^\flat \wedge \eta = 0$$

(i.e. $C^\flat$ is exterior recurrent with $(\phi C)^\flat$ as the recurerence form and $C^\flat$, $(\phi C)^\flat$ and η belong to the same ideal).

Any such a Kenmotsu manifold is foliated by 3 surfaces tangent to the distributions spanned by $\{\xi, C\}$, $\{\xi, \phi C\}$ and $\{C, \phi C\}$ respectively.

If Z it is any vector field orthogonal to the surface S tangent to the distribution spanned by $\{\xi, \phi C\}$, then $\mathcal{L}_C \mathcal{R}(C, Z)$ defines an infinitesimal conformal transformation for $g(Z, C)$.

In addition, C defines an infinitesimal relative conformal transformation of Ω, i.e.,

$$d(\mathcal{L}_C \Omega) = (d\rho + 2\rho\eta) \wedge \Omega, \quad \rho = 2f$$

and ϕC is a relatively integral invariant of Ω, i.e.

$$d(\mathcal{L}_{\phi C} \Omega) = 0.$$

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