Publ. Math. Debrecen
59 / 3-4 (2001), 389–406

Harmonicity, minimality, conformality, in terms of horizontal and vertical Lee forms

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Abstract. Let Φ be a holomorphic map of constant rank between almost Hermitian manifolds. We obtain a class of maps Φ for which harmonicity is equivalent with minimal fibres, by extending to higher dimensions a result of Baird–Eells, [2]. The holomorphic distributions $V = \operatorname{Ker} \Phi$ and $H = V^{\perp}$ allows us to attach the horizontal and vertical Lee forms, in terms of which we characterize: harmonic maps and morphisms, minimal fibres, the distribution H integrable and minimal. Obstructions to the existence of a conformal change of metric rendering Φ harmonic or with minimal fibres are provided.

1. Introduction

Large classes of maps between Riemannian manifolds for which harmonicity is equivalent with minimality of fibres are exhibited by Baird–Eells in [2] and Wood in [24]. In the case of holomorphic maps between almost Hermitian manifolds, we obtain a similar class in Theorem 5.3 which extends to higher dimensions a well-known result of Baird–Eells [2] (see Theorem 2.3 below). From [6], [1] it follows that any holomorphic map $\Phi : (M, g, J) \to (N, \widetilde{g}, \widetilde{J})$ of non-zero constant rank between almost Hermitian manifolds defines on M two complementary orthogonal holomorphic distributions, namely vertical $V = \operatorname{Ker} d\Phi$ and horizontal $H = V^{\perp}$, which are proper when the map is not immersive. (The distribution V is zero if and only if Φ is immersive.) As V is integrable, we give necessary and sufficient conditions for the integrability and minimality of H. Harmonic morphisms are a special subclass of harmonic maps

Mathematics Subject Classification: 53C20, 58E20, 53C55.
Key words and phrases: harmonic maps and morphisms, Lee forms, almost Hermitian manifolds.

[4], [10], [13]. We introduce horizontal and vertical Lee forms in terms of which we characterize harmonic maps, harmonic morphisms, minimality of fibres and cosymplectic manifolds.

As the holomorphicity of Φ and both distributions V and H are independent of the conformal change of metrics, we use the above one-forms to characterize the existence of a conformal change of g rendering V and H minimal. Some examples are provided at the end.

All data are assumed to be smooth. For any distributions D on manifold, $\Gamma(D)$ will denote the module of its vector fields.

2. Background

We review some basic notions from the theory of harmonic maps, [7]–[9]. Let Φ be a map between Riemannian manifolds and let $\nabla d\Phi$ denote its second fundamental form defined by $(\nabla d\Phi)(X,Y) = \nabla^{\Phi^{-1}TN}_{d\Phi(X)} d\Phi(Y) - d\Phi(\nabla^M_X Y)$, $\forall X, Y \in \Gamma(TM)$, where $\nabla^M, \nabla^N, \nabla^{\Phi^{-1}TN}$ denote respectively the Levi–Civita connections on M, N and the induced connection on the pull-back bundle $\Phi^{-1}TN$ from ∇^N. Φ is called totally geodesic (resp. harmonic) if $\nabla d\Phi$ (resp. the tension $\tau = \operatorname{trace} \nabla d\Phi$) vanishes on M.

A rich theory is devoted to a special class of harmonic maps, namely *harmonic morphisms*, [4], [10], [13]. Defined as a map $\Phi : (M, g) \to (N, \widetilde{g})$ between Riemannian manifolds, which pulls back any local harmonic function $f : U \to \mathbb{R}$ (on an open set $U \subset N$ with $\Phi^{-1}(U)$ non-empty) to a local harmonic function $f \circ \Phi : \Phi^{-1}(U) \to \mathbb{R}$, the concept of harmonic morphisms is characterized in [10], [13]. More precisely, the *harmonic morphisms* are the harmonic maps which are horizontally weakly conformal. To recall this notion, we first remark that any map $\Phi : (M, g) \to N$, from a Riemannian manifold, defines at each point $p \in M$ *the vertical space* $V_p = \operatorname{Ker} d\Phi_p$ and its g-orthogonal complement $H_p = V_p^{\perp}$, called *the horizontal space*. When Φ is of constant rank, *the vertical* (resp. *horizontal*) *distribution* is defined on M by $p \in M \to V_p$ (resp. H_p) $\subset T_pM$. Otherwise, we suppose Φ has constant rank on a subset (always open) $M \setminus C$ of M, where $C = \{p \in M \mid d\Phi_p = 0\}$.

Any submersion $\Phi : (M, g) \to (N, \widetilde{g})$ between Riemannian manifolds is called *horizontally conformal* if it is conformal (i.e. $\lambda g = \Phi^* \widetilde{g}$) on H, for a positive function λ, called *the conformal factor* or *dilation*. In particular, Φ is *horizontally homothetic* if $\operatorname{grad} \lambda \in V$. *A horizontally weakly conformal* *map* between Riemannian manifolds can be now defined as a map which is a horizontally conformal submersion on $M \setminus C$.

In the context of Riemannian manifolds, we recall some results of Baird–Eells [2] and J. C. Wood [24], concerning the relation between harmonic maps and minimal fibres. The vertical distribution of any map $\Phi : (M, g) \to N$ of constant rank on $M \setminus C$ (for C possible empty) is integrable. The leaves of the vertical foliation (called vertical leaves) are the connected components of the fibres of V. The fibres of Φ are called *minimal* if they have zero mean curvature.

The following Theorems 2.1 and 2.2 are essentially due to Baird–Eells [2] with a slight extension (which allow critical points) by J. C. Wood [24], [25], [4].

Theorem 2.1 [2]. *Let $\Phi : (M^m, g) \to (N^2, \widetilde{g})$ be a non–constant horizontally weakly conformal mapping to a Riemann surface. Then Φ is harmonic if and only if it has minimal fibres at regular points.*

The above result is extended for higher dimensions to:

Theorem 2.2 [2]. *Let $\Phi : (M, g) \to (N, \widetilde{g})$ be a non-constant horizontally weakly conformal map and* $\dim N > 2$. *Then any two of the following conditions imply the third:*

(i) *Φ is harmonic (and so a harmonic morphism);*

(ii) *the fibres of Φ are minimal at regular points;*

(iii) *Φ is horizontally homothetic.*

Another generalization of Theorem 2.1 is obtained by J. C. Wood [24], where it is shown that the equivalence Φ is harmonic $\Longleftrightarrow$ Φ has minimal fibres is valid for a larger class of maps Φ, namely those with holomorphic horizontal quadratic differential.

Several connections between harmonic and holomorphic maps are given in [3], [12], [17], [18], [23].

Corresponding to Theorem 2.1 for Riemannian case, we recall the following:

Theorem 2.3 [2]. *Let $\Phi : (M^m, g, J) \to (N^2, \widetilde{g}, \widetilde{J})$ be a non-constant holomorphic map from an almost Hermitian manifold to a Riemann surface. Then Φ is a harmonic morphism if and only if it has minimal fibres.*

Corresponding to the Riemannian case, where Theorem 2.1 was extended to higher dimensions, our aim is to obtain an extension of Theorem 2.3 in the almost Hermitian case.

3. Horizontal and vertical associated 2-forms

Some basic notions are fixed by the following:

Lemma 3.1 [1]. *Let $\Phi : (M^{2m}, J) \to (N, \widetilde{J})$ be a holomorphic map of rank k between almost complex manifolds and write* $\dim M = 2m$. *Then:*

(i) *$k = 2q$ for some $q \in N$ and the vertical distribution Φ is of dimension $2(m-q)$;*

(ii) *V is J-invariant, i.e. $J(V) = V$;*

(iii) *V is integrable and the vertical leaves are holomorphic submanifolds of M;*

(iv) *When the domain manifold (M, g, J) is almost Hermitian, the horizontal distribution H defined as the orthogonal complement of V is $2q$-dimensional and J-invariant. As H (resp. V) carries the induced Hermitian structure from M, then locally, one can always chose an orthonormal frame of H (resp. V) of the form*

$$\{e_i, Je_i : i = \overline{1, q}\}\ (\textit{resp. } \{\varepsilon_i, J\varepsilon_i : i = \overline{q+1, m}\}). \tag{3.1}$$

On the manifold M one can always choose a local orthonormal frame, denoted by:

$$\{Y_i, JY_i : i = \overline{1, m}\}. \tag{3.2}$$

From [6], any $2q$-dimensional distribution D on a $2m$-dimensional orientable manifold M^{2m} is called *holomorphic*, if the tangent bundle TM admits a reduction of the structure group to a product $U(m-q) \times U(q)$ of two unitary groups, with D corresponding to the $U(q)$ factor.

As a consequence of Lemma 3.1, the next statement is the starting point of our note.

Proposition 3.2. *Any holomorphic map of non-zero constant rank from an almost Hermitian manifold to an almost complex manifold is either immersive, or it defines on the domain manifold two proper complementary orthogonal holomorphic distributions.*

On an almost Hermitian manifold (M, g, J), *the associated 2-form* Ω is defined by:

$$\Omega(X, Y) = g(X, JY),\ \forall X, Y \in \Gamma(TM). \tag{3.3}$$

If $\Phi : (M, g, J) \to (N, \widetilde{g}, \widetilde{J})$ is a holomorphic map of constant rank, then corresponding to the horizontal and vertical distributions on M, we put:

$$X = hX + vX, \ \forall X \in \Gamma(TM), \tag{3.4}$$

where $hX \in \Gamma(H)$ and $vX \in \Gamma(V)$. It follows that:

$$\Omega = \Omega_h + \Omega_v. \tag{3.5}$$

Notation 3.3. Ω_h and Ω_v denote the horizontal and vertical associated 2-forms on M, defined by:

$$\begin{aligned} &\Omega_h(X, Y) = \Omega(hX, hY) \quad \text{and} \\ &\Omega_v(X, Y) = \Omega(vX, vY), \quad \text{respectively}, \ \forall X, Y \in \Gamma(TM). \end{aligned} \tag{3.6}$$

or equivalently:

$$\begin{aligned} &\Omega_h(X, Y) = \Omega(hX, Y) = \Omega(X, hY) \quad \text{and} \\ &\Omega_v(X, Y) = \Omega(vX, Y) = \Omega(X, vY), \end{aligned} \tag{3.7}$$

respectively $\forall X, Y \in \Gamma(TM)$.

$$\begin{aligned} &H = \{X \in \Gamma(TM) \mid i_X \Omega_v = 0\} \quad \text{and} \\ &V = \{X \in \Gamma(TM) \mid i_X \Omega_h = 0\}. \end{aligned} \tag{3.8}$$

The horizontal conformality, horizontal homothety and weakly conformality, defined before by restricting to H, can be expressed globally, in terms of Ω_h and Ω_v as follows:

Lemma 3.4. *Let $\Phi : (M, g, J) \to (N, \widetilde{g}, \widetilde{J})$ be a holomorphic map and let $\widetilde{\Omega}$ be the associated 2-form on N.*

(a) *When* rank Φ *is constant, Φ is an isometric immersion if and only if $\Omega_v = 0$ and $\Omega_h = \Phi^*\widetilde{\Omega}$;*

(b) *Suppose that Φ is submersive. Then:*
   (i) *Φ is a Riemannian submersion if and only if $\Omega_h = \Phi^*\widetilde{\Omega}$;*
   (ii) *Φ is a horizontally conformal (resp. horizontally homothetic) submersion if and only if $\lambda\Omega_h = \Phi^*\widetilde{\Omega}$ for a positive function (resp. positive function with vertical gradient) λ on M;*

(c) *Suppose that Φ is submersive on $M \setminus C$. Then Φ is horizontally weakly conformal if and only if $\lambda\Omega_h = \Phi^*\widetilde{\Omega}$ for a positive function λ on $M \setminus C$.*

The derivatives of Ω_h and Ω_v are given by the following:

Formulas 3.5. For any $A, B, C \in \Gamma(H)$ and $U, W \in \Gamma(V)$, we have:

$$d\Omega_h(A,B,C) = d\Omega(A,B,C) \text{ and } d\Omega_v(A,B,C) = 0; \tag{3.9}$$

$$d\Omega_h(A,B,U) = d\Omega(A,B,U) + \Omega(v[A,B],U) \text{ and} \tag{3.10}$$
$$d\Omega_v(A,B,U) = -\Omega([A,B],U);$$

$$d\Omega_h(U,W,\cdot) = 0 \quad \text{and } d\Omega_v(U,W,\cdot) = d\Omega(U,W,\cdot). \tag{3.11}$$

As a consequence, we obtain:

Proposition 3.6. *For any holomorphic map of constant rank from an almost Hermitian manifold to an almost complex manifold, H is integrable if and only if*

$$d\Omega_v(A,B,\cdot) = 0 \quad \textit{or equivalently} \tag{3.12}$$

$$d\Omega_h(A,B,\cdot) = d\Omega(A,B,\cdot), \quad \forall A, B \in \Gamma(H). \tag{3.13}$$

Remark 3.7. Thus the restriction of $d\Omega_v$ to $H \times H \times TM$ measures the obstruction to integrability of the horizontal distribution. When H is integrable, the horizontal leaves are holomorphic submanifolds.

4. Harmonicity in terms of horizontal and vertical Lee forms

On any $2m$-dimensional almost Hermitian manifold (M^{2m}, g, J) with Levi–Civita connection denoted by ∇, *the divergence* of J is defined by

$$\delta J = \sum_{i=1}^{m} \{(\nabla_{Y_i} J)Y_i + (\nabla_{JY_i} J)JY_i)\}, \tag{4.1}$$

and from [22], [12], modulo a constant, the vector field $J\delta J$ is the dual to the Lee form $\alpha \in \Gamma(T^*M)$, [16] defined by $\alpha = 0$ if $m = 1$ and otherwise by

$$\alpha_p(X) = \frac{1}{2(m-1)} \sum_{i=1}^{m} d\Omega(Y_i, JY_i, X), \quad \forall X \in T_p(M),\ p \in M. \tag{4.2}$$

Any almost Hermitian manifold whose Lee form (or equivalently δJ) vanishes, is called *cosymplectic* as in [12, p. 188], [20], or *semi-Kähler* as in [11] (or like in [15]).

Let $\Phi : (M^{2m}, g, J) \to (N, \widetilde{J})$ be a holomorphic map of constant rank from an almost Hermitian manifold to an almost complex manifold. To exclude the case when either the horizontal or the vertical distribution is trivial, we work under the following:

Hypothesis 4.1. Throughout the rest of the paper, the maps are neither constant, nor immersive. Then the Lee form has the following splitting:

$$2(m-1)\alpha = \alpha_h + \alpha_v, \tag{4.3}$$

where:

Notation 4.2. α_h and α_v denote the horizontal and vertical Lee forms on M, defined respectively by:

$$\begin{aligned} \alpha_v(X) &= \sum_{i=1}^{m} d\Omega_v(Y_i, JY_i, X), \quad \text{and} \\ \alpha_h(X) &= \sum_{i=1}^{m} d\Omega_h(Y_i, JY_i, X), \quad \forall X \in T_pM,\ p \in M. \end{aligned} \tag{4.4}$$

Remark 4.3.

(i) Defined pointwise by (4.4), α_h and α_v are 1-forms and (4.4) is independent of the choice of orthonormal frame.

(ii) Writing $\operatorname{rank}\Phi = 2q$, then from (3.9), (3.11) we obtain the following condition equivalent to (4.4):

$$\alpha_h(X) = \sum_{i=1}^{q} d\Omega_h(e_i, Je_i, X), \qquad \text{and} \tag{4.5}$$

$$\begin{aligned} \alpha_v(X) &= \sum_{i=1}^{q} d\Omega_v(e_i, Je_i, X) + \sum_{i=q+1}^{m} d\Omega_v(\varepsilon_i, J\varepsilon_i, X) \\ &= \sum_{i=1}^{q} d\Omega_v(e_i, Je_i, X) + \sum_{i=q+1}^{m} d\Omega(\varepsilon_i, J\varepsilon_i, X),\ \forall X \in \Gamma(TM) \end{aligned}$$

In particular, for horizontal vector fields, (4.5) becomes

$$\begin{aligned}\alpha_h(A) &= \sum_{i=1}^{q} d\Omega_h(e_i, Je_i, A) = \sum_{i=1}^{q} d\Omega(e_i, Je_i, A) \quad \text{and}\\ \alpha_v(A) &= \sum_{i=q+1}^{m} d\Omega_v(\varepsilon_i, J\varepsilon_i, A) = \sum_{i=q+1}^{m} d\Omega(\varepsilon_i, J\varepsilon_i, A),\\ &\forall A \in \Gamma(H).\end{aligned} \tag{4.6}$$

(iii) In a similar way to (4.3), the divergence δJ splits:

$$\delta J = \delta_h J + \delta_v J, \quad \text{where} \tag{4.7}$$

$$\delta_h J = \sum_{i=1}^{q}\{(\nabla_{e_i} J)e_i + (\nabla_{Je_i} J)Je_i\} \quad \text{and} \tag{4.8}$$

$$\delta_v J = \sum_{i=q+1}^{m}\{(\nabla_{\varepsilon_i} J)\varepsilon_i + (\nabla_{J\varepsilon_i} J)J\varepsilon_i\}.$$

On H the following dualities hold:

$$\begin{aligned}\alpha_h(A) &= g(J\delta_h J, A) = \Omega(A, \delta_h J) \text{ and}\\ \alpha_v(A) &= g(J\delta_v J, A) = \Omega(A, \delta_v J), \quad \forall A \in \Gamma(H).\end{aligned} \tag{4.9}$$

The $(1,2)$-symplectic manifolds [12, p. 188], [20] are also called *quasi-Kähler* [11] (or see [15]). An almost Hermitian manifold is $(1,2)$-symplectic if and only if [1]

$$d\Omega(X, JX, Y) = 0, \quad \forall X, Y \in \Gamma(TM). \tag{4.10}$$

Lemma 4.4. *If $\Phi : (M, g, J) \to (N, \widetilde{g}, \widetilde{J})$ is a holomorphic map of constant rank and the target $(N, \widetilde{g}, \widetilde{J})$ is $(1,2)$-symplectic, then Φ is harmonic if and only if*

$$\alpha_h = -\alpha_v \text{ on } H. \tag{4.11}$$

Proof. The relation (4.3) yields the following equivalence:

The relation (4.11) $\Longleftrightarrow \alpha = 0 \quad \text{on} \quad H \Longleftrightarrow J\delta J \in V$.

Now the assertion follows from a result of [12], stating that under the above conditions, Φ is harmonic if and only if $d\Phi(J\delta J) = 0$. $\square$

Definition 4.5. A one-form is called *D-annihilator*, if it vanishes on any vector field of a distribution D.

Lemma 4.6. *A holomorphic map $\Phi : (M^{2m}, g, J) \to (N^2, \widetilde{g}, \widetilde{J})$ from an almost Hermitian manifold to a Riemann surface is a harmonic morphism if and only if α_v is an H-annihilator at any regular point of M.*

Proof. As Φ is non-constant (from Hypothesis 4.1) and holomorphic, then it is horizontally weakly conformal. Since on regular points H is 2-dimensional, we obtain from (4.6) that $\alpha_h = 0$ on H and the statement follows from Lemma 4.4. $\square$

Under Hypothesis 4.1, we have:

Proposition 4.7. *Let $\Phi : (M, g, J) \to (N, \widetilde{g}, \widetilde{J})$ be a horizontally homothetic holomorphic map onto N. Then N is cosymplectic if and only if α_h is an H-annihilator on any regular point of M.*

Proof. As Φ is horizontally homothetic, it is submersive on $M \setminus C$ and hence $\dim N = \operatorname{rank} \Phi$ $(= 2n)$. If $\widetilde{\Omega}$ (resp. $\widetilde{\alpha}$) denotes the associated 2-form (resp. the Lee form) on N, then from Lemma 3.4 the following equivalences hold on $M \setminus C$:

$$\alpha_h = 0 \text{ on } H \Longleftrightarrow \sum_{i=1}^{n} d\Omega_h(e_i, Je_i, A) = 0,\ \forall A \in \Gamma(H) \Longleftrightarrow$$

$$\Longleftrightarrow \sum_{i=1}^{n} d\widetilde{\Omega}(d\Phi(e_i), \widetilde{J} d\Phi(e_i), d\Phi(A)) = 0,\ \forall A \in \Gamma(H) \Longleftrightarrow$$

$$\Longleftrightarrow \widetilde{\alpha} = 0 \text{ on } \Phi(M \setminus C),$$

since $\{d\Phi(e_i),\ \widetilde{J}\Phi(e_i) : i = \overline{1, n}\}$ is a multiple of an orthonormal frame on $\Phi(M \setminus C)$.

If we suppose N be cosymplectic, i.e. $\widetilde{\alpha} = 0$ on N, then the above equivalence yields $\alpha_h = 0$ on H at any point of $M \setminus C$.

Conversely, if we suppose α_h be an H-annihilator on $M \setminus C$, which is equivalent to $\widetilde{\alpha} = 0$ on $\Phi(M \setminus C)$, then by Sard, $\Phi(M \setminus C)$ is dense in $\Phi(M) = N$ so by continuity, $\widetilde{\alpha} = 0$ on the whole of N, i.e. N is cosymplectic. $\square$

Remark 4.8. In Proposition 4.7 (resp. Lemma 4.6) the restriction of α_h (resp. α_v) to H measures the obstruction of N to be cosymplectic (resp. the obstruction of Φ to be a harmonic morphism).

5. Minimal foliations

If D is a k-dimensional distribution on a Riemannian manifold (M, g), then its mean curvature vector field is defined by [5, p. 7]:

$$\mu_D = \frac{1}{k} \sum_{i=1}^{k} \mathrm{nor}(\nabla_{Z_i} Z_i), \tag{5.1}$$

where $\{Z_i : i = \overline{1, k}\}$ is any local orthonormal frme of D, ∇ denotes the Levi–Civita connection and $\mathrm{nor}(\nabla_X Y)$ denotes the component of $\nabla_X Y$ orthogonal to D, $\forall X, Y \in \Gamma(D)$.

When D is integrable, then its foliation is called *minimal (resp. totally geodesic)* provided its leaves are minimal (resp. totally geodesic) submanifolds of M, or equivalently if $\mu_D = 0$ (resp. $\mathrm{nor}(\nabla_X Y) = 0, \forall X, Y \in \Gamma(D)$). Minimal foliations are also called harmonic foliations [14].

In this section, our goal is to characterize the minimality of H, V in terms of α_h, α_v.

Under the Hypothesis 4.1, we have:

Proposition 5.1. *Any holomorphic map of constant rank $\Phi : (M, g, J) \to (N, \widetilde{J})$ has minimal fibres if and only if α_v is an H-annihilator.*

In other words, the restriction of α_v to H measures the obstruction of Φ to have minimal fibres.

Proof. If we denote rank $\Phi = 2q$, then the mean curvature vector field of V is given by:

$$\mu_V = \frac{1}{m-q} \sum_{i=q+1}^{m} \mathrm{nor}(\nabla_{\varepsilon_i} \varepsilon_i + \nabla_{J\varepsilon_i} J\varepsilon_i). \tag{5.2}$$

From (4.8) we obtain:

$$\begin{aligned} J\delta_v J &= \sum_{i=q+1}^{m} \{J(\nabla_{\varepsilon_i} J)\varepsilon_i + J(\nabla_{J\varepsilon_i} J)J\varepsilon_i\} \\ &= \sum_{i=q+1}^{m} \{\nabla_{\varepsilon_i}\varepsilon_i + \nabla_{J\varepsilon_i} J\varepsilon_i + J[\varepsilon_i, J\varepsilon_i]\}. \end{aligned}$$

As V is an integrable J-invariant distribution, from (4.9) we obtain:

$$\alpha_v(A) = g(J\delta_v J, A) = (m-q)g(\mu_V, A), \quad \forall A \in \Gamma(H),$$

which completes the proof. □

We may give another proof of Proposition 5.1, by using the Rummler–Sullivan criterion. On a Riemannian manifold M, any k-dimensional foliation F $(0 < k < \dim M)$ is minimal if and only if the local volume form along the leaves is the restriction of a k-form χ on M which is relatively closed, i.e. $d\chi(X_1, \dots, X_{k+1}) = 0$ whenever k of the vector fields $X_1, \dots, X_{k+1}$ are tangent to F, [19], [21]. Assuming the hypothesis of Proposition 5.1, we write $\dim M = 2m$ and $\operatorname{rank}\Phi = 2q$. If we take the characteristic form $\chi = \Omega\Lambda\dots\Lambda\Omega$ to be $(m-k)$ times the exterior product of the associated 2-form Ω, then its restriction to V is $\chi|_V = \Omega_v \wedge \dots \wedge \Omega_v$, which gives the volume form on the vertical leaves. χ is relatively closed $\iff d\chi(\varepsilon_{q+1}, J\varepsilon_{q+1}, \dots, \varepsilon_m, J\varepsilon_m, X) = 0$, $\forall X \in \Gamma(TM)$. Being trivial for $X \in \Gamma(V)$, the last relation is equivalent to $d\chi(\varepsilon_{q+1}, J\varepsilon_{q+1}, \dots, \varepsilon_m, J\varepsilon_m, A) = 0$, $\forall A \in \Gamma(H)$. For any $A \in \Gamma(H)$, the equivalence follows:

$$d\chi(\varepsilon_{q+1}, J\varepsilon_{q+1}, \dots, \varepsilon_m, J\varepsilon_m, A) = 0 \iff$$

$$\iff (d\Omega\Lambda\Omega\Lambda...\Lambda\Omega)(\varepsilon_{q+1}, J\varepsilon_{q+1}, \dots, \varepsilon_m, J\varepsilon_m, A) = 0$$

$$\iff \text{(from (3.1), (3.3))} \sum_{i=q+1}^{m} d\Omega(\varepsilon_i, J\varepsilon_i, A) = 0 \iff \text{(from (4.6))}\ \alpha_v(A) = 0$$

and Proposition 5.1 is proved again.

Remark 5.2. From Lemma 4.6 and Proposition 5.1, we recover the result of Baird–Eells contained in Theorem 2.3.

Based on Lemma 4.4 and Proposition 5.1, we provide now an extension of Theorem 2.3 in higher dimensions, under Hypothesis 4.1:

Theorem 5.3. *Let $\Phi : (M, g, J) \to (N, \widetilde{g}, \widetilde{J})$ be a holomorphic map of constant rank and suppose that N is $(1,2)$-symplectic, with $\dim N > 2$. Then any two of the following conditions imply the other:*

(i) *Φ is a harmonic map;*

(ii) *Φ has minimal fibres;*

(iii) *α_h is an H-annihilator.*

Remark 5.4. When $\dim N = 2$ (iii) is obviously satisfied and the resulting equivalence (i) $\iff$ (ii) gives Theorem 2.3.

In [24, Theorem 2.9], a relation between harmonicity and minimality of fibres is given for $\dim N = \operatorname{rank}\Phi$.

Lemma 5.5. *Any holomorphic map of constant rank* $\Phi : (M, g, J) \to (N, \widetilde{J})$ *defines a minimal horizontal foliation if and only if* (3.12) *holds and* α_h *is a* V*-annihilator.*

Proof. From Proposition 3.6, the integrability of H is equivalent to (3.12), in which case the horizontal foliation is minimal if and only if the mean curvature vector field of H:

$$\mu_H = \frac{1}{q}\sum_{i=1}^{q} \operatorname{nor}(\nabla_{e_i} e_i + \nabla_{Je_i} Je_i) \tag{5.3}$$

is identically zero. From (4.8) it follows:

$$J\delta_h J = \sum_{i=1}^{q}\{J(\nabla_{e_i} J)e_i + J(\nabla_{Je_i} J)e_i\} = \sum_{i=1}^{q}\{\nabla_{e_i} e_i + \nabla_{Je_i} Je_i + J[e_i, Je_i]\}.$$

Since (3.12) is equivalent to (3.13) and H is J-invariant then from (4.5) and the definition of Levi–Civita connection, it follows:

$$\begin{aligned} qg(\mu_H, U) = g(J\delta_h J, U) &= g\left(\sum_{i=1}^{q}(\nabla_{e_i} e_i + \nabla_{Je_i} Je_i), U\right) \\ &= \sum_{i=1}^{q} d\Omega(e_i, Je_i, U) = \sum_{i=1}^{q} d\Omega_h(e_i, Je_i, U) = \alpha_h(U), \ \forall U \in \Gamma(V), \end{aligned}$$

which complete the proof. □

From Proposition 5.1 and Lemma 5.5, we obtain:

Corollary 5.6. *Any holomorphic map of constant* rank $\Phi : (M, g, J) \to (N, \widetilde{J})$ *defines transversal minimal horizontal and vertical foliations if and only if* (3.12) *holds,* α_h *is a* V*-annihilator and* α_v *is an* H*-annihilator.*

In low dimensions, it follows:

Proposition 5.7. *Any holomorphic submersion from a 4-dimensional almost Hermitian manifold to a Riemann surface defines a minimal horizontal foliation if and only if α_h and α_v are V-annihilators.*

Proof. The domain manifold being 4-dimensional, then (3.12) is equivalent to $\alpha_v = 0$ on V. Then the statement comes from Lemma 5.5. □

6. Conformal and bi-conformal change of metric

On an almost Hermitian manifold (M, g, J), any conformal change of g, being compatible with J, provides an almost Hermitian structure on M, as an element of the family $\rho = \{(\lambda g, J) : \lambda > 0\}$ indexed by the conformal factor λ. For any almost complex manifold $(N, \widetilde{J})$, the map $\Phi : M \to (N, \widetilde{J})$ is holomorphic and in the case when Φ is of constant rank, the horizontal and vertical distributions on M do not depend on the choice of the almost Hermitian structure in ρ.

As any holomorphic map $\Phi : (M, g, J) \to (N, \widetilde{J})$ of constant rank induces the horizontal H and vertical V distributions on M, then any bi-conformal change of g is defined by:

$$\widehat{g}(X, Y) = \lambda g(hX, hY) + \mu g(vX, vY), \quad \forall X, Y \in \Gamma(TM), \tag{6.1}$$

for some positive functions λ, μ on M. Note that after any bi-conformal change $\widehat{g}$ of g, the holomorphic map $\Phi : (M, \widehat{g}, J) \to (N, \widetilde{J})$ defines the same horizontal and vertical distributions H and V on M. Obviously, any conformal change of g is a special sort of bi-conformal change with $\lambda = \mu$.

By a horizontally (resp. vertically) exact one-form ω, we mean that ω restricted to H (resp. V) is exact i.e. $\omega = df$ on H (resp. V), for a function f on M.

Lemma 6.1. *If $\Phi : (M, g, J) \to (N, \widetilde{J})$ is a holomorphic map of constant rank, then the following assertions are equivalent:*

(i) *α_h is a V-annihilator;*

(ii) *there is a conformal change $\widehat{g} = \lambda g$ ($\lambda > 0$) of g with respect to which the horizontal Lee form $\widehat{\alpha}_h$ is vertically exact;*

(iii) *there is a family of bi-conformal changes* $\{\widehat{g} = \lambda g|_H + \mu g|_V : \mu > 0\}$ *of* g, *for a certain* $\lambda > 0$, *with respect to which* $\widetilde{\alpha}_h$ *is vertically exact. (Here* $g|_H$ *and* $g|_V$ *denote the restrictions of* g *to* H *and* V, *respectively).*

A similar statement is valid to characterize α_v as an H-annihilator.

PROOF. Any bi-conformal change $\widehat{g}$ of g given by (6.1), induces $\widehat{\Omega} = \widehat{\Omega}_h + \widehat{\Omega}_v$, where $\widehat{\Omega}_h = \lambda\Omega_h$ and $\widehat{\Omega}_v = \mu\Omega_v$ from which $d\widehat{\Omega}_h = d\lambda \wedge \Omega_h + \lambda d\Omega_h$ and $d\widehat{\Omega}_v = d\mu \wedge \Omega_v + \mu d\Omega_v$. The local orthonormal frames with respect to $\widehat{g}$, which correspond to those given by (3.1), are obtained by taking $\widehat{e}_i = \lambda^{-1/2} e_i$, $\widehat{\varepsilon}_j = \mu^{-1/2}\varepsilon_j$ for $i = \overline{1, q}$, $j = \overline{q+1, m}$, where $\operatorname{rank}\Phi = 2q$ and $\dim M = 2m$. As in (4.5), we deduce:

$$
\begin{aligned}
(6.2)\ \widehat{\alpha}_h(X) &= \sum_{i=1}^{q} d\widehat{\Omega}_h(\widehat{e}_i, J\widehat{e}_i, X) \\
&= \sum_{i=1}^{q} d(\ln\lambda) \wedge \Omega_h(e_i, Je_i, X) + \alpha_h(X), \quad \forall X \in \Gamma(TM).
\end{aligned}
$$

By (3.7) it follows that:

$$
\widehat{\alpha}_h(U) = -q\, d(\ln\lambda)(U) + \alpha_h(U), \quad \forall U \in \Gamma(V). \tag{6.3}
$$

From (4.6), in a similar way, we obtain:

$$
\widehat{\alpha}_v(A) = (q-m)\, d(\ln\mu)(A) + \alpha_v(A), \quad \forall A \in \Gamma(H), \tag{6.4}
$$

which complete the proof. □

As a consequence of Proposition 5.1 and Lemma 6.1, under Hypothesis 4.1, it follows:

Proposition 6.2. *Let* $\Phi : (M, g, J) \to (N, \widetilde{J})$ *be a holomorphic map of constant rank. Then the existence of a conformal change of* g *rendering the fibres minimal is equivalent to the condition that* α_v *be horizontally exact.*

From Theorem 5.3, a particular case of Proposition 6.2 states:

Proposition 6.3. *A holomorphic map* $\Phi : (M^{2m}, g, J) \to (N^2, \widetilde{g}, \widetilde{J})$ *into a Riemann surface is a harmonic morphism with respect to a conformal change of* g *if and only if* α_v *is horizontally exact at regular points.*

Since the horizontal distribution is independent of any conformal change of metric, then from Lemmas 5.5 and 6.1, we obtain:

Proposition 6.4. *Let $\Phi : (M, g, J) \to (N, \widetilde{J})$ be a holomorphic map of constant rank defining an integrable horizontal distribution. Then there is a conformal change of g, rendering the horizontal foliation minimal if and only if α_h is vertically exact.*

By means of bi-conformal changes of metric, as a direct consequence of Proposition 5.1 and Lemmas 5.5 and 6.1, it follows that the statements of Propositions 6.2 and 6.4 can be combined as follows:

Corollary 6.5. *Let $\Phi : (M, g, J) \to (N, \widetilde{J})$ be a holomorphic map of constant rank defining an integrable horizontal distribution. Then there is a bi-conformal change of g rendering the horizontal and vertical foliation minimal if and only if α_h and α_v are respectively, vertically and horizontally exact.*

7. Examples

We show here some particular cases when the associated Lee forms α_h and α_v are annihilators.

Example 7.1. The associated Lee forms α_h and α_v are both H-annihilators, when they are defined by a holomorphic map $\Phi : (M, g, J) \to (N, \widetilde{J})$ of constant rank, from an almost Kähler manifold, as we deduce from (4.6).

Example 7.2. Even if the associated Lee forms α_h and α_v are both H-annihilators, the horizontal distribution H may not be integrable, as follows.

Let $K = \Gamma \setminus G$ be Kodaira–Thurston manifold, where G is the real Lie group of complex matrices expressed by:

$$\begin{pmatrix} 1 & \bar{z}_1 & z_2 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix}$$

(with $z_k = x_k + iy + k \in \mathbb{C},\ k \in \overline{1, 2}$) and Γ denotes the subgroup of G containing all matrices of G whose entries are Gaussian integers. Note that the Lie algebra of the Kodaira–Thurston manifold may be described as the 4-dimensional Lie algebra with basis $\{E_1, E_2, E_3, E_4\}$ expressed by:

$$E_1 = \begin{pmatrix} A & 0_3 \\ 0_3 & A \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0_3 & B \\ -B & 0_3 \end{pmatrix},$$
$$E_3 = \begin{pmatrix} 0_3 & C \\ -C & 0_3 \end{pmatrix}, \quad E_4 = \begin{pmatrix} C & 0_3 \\ 0_3 & C \end{pmatrix}$$

where 0_3 is the zero-matrix of order 3,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A left invariant almost complex structure on this manifold may be defined by:

$$JE_1 = E_2, \quad JE_2 = -E_1, \quad JE_3 = E_4, \quad JE_4 = -E_3.$$

Let g be a Riemannian metric compatible with J. Then (K, g, J) is almost Hermitian. Define an equivalent relation on $\mathbb{C}$ by $u \sim v \iff u - v$ is a Gaussian integer, so that $\mathbb{C}/\sim$ is a torus. Define a map $\Phi : K \to \mathbb{C}/\sim$ by:

$$\begin{pmatrix} 1 & \bar{z}_1 & z_2 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow [z_1].$$

Then we obtain that Φ is holomorphic.

The vertical distribution V is spanned by $\{E_3, E_4\}$ and the horizontal distribution H is spanned by $\{E_1, E_2\}$. We have $[E_1, E_2] = E_3$ and all the other brackets are trivial. By direct calculation $\alpha_h = 0$ on M, $\alpha_v(E_4) = 1$ and $\alpha_v(E_i) = 0$, $i = \overline{1,3}$. As α_v is an H-annihilator, then from Proposition 5.1 it follows that the vertical foliation is minimal (moreover it is totally geodesic). This agrees with [6], where the authors obtained that on the Kodaira–Thurston manifold, the distribution spanned by $\{E_3, E_4\}$ is totally geodesic.

From Theorem 2.3 it follows that Φ is a harmonic morphism. Since (3.12) is not verified, the horizontal distribution M is not integrable which agrees with [6], where the non-integrability of the distribution spanned by $\{E_1, E_2\}$ was shown.

Example 7.3. This is to provide an example for Propositions 4.7, 5.1 and Theorem 5.3. Let $M = S^{2p+1} \times S^{2q+1}$ be the Calabi–Eckmann manifold, where $S^{2p+1} \subset \mathbb{C}^{p+1}$ and $S^{2q+1} \subset \mathbb{C}^{q+1}$ are unit spheres with their standard metrics.

If $\Phi : M \to CP^p \times CP^q$ is the holomorphic map as in [12], we obtain that α_h and α_v are H-annihilators. However, they are not both V-annihilators. By Propositions 4.7, 5.1 and Theorem 5.3 we reobtain here a result of [12], namely that Φ is harmonic with minimal fibres.

Acknowledgements. The authors are deeply indebted to Professor John C. Wood and the referee for their comments which led to an improvement of the original manuscript. This work was done while the first author was visiting Département de Mathématiques, UBO, Brest, France.

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(Received January 18, 2001; revised June 22, 2001)