Publ. Math. Debrecen
59 / 3-4 (2001), 407–427

Frobenius functors and transfer

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Abstract. If the functor F is both a left and a right adjoint of G, then one may define transfer maps associated to F and G. We use these natural transformations to generalize Higman's Criterion for relative projectivity. We also give several applications to situations involving ring extensions and Hopf algebra actions.

Introduction

The functors $F:\mathcal{A}\to\mathcal{B}$ and $G:\mathcal{B}\to\mathcal{A}$ are said to form a Frobenius pair if G is both a left and a right adjoint of F. This concept was introduced in [6], and a general study was done in [10], but of course, various Frobenius functors have been studied long before.

If (F,G) is a Frobenius pair, then one may define the natural transformations

$$\mathrm{Tr}_F:\mathrm{Hom}_{\mathcal{B}}(F(-),F(-))\to\mathrm{Hom}_{\mathcal{A}}(-,-),$$
$$\mathrm{Tr}_G:\mathrm{Hom}_{\mathcal{A}}(G(-),G(-))\to\mathrm{Hom}_{\mathcal{B}}(-,-)$$

(see [11] and the references given there, and also [2]). These transformations are investigated in Section 1, and it turns out they satisfy the usual properties of the trace map Tr_H^G from group representation theory. In Section 2 we discuss relatively F-projective objects of $\mathcal{A}$, and we give a generalization of Higman's Criterion. We include full proofs for convenience. There is a close connection with the notion of separable functors

Mathematics Subject Classification: 18A40, 16W30, 16W50.
Key words and phrases: adjoint functors, Frobenius functors, relative projectivity, transfer, Frobenius extensions, group graded algebras, Hopf algebra actions.

introduced in [17], but there is no implication relationship between the two properties. However many examples of separable functors are Frobenius functors.

Section 3 is devoted to applications of this general setting. We calculate the transfer map in some particular cases, obtaining generalizations of results of [8], [12], [4]. We prove an essential version of Maschke's theorem, and we give another characterization of Frobenius functors between module categories.

Although not always needed, all our categories and functors will be additive. Rings are associative with unity, and modules are unital and left, unless otherwise specified. Our presentation of Higman's Criterion in Section 2 is inspired by [1, Section 3.6]. Some of the examples in the last section are concerned with group graded algebras and Hopf algebras; our references for these topics are [14], [16], [12] and [13].

1. The transfer map

1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories, $F:\mathcal{A}\to\mathcal{B}$ an (additive) functor and $G:\mathcal{B}\to\mathcal{A}$ a right adjoint of F. Denote by

$$\alpha_{-,-}:\mathrm{Hom}_{\mathcal{B}}(F,\mathrm{id}_{\mathcal{B}})\to\mathrm{Hom}_{\mathcal{A}}(\mathrm{id}_{\mathcal{A}},G)$$

the adjunction isomorphism, and let the unit and the counit of this adjunction be

$$\eta:\mathrm{id}_{\mathcal{A}}\to G\circ F,\quad \eta_A=\alpha_{A,F(A)}(\mathrm{id}_{F(A)}):A\to(G\circ F)(A)$$
$$\varepsilon:F\circ G\to\mathrm{id}_{\mathcal{B}},\quad \varepsilon_B=\alpha^{-1}_{G(B),B}(\mathrm{id}_{G(B)}):(F\circ G)(B)\to B,$$

where $A\in\mathcal{A}$ and $B\in\mathcal{B}$.

1.2. If G is also a left adjoint of F, then we say that F and G are *Frobenius functors* and (F,G) is a *Frobenius pair*. In this case we also have the adjunction isomorphism

$$\gamma_{-,-}:\mathrm{Hom}_{\mathcal{A}}(G,\mathrm{id}_{\mathcal{A}})\to\mathrm{Hom}_{\mathcal{B}}(\mathrm{id}_{\mathcal{B}},F)$$

with unit and counit

$$\xi:\mathrm{id}_{\mathcal{B}}\to F\circ G,\quad \xi_B=\gamma_{B,G(B)}(\mathrm{id}_{G(B)}):B\to(F\circ G)(B),$$
$$\tau:G\circ F\to\mathrm{id}_{\mathcal{A}},\quad \tau_A=\gamma^{-1}_{F(A),A}(\mathrm{id}_{F(A)}):(G\circ F)(A)\to A$$

for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

1.3. Assume that (F, G) is a Frobenius pair. Then we have the following well known equalities.

(1) $\alpha_{A,B}(f) = G(f) \circ \eta_A$ for every $f \in \mathrm{Hom}_{\mathcal{B}}(F(A), B)$;

(2) $\alpha^{-1}_{A,B}(h) = \varepsilon_B \circ F(h)$ for every $h \in \mathrm{Hom}_{\mathcal{A}}(A, G(B))$;

(3) $\gamma_{B,A}(l) = F(l) \circ \xi_B$ for every $l \in \mathrm{Hom}_{\mathcal{A}}(G(B), A)$;

(4) $\gamma^{-1}_{B,A}(g) = \tau_A \circ G(g)$, for every $g \in \mathrm{Hom}_{\mathcal{B}}(B, F(A))$.

Consequently, we have:

(a) $\varepsilon_{F(A)} \circ F(\eta_A) = \mathrm{id}_{F(A)}$;

(b) $G(\varepsilon_B) \circ \eta_{G(B)} = \mathrm{id}_{G(B)}$;

(c) $F(\tau_A) \circ \xi_{F(A)} = \mathrm{id}_{F(A)}$;

(d) $\tau_{G(B)} \circ G(\xi_B) = \mathrm{id}_{G(B)}$.

1.4. The functors F and G induce the natural transformations

$$\mathrm{Res}_F = \mathrm{Res}_F(-,-) : \mathrm{Hom}_{\mathcal{A}}(-,-) \to \mathrm{Hom}_{\mathcal{B}}(F(-), F(-)), \quad f \mapsto F(f),$$

$$\mathrm{Res}_G = \mathrm{Res}_G(-,-) : \mathrm{Hom}_{\mathcal{B}}(-,-) \to \mathrm{Hom}_{\mathcal{A}}(G(-), G(-)), \quad g \mapsto G(g),$$

for any morphism f in $\mathcal{A}$, and for any morphism g in $\mathcal{B}$.

If F and G are Frobenius functors, we may define natural transformations in the opposite direction.

1.5. Lemma. *If (F, G) is a Frobenius pair, then for all $A, A' \in \mathcal{A}$ the following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(F(A), F(A')) \arrow[r, "\alpha_{A,F(A')}"] \arrow[d, "\gamma^{-1}_{F(A),A'}"'] & \mathrm{Hom}_{\mathcal{A}}(A, (G \circ F)(A')) \arrow[d, "\mathrm{Hom}_{\mathcal{A}}(\mathrm{id}_A, \tau_{A'})"] \\
\mathrm{Hom}_{\mathcal{A}}((G \circ F)(A), A') \arrow[r, "\mathrm{Hom}_{\mathcal{A}}(\eta_A, \mathrm{id}_{A'})"] & \mathrm{Hom}_{\mathcal{A}}(A, A')
\end{tikzcd}$$

Proof. Let $g : F(A) \to F(A')$ be a morphism in G. Then by 1.3 (1) we have that

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{A}}(\mathrm{id}_{\mathcal{A}}, \tau_{A'}) \circ \alpha_{A,F(A')}(g) &= \mathrm{Hom}_{\mathcal{A}}(\mathrm{id}_{\mathcal{A}}, \tau_{A'})(G(g) \circ \eta_A) \\
&= \tau_{A'} \circ G(g) \circ \eta_A.
\end{aligned}$$

Similarly

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}}(\eta_A, \mathrm{id}_{A'}) \circ \gamma^{-1}_{F(A),A'}(g) &= \mathrm{Hom}_{\mathcal{A}}(\eta_A, \mathrm{id}_{A'})(\tau_{A'} \circ G(g)) \\ &= \tau_{A'} \circ G(g) \circ \eta_A. \qquad \square\end{aligned}$$

1.6. Proposition. *Assume that G is a right adjoint of F.*

a) *There are isomorphisms*

$$\beta : \mathrm{Nat}(G \circ F, \mathrm{id}_{\mathcal{A}}) \to \mathrm{Nat}(\mathrm{Hom}_{\mathcal{B}}(F, F), \mathrm{Hom}_{\mathcal{A}}(-,-)),$$
$$\delta : \mathrm{Nat}(\mathrm{id}_{\mathcal{B}}, F \circ G) \to \mathrm{Nat}(\mathrm{Hom}_{\mathcal{A}}(G, G), \mathrm{Hom}_{\mathcal{B}}(-,-)).$$

b) *(F, G) is a Frobenius pair is and only if there exist natural transformations*

$$\mathrm{Tr}_F : \mathrm{Hom}_{\mathcal{B}}(F, F) \to \mathrm{Hom}_{\mathcal{A}}(-,-),$$
$$\mathrm{Tr}_G : \mathrm{Hom}_{\mathcal{A}}(G, G) \to \mathrm{Hom}_{\mathcal{B}}(-,-)$$

such that

$$F(\mathrm{Tr}_F(\varepsilon_{F(A)})) \circ \mathrm{Tr}_G(\eta_{GF(A)}) = \mathrm{id}_{F(A)}, \tag{1.6.1}$$

$$\mathrm{Tr}_F(\varepsilon_{FG(B)}) \circ G(\mathrm{Tr}_G(\eta_{G(B)})) = \mathrm{id}_{G(B)}\,. \tag{1.6.2}$$

Proof. a) Let $\nu : G \circ F \to \mathrm{id}_{\mathcal{A}}$ be a natural transformation, and define

$$\beta(\nu) : \mathrm{Hom}_{\mathcal{B}}(F, F) \to \mathrm{Hom}_{\mathcal{A}}(-,-),$$
$$\beta(\nu)_{A,A'}(g) = \nu_{A'} \circ G(g) \circ \eta_A$$

for all morphisms $g : F(A) \to F(A')$ in $\mathcal{B}$. It can be easily seen that the naturality of $\beta(\nu)$ comes down to the following statement. For all morphisms $u : A_1 \to A_2$, $u' : A'_1 \to A'_2$ in $\mathcal{A}$ and $g_1 : F(A_1) \to F(A'_1)$, $g_2 : F(A_2) \to F(A'_2)$ in $\mathcal{B}$, if the first diagram below is commutative, then the second diagram is also commutative.

$$\begin{tikzcd} F(A_1) \arrow[r, "g_1"] \arrow[d, "F(u)"'] & F(A'_1) \arrow[d, "F(u')"] \\ F(A_2) \arrow[r, "g_2"'] & F(A'_2) \end{tikzcd} \qquad \begin{tikzcd} A_1 \arrow[r, "\beta(\nu)_{A_1,A'_1}(g_1)"] \arrow[d, "u"'] & A'_1 \arrow[d, "u'"] \\ A_2 \arrow[r, "\beta(\nu)_{A_2,A'_2}(g_2)"'] & A'_2 \end{tikzcd}$$

This follows by straightforward verification, using 1.3 and the naturality of ν and η.

Conversely, if $\theta : \mathrm{Hom}_{\mathcal{B}}(F, F) \to \mathrm{Hom}_{\mathcal{A}}(-,-)$ is a natural transformation, define

$$\beta^{-1}(\theta) : G \circ F \to \mathrm{id}_{\mathcal{A}},$$
$$\beta^{-1}(\theta)_A = \theta_{GF(A),A}(\varepsilon_{F(A)}) : (G \circ F)(A) \to A.$$

For the naturality of $\beta^{-1}(\theta)$, let $u : A_1 \to A_2$ be a morphism in $\mathcal{A}$, and consider the following diagrams.

$$\begin{tikzcd} FGF(A_1) \arrow[r, "\varepsilon_{F(A_1)}"] \arrow[d, "FGF(u)"'] & F(A_1) \arrow[d, "F(u)"] \\ FGF(A_2) \arrow[r, "\varepsilon_{F(A_2)}"'] & F(A_2) \end{tikzcd} \qquad \begin{tikzcd} GF(A_1) \arrow[r, "\theta_{GF(A_1),A_1}(\varepsilon_{F(A_1)})"] \arrow[d, "GF(u)"'] & A_1 \arrow[d, "u"] \\ GF(A_2) \arrow[r, "\theta_{GF(A_2),A_2}(\varepsilon_{F(A_2)})"'] & A_2 \end{tikzcd}$$

Then the first diagram above commutes by the naturality of ε, so the second commutes by the naturality of θ.

To show that β^{-1} is indeed the inverse of β, let $\nu : G \circ F \to \mathrm{id}_{\mathcal{A}}$ be a natural transformation and A an object of $\mathcal{A}$. Then by 1.3 (b) we have

$$\beta^{-1}(\beta(\nu))_A = \beta(\nu)_{GF(A),A}(\varepsilon_{F(A)}) = \nu_A \circ G(\varepsilon_{F(A)}) \circ \eta_{F(A)} = \nu_{F(A)}.$$

On the other hand, let $\theta : \mathrm{Hom}_{\mathcal{B}}(F, F) \to \mathrm{Hom}_{\mathcal{A}}(-,-)$. Then for all morphisms $g : F(A) \to F(A')$ in $\mathcal{B}$ we have

$$\begin{aligned} \beta(\beta^{-1}(\theta))_{A,A'}(g) &= \theta_{GF(A'),A'}(\varepsilon_{F(A')}) \circ G(g) \circ \eta_A \\ &= \theta_{GF(A'),A'}(\varepsilon_{F(A')}) \circ \alpha_{A,F(A')}(g). \end{aligned}$$

By 1.3 (2) we have that

$$\varepsilon_{F(A')} \circ F(\alpha_{A,F(A')}(g)) = \alpha^{-1}_{A,F(A')}(\alpha_{A,F(A')}(g)) = g,$$

and by the naturality of θ we obtain

$$\theta_{GF(A'),A'}(\varepsilon_{F(A')}) \circ \alpha_{A,F(A')}(g) = \theta_{A,A'}(g),$$

that is, $\beta(\beta^{-1}(\theta)) = \theta$.

Next, given the natural transformation $\zeta : \mathrm{id}_{\mathcal{B}} \to F \circ G$ and the morphism $f : G(B) \to G(B')$ in $\mathcal{A}$, define

$$\delta(\zeta) : \mathrm{Hom}_{\mathcal{A}}(G, G) \to \mathrm{Hom}_{\mathcal{B}}(-,-),$$
$$\delta(\zeta)_{B,B'}(f) = \varepsilon_{B'} \circ F(f) \circ \zeta_B.$$

Then similar arguments show that δ is an isomorphism.

b) Assume that (F, G) is a Frobenius pair, and let $\mathrm{Tr}_F = \beta(\tau)$ and $\mathrm{Tr}_G = \delta(\xi)$. Then (1.6.1) and (1.6.2) follow immediately from 1.3 (c) and (d).

Conversely, assume that Tr_F and Tr_G are natural transformations satisfying (1.6.1) and (1.6.2), and let $\tau = \beta^{-1}(\mathrm{Tr}_F)$ and $\xi = \delta^{-1}(\mathrm{Tr}_G)$. Then it is easy to see that τ and ξ satisfy 1.3 (c) and (d), and therefore G is a left adjoint of F. $\square$

1.7. Definition. Let (F, G) be a Frobenius pair. The natural transformation

$$\mathrm{Tr}_F = \beta(\tau) : \mathrm{Hom}_{\mathcal{B}}(F, F) \to \mathrm{Hom}_{\mathcal{A}}(-,-),$$
$$(g : F(A \to F(A')) \mapsto \mathrm{Tr}_F(g) = \tau_{A'} \circ G(g) \circ \eta_A$$

is the *transfer* (or trace) map associated to F. Similarly,

$$\mathrm{Tr}_G = \delta(\xi) : \mathrm{Hom}_{\mathcal{A}}(G, G) \to \mathrm{Hom}_{\mathcal{B}}(-,-),$$
$$(f : G(B) \to G(B')) \mapsto \mathrm{Tr}_G(f) = \varepsilon_{B'} \circ F(f) \circ \xi_B$$

is the transfer map associated to G.

In the next proposition we collect the main properties of the trace map.

1.8. Proposition. a) *For all $f : A_1 \to A_2$ in $\mathcal{A}$ and $u : F(A_2) \to F(A_3)$ in $\mathcal{B}$ we have $\mathrm{Tr}_F(u \circ F(f)) = \mathrm{Tr}_F(u) \circ f$, and for all morphisms $u : F(A_1) \to F(A_2)$ in $\mathcal{B}$ and $f : A_2 \to A_3$ in $\mathcal{A}$ we have $\mathrm{Tr}_F(F(f) \circ u) = f \circ \mathrm{Tr}_F(u)$. In particular, $\mathrm{Im}\,\mathrm{Tr}_F$ is an "ideal" of $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$.*

b) *For each object A of $\mathcal{A}$ consider the endomorphism $e_A = \tau_A \circ \eta_A$ of A. Then $e_A = (\mathrm{Tr}_F \circ \mathrm{Res}_F)(\mathrm{id}_A)$ is a central element of $\mathrm{End}_{\mathcal{A}}(A)$, and for any morphism $f : A_1 \to A_2$ in $\mathcal{A}$ we have*

$$(\mathrm{Tr}_F \circ \mathrm{Res}_F)(f) = e_{A_2} \circ f = f \circ e_{A_1}.$$

c) (Transitivity) *Let* (F,G) *and* (F',G') *be Frobenius pairs, where* $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{F'} \mathcal{C}$ *and* $\mathcal{C} \xrightarrow{G'} \mathcal{B} \xrightarrow{G} \mathcal{A}$. *Then* $(F' \circ F, G \circ G')$ *is a Frobenius pair, and* $\mathrm{Tr}_{F' \circ F} = \mathrm{Tr}_F \circ \mathrm{Tr}_{F'}$.

Proof. a) By the naturality of η we have that

$$\begin{aligned}\mathrm{Tr}_F(u \circ F(f)) &= \tau_{A_3} \circ G(u \circ F(f)) \circ \eta_{A_1} = (\tau_{A_3} \circ G(u) \circ \eta_{A_2}) \circ f\\ &= \mathrm{Tr}_F(u) \circ f.\end{aligned}$$

The second statement can be proved similarly.

Observe that the naturality of $\mathrm{Tr}_F = \beta(\tau)$ follows from a): if $F(u') \circ g_1 = g_2 \circ F(u)$, then $\mathrm{Tr}_F(F(u') \circ g_1) = \mathrm{Tr}_F(g_2 \circ F(u))$, and by a) we get $u' \circ \mathrm{Tr}_F(g_1) = \mathrm{Tr}_F(g_2) \circ u$. Conversely, a) also follows from the naturality of Tr_F: letting $u' = u \circ F(f)$, the naturality of Tr_F implies $\mathrm{Tr}_F(u') = \mathrm{Tr}_F(u) \circ f$.

b) By definitions we have that

$$(\mathrm{Tr}_F \circ \mathrm{Res}_F)(\mathrm{id}_A) = \mathrm{Tr}_F(\mathrm{id}_{F(A)}) = \tau_A \circ G(\mathrm{id}_{F(A)}) \circ \eta_A = \tau_A \circ \eta_A.$$

The naturality of η and τ implies that e_A is central.

Now let $f : A_1 \to A_2$ and take $u = \mathrm{id}_{F(A_2)}$ in a). Then

$$\begin{aligned}(\mathrm{Tr}_F \circ \mathrm{Res}_F)(f) &= \mathrm{Tr}_F(\mathrm{id}_{F(A_2)}) \circ f = \mathrm{Tr}_F(F(\mathrm{id}_{A_2})) \circ f\\ &= (\mathrm{Tr}_F \circ \mathrm{Res}_F)(\mathrm{id}_{A_2}) \circ f = e_{A_2} \circ f.\end{aligned}$$

c) Clearly, $(F' \circ F, G \circ G')$ is a Frobenius pair. By the definition of Tr we have that $\mathrm{Tr}_{F' \circ F}(f) = \tau'' \circ (G \circ G')(f) \circ \eta''$, where

$$\eta'' = \alpha''(\mathrm{id}) = (\alpha \circ \alpha')(\mathrm{id}) = \alpha(G(\mathrm{id}) \circ \eta') = \alpha(\eta') = G(\eta') \circ \eta.$$

Analogously, $\tau'' = \tau \circ G(\tau')$, and it follows that

$$\begin{aligned}\mathrm{Tr}_{F' \circ F}(f) &= \tau \circ G(\tau') \circ (G \circ G'(f)) \circ G(\eta') \circ \eta\\ &= \tau \circ G(\mathrm{Tr}_{F'}(f)) \circ \eta = \mathrm{Tr}_F(\mathrm{Tr}_{F'}(f)).\end{aligned} \qquad \square$$

1.9. Notice that the existence of the transfer maps is related to the separability of F and G. Recall that according to [17], an arbitrary covariant functor $F : \mathcal{A} \to \mathcal{B}$ is called *separable*, if there is a natural transformation

$$\phi : \mathrm{Hom}_{\mathcal{B}}(F, F) \to \mathrm{Hom}_{\mathcal{A}}(-, -)$$

such that $\phi \circ \mathrm{Res}_F$ is the identity natural transformation on $\mathrm{Hom}_{\mathcal{A}}(-,-)$. Indeed, condition (SF1) in [17] is the splitness property, while (SF2) is the naturality.

When F is a left adjoint of G, then Proposition 1.6 a) can be used to give an alternative proof of RAFAEL's theorem [18].

1.10. Proposition [18, 19]**.** *Assume that (F, G) is an adjoint pair. Then F is separable if and only if the unit of the adjunction splits, and G is separable if and only if the counit of the adjunction splits.*

PROOF. Assume that F is a separable functor and let $\nu = \beta^{-1}(\phi) : G \circ F \to \mathrm{id}_{\mathcal{A}}$. By the naturality of ϕ we obtain

$$\begin{aligned}\nu_A \circ \eta_A &= \phi_{GF(A),A}(\varepsilon_{F(A)}) \circ \eta_A = \phi_{A,A}(\varepsilon_{F(A)} \circ F(\eta_A)) \\ &= \phi_{A,A}(\mathrm{id}_{F(A)}) = \phi_{A,A}(\mathrm{Res}_F(\mathrm{id}_A)) = \mathrm{id}_A\,.\end{aligned}$$

Conversely, assume that $\nu : G \circ F \to \mathrm{id}_{\mathcal{A}}$ is a natural transformation such that $\nu \circ \eta = \mathrm{id}_{\mathcal{A}}$ and let $\phi = \beta(\nu)$. Then, for any morphism $f : A \to A'$ in $\mathcal{A}$,

$$\phi_{A,A'}(F(f)) = \nu_{A'} \circ G(F(f)) \circ \eta_A = f \circ \nu_A \circ \eta_A = f,$$

hence ϕ splits Res_F, and F is separable. $\square$

2. Relative projectivity and Higman's criterion

We keep the notations of the preceding section, and assume in addition that $\mathcal{A}$ and $\mathcal{B}$ are abelian categories. In order to state Higman's theorem in this context we need more definitions.

2.1. Definition. Let $F : \mathcal{A} \to \mathcal{B}$ an additive functor.

a) The object A of $\mathcal{A}$ is called *relatively F-projective* if whenever $f : A' \to A''$ is an epimorphism in $\mathcal{A}$ and $g : A \to A''$ is a morphism such that there is a morphism $h : F(A) \to F(A')$ in $\mathcal{B}$ with $F(f) \circ h = F(g)$, there is a morphism $\bar{h} : A \to A'$ in $\mathcal{A}$ such that $f \circ \bar{h} = g$.

b) Dually, A is called *relatively F-injective* if whenever $f : A' \to A''$ is a monomorphism in $\mathcal{A}$ and $g : A' \to A$ is a morphism such that there is a morphism $h : F(A'') \to F(A)$ in $\mathcal{B}$ with $h \circ F(f) = F(g)$, there is a morphism $\bar{h} : A'' \to A$ such that $\bar{h} \circ f = g$.

c) The short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$-modules is called *F-split*, if $0 \to F(A') \to F(A) \to F(A'') \to 0$ splits in $\mathcal{B}$-modules.

2.2. Theorem (Higman's Criterion)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and (F, G) a Frobenius pair. Let A be an object of $\mathcal{A}$ and consider the following statements.*

(1) *A is a direct summand of $G(F(A))$.*

(2) *There is an object B of $\mathcal{B}$ such that A is a direct summand of $G(B)$.*

(3) $\mathrm{Tr}_F : \mathrm{End}_{\mathcal{B}}(F(A)) \to \mathrm{End}_{\mathcal{A}}(A)$ *is surjective.*

(4) *$\tau_A : G(F(A)) \to A$ has a section.*

(5) *$\eta_A : A \to G(F(A))$ has a retraction.*

(6) *A is relatively F-projective.*

(7) *A is relatively F-injective.*

(8) *If $f : A \to A''$ is an F-split epimorphism, then f splits.*

(9) *If $f : A' \to A$ is an F-split monomorphism, then f splits.*

(10) *e_A is an invertible element of $\mathrm{End}_{\mathcal{A}}(A)$.*

(11) *There is $f \in \mathrm{End}(A)$ such that $\mathrm{Tr}_F(F(f)) = \mathrm{id}_A$.*

Then statements (1) *to* (5) *are equivalent,* (10) *to* (11) *are equivalent, and the implications* (3) $\Longrightarrow$ (6) $\Longrightarrow$ (8), (3) $\Longrightarrow$ (7) $\Longrightarrow$ (9) *and* (10) $\Longrightarrow$ (1) *also hold.*

If in addition η_A is monomorphism, then (9) $\Longrightarrow$ (1) *holds, and if τ_A is epimorphism, then* (8) $\Longrightarrow$ (1) *holds. In particular, if F is faithful, then statements* (1) *to* (9) *are equivalent.*

Proof. The implications (1) $\Longrightarrow$ (2), (4) $\Longrightarrow$ (1), (5) $\Longrightarrow$ (1) and (10) $\Longrightarrow$ (1) are trivial.

(2) $\Longrightarrow$ (3) Assume that A is a direct summand of $G(B)$ and let $p : G(B) \to A$ and $q : A \to G(B)$ be such that $p \circ q = \mathrm{id}_A$. Additionally, let

$$\pi_B = \xi_B \circ \varepsilon_B : F(G(B)) \to F(G(B)).$$

Then by 1.3 (b) and (d) we have that

$$\begin{aligned}\mathrm{Tr}_F(\pi_B) &= \tau_{G(B)} \circ G(\pi_B) \circ \eta_{G(B)} \\ &= \tau_{G(B)} \circ G(\xi_B) \circ G(\varepsilon_B) \circ \eta_{G(B)} = \mathrm{id}_{G(B)}\,.\end{aligned}$$

Consequently, by Proposition 1.8 a) we obtain

$$\mathrm{Tr}_F(F(p) \circ \pi_B \circ F(q)) = p \circ \mathrm{Tr}_F(\pi_B) \circ q = \mathrm{id}_A\,.$$

(3) $\Longrightarrow$ (4), (5) If $k \in \mathrm{End}_{\mathcal{B}}(F(A))$ is such that $\mathrm{Tr}_F(k) = \mathrm{id}_A$, then $\tau_A \circ G(k) \circ \eta_A = \mathrm{id}_A$.

(3) $\Longrightarrow$ (6) Let $k \in \mathrm{End}_{\mathcal{B}}(F(A))$ be as above, and with the notations of Definition 2.1 a) let $\bar{h} = \mathrm{Tr}_F(h \circ k)$. Then

$$\begin{aligned} f \circ \bar{h} &= f \circ \mathrm{Tr}_F(h \circ k) = \mathrm{Tr}_F(f \circ h \circ k) \\ &= \mathrm{Tr}_F(g \circ k) = g \circ \mathrm{Tr}_F(k) = g. \end{aligned}$$

A dual proof works for (3) $\Longrightarrow$ (7).

(6) $\Longrightarrow$ (8) and (7) $\Longrightarrow$ (9) follow easily by Definition 2.1.

(10) $\Longleftrightarrow$ (11) Suppose there is $f \in \mathrm{End}(A)$ an inverse of $e_A = \tau_A \circ \eta_A$. Then by Proposition 1.8 we have $\mathrm{Tr}_F(F(f)) = e_A \circ u = \mathrm{id}_A$. Conversely, if there is $f \in \mathrm{End}(A)$ such that $\mathrm{Tr}_F(F(f)) = \mathrm{id}_A$, using the definition of Tr_F, we have that $\tau_A \circ G(F(f)) \circ \eta_A = \mathrm{id}_A$. By the naturality of τ_A we get $f \circ \tau_A \circ \eta_A = \mathrm{id}_A$, while the naturality of η_A gives $\tau_A \circ \eta_A \circ f = \mathrm{id}_A$.

(8) $\Longrightarrow$ (1) and (9) $\Longrightarrow$ (1) Observe that by 1.3 (a) the monomorphism η_A is F-split, and by 1.3 (c) the epimorphism τ_A is F-split. Recall also that if F is faithful, then it reflects monomorphisms and epimorphisms. $\square$

Due to the next proposition one can define a notion of *relatively F-projective morphism.*

2.3. Proposition. *Let A and A' be objects of $\mathcal{A}$ and $f \in \mathrm{Hom}_{\mathcal{A}}(A, A')$. The following statements are equivalent:*

(1) *f belongs to the image of $\mathrm{Tr}_F : \mathrm{Hom}_{\mathcal{B}}(F(A), F(A')) \to \mathrm{Hom}_{\mathcal{A}}(A, A')$.*
(2) *There is $g : G(F(A)) \to A'$ such that $f = g \circ \eta_A$.*
(3) *There is $h : A \to G(F(A'))$ such that $f = \tau_{A'} \circ h$.*
(4) *f factorizes through an object satisfying conditions* (1) *to* (5) *of Theorem 2.2.*

Proof. (1) $\Longrightarrow$ (2), (3) If $f = \mathrm{Tr}(k)$, then by Lemma 1.5,

$$f = \gamma^{-1}_{F(A),A'}(k) \circ \eta_A = \tau_{A'} \circ \alpha_{A,F(A')}(k).$$

(4) $\Longrightarrow$ (1) If $\mathrm{id}_A = \mathrm{Tr}_F(k)$ and $f = f_2 \circ f_1$, then by Proposition 1.8 a)

$$f = f_2 \circ \mathrm{Tr}_F(k) \circ f_1 = \mathrm{Tr}_F(f_2 \circ k \circ f_1) \in \mathrm{Im}\, \mathrm{Tr}_F\, . \qquad \square$$

3. Applications

The papers in the list of references contain many examples of Frobenius and separable functors. In this section we calculate the transfer map in some of the most usual situations.

In the first two examples we deal with graded rings. Various functors between categories of graded modules have been considered in [17]–[19], [15], [9] and [10].

3.1. *Strongly graded rings.* The original theorem of Higman was concerned with group algebras, and this situation generalizes easily to strongly graded rings.

Let G be a group, $R = \bigoplus_{g\in G} R_g$ a strongly G-graded ring, and H a subgroup of G. Let $\mathcal{F} = \mathrm{Res}_H^G : R\text{-Mod} \to R_H\text{-Mod}$ be the restriction functor and $\mathcal{G} = \mathrm{Ind}_H^G : R_H\text{-Mod} \to R\text{-Mod}$ the induction functor. Then the functor $\mathcal{G}$ is a left adjoint of $\mathcal{F}$. For $M \in R$-Mod and $N \in R_H$-Mod, we have the natural isomorphism

$$\gamma_{N,M} : \mathrm{Hom}_R(R \otimes_{R_H} N, M) \to \mathrm{Hom}_{R_H}(N, M)$$

defined by $\gamma_{N,M}(f)(n) = f(1 \otimes n)$, with inverse $\gamma^{-1}_{N,M}(f')(r \otimes n) = rf'(n)$ for all $n \in N$, $r \in R$. The unit and the counit of this adjunction are

$$\xi_N : N \to \mathrm{Res}_H^G(R \otimes_{R_H} N), \qquad \xi_N(n) = 1 \otimes n,$$
$$\tau_M : R \otimes_{R_H} \mathrm{Res}_H^G(M) \to M, \qquad \tau_M(r \otimes m) = rm.$$

If G/H is finite, then $\mathcal{G}$ is also a right adjoint of $\mathcal{F}$. We choose a system $[G/H] = \{g_1, \ldots, g_l\}$ of representatives for the left cosets of H in G, and for each g_i, let $r^i_1, \ldots, r^i_{t_1} \in R_{g_i}$, $s^i_1, \ldots, s^i_{t_i} \in R_{g_i^{-1}}$ such that $\sum_{j=1}^{t_i} r^i_j s^i_j = 1$.

We have the isomorphism

$$\alpha_{M,N} : \mathrm{Hom}_{R_H}(\mathrm{Res}_H^G M, N) \to \mathrm{Hom}_R(M, R \otimes_{R_H} N),$$
$$\alpha_{M,N}(f)(m) = \sum_{i=1}^{l} \sum_{j=1}^{t_i} r^i_j \otimes_{R_H} f(s^i_j m),$$

with inverse $\alpha^{-1}(f')(m) = f'(m)_H$, for all $m \in M$. Then the unit and the counit associated to α are

$$\eta_M : M \to R \otimes_{R_H} \operatorname{Res}^G_H M, \qquad \eta_M(m) = \sum_{i=1}^{l} \sum_{j=1}^{t_i} r^i_j \otimes_{R_H} s^i_j m,$$

$$\varepsilon_N : \operatorname{Res}^G_H(R \otimes_{R_H} N) \to N, \qquad \varepsilon_N(r \otimes_{R_H} n) = r_H n,$$

where $r_H = \sum_{h \in H} r_h \in R_H$ and $n \in N$. (We have used that $(R \otimes_{R_H} N)_H = R_H \otimes_{R_H} N \simeq N$.)

Observe that $\varepsilon_N \circ \xi_N = \mathrm{id}_N$, hence $\mathcal{G}$ is separable, and $\tau_M \circ \eta_M = [G : H] \,\mathrm{id}_M$. The transfer map $\mathrm{Tr}_{\mathcal{F}} : \mathrm{Hom}_{R_H}(\mathcal{F}(M), \mathcal{F}(M')) \to \mathrm{Hom}_R \times (M, M')$ is given by

$$\mathrm{Tr}_{\mathcal{F}}(f)(m) = \sum_{i=1}^{l} \sum_{j=1}^{t_i} r^i_j f(s^i_j m),$$

for all $m \in M$, while $\mathrm{Tr}_{\mathcal{G}} : \mathrm{Hom}_R(\mathcal{G}(N), \mathcal{G}(N')) \to \mathrm{Hom}_{R_H}(N, N')$ is given by

$$\mathrm{Tr}_{\mathcal{G}}(h)(n) = \varepsilon_{N'}(h(1 \otimes_{R_H} n)) = h(1 \otimes_{R_H} n)_H,$$

for all $n \in N$.

3.2. *The grade forgetting functor.* This is in essence the dualization of the previous example. Let $R = \bigoplus_{g \in G} R_g$ be an arbitrary G-graded ring, and fix two subgroups $K \leq H$ of G. The grade forgetting functor

$$\mathcal{G} : (G/K, R)\text{-gr} \to (G/H, R)\text{-gr}$$

(usually denoted by $\mathcal{U}$) is defined as follows: for $M = \bigoplus_{x \in G/K} M_x \in (G/K, R)$-gr we have $\mathcal{G}(M) = \bar{M} = \bigoplus_{y \in G/H} \bar{M}_y$, where $\bar{M} = M$ (as R-module), and $\bar{M}_y = \bigoplus_{x \subseteq y} M_x$ for all $y \in G/H$, and obviously, $\mathcal{G}(f) = f$ for every morphism $f : M \to M'$ in $(G/K, R)$-gr.

Then $\mathcal{G}$ has a right adjoint $\mathcal{F} : (G/H, R)\text{-gr} \to (G/K, R)$-gr defined as follows: for $N = \bigoplus_{y \in G/H} N_y \in (G/H, R)$-gr we have

$$\mathcal{F}(N) = \tilde{N} = \bigoplus_{x \in G/K} \tilde{N}_x, \qquad \tilde{N}_x = N_{xH},$$

with multiplication by scalars given by $r_g\tilde{n}_x = r_g n_y \in N_{gx}$ where $y = xH$, $\tilde{n}_x = n_y \in N_y, r_g \in R_g,\ g \in G$. If $f : N \to N'$ is morphism in $(G/H, R)$-gr, then $\tilde{f} = \mathcal{F}(f) : \tilde{N} \to \tilde{N}'$ is given by $\tilde{f}(\tilde{n}_x) = f(n_y) \in \tilde{N}_x = N_y$, with $y = xH$ and $\tilde{n}_x = n_y$ as above.

The isomorphism

$$\gamma_{M,N}^{-1} : \mathrm{Hom}_{(G/K,R)\text{-gr}}(M, \mathcal{F}(N)) \to \mathrm{Hom}_{(G/H,R)}(\mathcal{G}(M), N),$$

is given by

$$\gamma_{M,N}^{-1}(g)(m_x) = g(m_x) \in \mathcal{F}(N)_x = N_{xH}$$

for all $g : M \to \mathcal{F}(N)$ and $m_x \in M_x = \mathcal{G}(M)_{xH}$, and we have $\gamma_{M,N}(f)(m_x) = f(m_x) \in \mathcal{F}(N)_x = N_{xH}$ for all $f : \mathcal{U}(M) \to N$ and $m_x \in M_x$.The unit and the counit of this adjunction are

$$\begin{aligned} &\xi_M : M \to \mathcal{F}(\mathcal{G}(M)), && \xi_M(m_x) = m_x \in \mathcal{F}(\mathcal{G}(M))_x,\\ &\tau_N : \mathcal{G}(\mathcal{F}(N)) \to N, && \tau_N(\tilde{n}_x) = \tilde{n}_x \in N_{xH} \end{aligned}$$

for all $x \in G/K$, $m_x \in M_x$ and $\tilde{n}_x = n_y \in N_y$, $y = xH$.

If H/K is finite then $\mathcal{F}$ is also a left adjoint of $\mathcal{G}$. We have the isomorphism

$$\begin{aligned} &\alpha_{N,M} : \mathrm{Hom}_{(G/K,R)\text{-gr}}(\mathcal{F}(N), M) \to \mathrm{Hom}_{(G/H,R)}(N, \mathcal{G}(M)),\\ &\alpha_{N,M}(f)(n_y) = \sum_{\substack{x\in G/K\\ x\subseteq y}} f(\tilde{n}_x), \quad \text{and} \quad \alpha_{N,M}^{-1}(g)(\tilde{n}_x) = g(n_x)_x, \end{aligned}$$

for all $f : \mathcal{F}(M) \to N$ and $n_y \in N_y$, with $\tilde{n}_x = n_y \in \mathcal{F}(N)_x$, and for all $g : N \to \mathcal{G}(M)$ and $\tilde{n}_x = n_y \in \mathcal{F}(N)_x = N_{xH}$, $y = xH$ where $g(\tilde{n}_x)$ is the x-th component of $g(\tilde{n}_x) \in \bigoplus_{x\in G/K} M_x$.

The unit and the counit of this adjunction are defined by

$$\begin{aligned} &\eta_N : N \to \mathcal{G}(\mathcal{F}(N)), \quad \eta_N(n_y) = \sum_{\substack{x\in G/K\\ x\subseteq y}} \tilde{n}_x,\\ &\varepsilon_M : \mathcal{F}(\mathcal{G}(M)) \to M, \quad \varepsilon_M({}^{x'}m_x) = \begin{cases} m_x, & \text{if } x = x'\\ 0, & \text{if } x \neq x', \end{cases} \end{aligned}$$

where $\tilde{n}_x = n_y \in N_y$, $x, x' \in G/K,\ xH = x'H$ and ${}^{x'}m_x \in \mathcal{F}(\mathcal{U}(M))_{x'}$.

Observe that $\varepsilon_M \circ \xi_M = \mathrm{id}_M$, hence $\mathcal{G}$ is separable (see also [19, Section 4] for a discussion of more general functors), and $\tau_N \circ \eta_N = [H:K] \cdot \mathrm{id}_N$.

Finally, for any $f \in \mathrm{Hom}_{(G/K,R)\text{-}\mathrm{Gr}}(\mathcal{G}(M), \mathcal{G}(M')$ we have

$$\mathrm{Tr}_{\mathcal{G}}(f)(m_x) = (\varepsilon_{M'} \circ \mathcal{F}(f) \circ \xi_M)(m_x) = f(m_x)_x,$$

and for any $g \in \mathrm{Hom}_{(G/H,R)\text{-}\mathrm{Gr}}(\mathcal{F}(N), \mathcal{F}(N'))$ we have

$$\mathrm{Tr}_{\mathcal{F}}(g)(n_y) = (\tau_{N'} \circ \mathcal{G}(g) \circ \eta_N)(n_y) = [H/K]g(n_y),$$

where $m_x \in M_x$, $x \in G/K$, and $n_y \in N_y$, $y \in G/H$.

3.3. *Frobenius extensions of rings.* Let $\phi : B \to A$ be a ring homomorphism (so A becomes a (B,B)-bimodule via ϕ), $\mathcal{A} = A$-Mod, $\mathcal{B} = B$-Mod, and consider the scalar restriction functor $F = \phi_* : A\text{-Mod} \to B\text{-Mod}$ and the induction functor $G = A \otimes_B - : B\text{-Mod} \to A\text{-Mod}$. Then G is a left adjoint of F, and (with the notations of 1.2) we have the isomorphism

$$\gamma_{M,N}^{-1} : \mathrm{Hom}_B(N, {}_BM) \to \mathrm{Hom}_A(A \otimes_R N, M),$$
$$\gamma_{M,N}^{-1}(f)(a \otimes_B n) = af(n), \quad \gamma_{M,N}(g)(n) = g(1 \otimes_B n)$$

for any $N \in \mathcal{A}$, $M \in \mathcal{B}$, $a \in A$ and $n \in N$. The unit an the counit of this adjunction are

$$\xi_N : N \to A \otimes_B N, \qquad \xi_N(n) = 1 \otimes n,$$
$$\tau_M : A \otimes_B M \to M, \qquad \tau_M(a \otimes_B m) = am.$$

The separability of F and G was discussed in [17, Proposition 1.3] and [9, Corollary 1.5]. Clearly, (F,G) is a Frobenius pair if and only if the functors $A \otimes_B -$ and $\mathrm{Hom}_B(A, -)$ are isomorphic. Equivalent conditions have been given by Nakayama and Tsuzuku (see [15] and the references given there). By [4, Theorem 2.4], (F,G) is a Frobenius pair if and only if there is an (B,B)-bimodule map $\eta : A \to B$ and an (A,A)-bimodule map $\delta : A \to A \otimes_B A$ such that

$$(\mathrm{id}_A \otimes_B \eta) \circ \delta = \delta \circ (\eta \otimes_B \mathrm{id}_A) = \mathrm{id}_A\,. \tag{3.3.1}$$

We shall use this characterization to compute the transfer map

$$\mathrm{Tr}_F : \mathrm{Hom}_B(M, M') \to \mathrm{Hom}_A(M, M')$$

for the A-modules M and M'. First, we have the adjunction isomorphism

$$\alpha_{M,N} : \mathrm{Hom}_B({}_BM, N) \to \mathrm{Hom}_A(M, A \otimes_R N),$$

where for $f : {}_BM \to N$, $\alpha_{M,N}(f)$ is the composition

$$M \xrightarrow{\simeq} A \otimes_A M \xrightarrow{\delta\otimes\mathrm{id}_M} A \otimes_B A \otimes_A M \xrightarrow{\simeq} A \otimes_B M \xrightarrow{\mathrm{id}_A\otimes f} A \otimes_B N.$$

To give an explicit expression of α, let $\delta(1_A) = \sum_i a_i \otimes a_i' \in A \otimes_B A$; then $\alpha_{M,N}(f)(m) = \sum_i a_i \otimes_B f(a_i'm)$. Further, if $g : M \to A \otimes_B N$, then $\alpha^{-1}(g)$ is the composition

$$M \xrightarrow{g} A \otimes_B N \xrightarrow{\eta} B \otimes_B M \xrightarrow{\simeq} N.$$

The unit and the counit of this adjunction are

$$\eta_M : M \to A \otimes_B M, \qquad \eta_M(m) = \sum_i a_i \otimes_B a_i'm,$$
$$\varepsilon_N : {}_BA \otimes_B N \to N, \qquad \varepsilon_N(a \otimes n) = \eta(a)n.$$

Observe that $e_M(m) = (\tau_M \circ \eta_M)(m) = (\sum_i a_i a_i')m = e_A(1_A)m$, and $e_N(n) = (\varepsilon_N \circ \xi_N)(n) = \eta(1_A)n$. In particular, F is separable if $\sum_i a_i a_i'$ is invertible in A, and G is separable if $\eta(1)$ is invertible in B.

Finally, if $M, M' \in A$-Mod and $f \in \mathrm{Hom}_B(M, M')$, then $\mathrm{Tr}_F(f)(m) = \sum_i a_i f(a_i'm)$ for all $m \in M$.

We also have the following "essential" version of Maschke's theorem (see for instance [12, Theorem 4]).

3.4. Theorem. *Let M be an A-module and M' an A-submodule of M. Assume that M has no $e_A(1)$-torsion, and that ${}_BM'$ is a direct summand of ${}_BM$. Then there is an A-submodule N of M such that $M' \oplus N$ is an essential A-submodule of M.*

Proof. Let $\iota : M' \to M$ be the inclusion, and let $p : M \to M'$ be an R-module map such that $p \circ \iota = \mathrm{id}_{M'}$. Then for all $m \in M'$

$$\begin{aligned}\mathrm{Tr}_F(p)(m) &= (\mathrm{Tr}_F(p) \circ \iota)(m) = \mathrm{Tr}_F(p \circ F(\iota))(m) \\ &= \mathrm{Tr}_F(\mathrm{id}_{M'})(m) = e_{M'}(m) = e_A(1) \cdot m.\end{aligned}$$

Let $N = \operatorname{Ker}\operatorname{Tr}_F(p)$. Then N is an A-submodule of M, and $N \cap M' = 0$ since M has no $e_A(1)$-torsion. Moreover,

$$\begin{aligned}
&\operatorname{Tr}_F(p)(e_A(1)m - \operatorname{Tr}_F(p)(m)) \\
&\quad = e_A(1)\operatorname{Tr}_F(p)(m) - (\operatorname{Tr}_F(p) \circ \iota \circ \operatorname{Tr}_F(p))(m) \\
&\quad = e_A(1)\operatorname{Tr}_F(p)(m) - (\operatorname{Tr}_F(p) \circ \iota)(\operatorname{Tr}_F(p)(m)) = 0,
\end{aligned}$$

hence $e_A(1) \cdot M \subseteq N + M'$, and therefore $N + M'$ is an essential A-submodule of M. $\square$

Next we present, following [12] and [13], two particular examples of Frobenius extensions of rings. Many other examples involving Hopf algebra actions can be found in [3]–[7] and [10].

3.5. *Hopf-Galois extensions.* Let H be a Hopf k-algebra with comultiplication Δ, counit ϵ and antipode S. We assume that H is finitely generated and projective over the commutative ring k. Let further A be a right H-comodule algebra with structural map

$$\varepsilon_A : A \to A \otimes H, \quad \varepsilon_A(a) = \sum a_0 \otimes a_1,$$

and consider the subalgebra $B = \{a \in A \mid \varepsilon_A(a) = a \otimes 1\}$ of H-coinvariant elements. We assume that A/B is an H-Galois extension, that is, the map

$$\beta : A \otimes_B A \to A \otimes H, \quad a' \otimes_B a \mapsto \sum a' a_0 \otimes a_1$$

is bijective. Finally, let H^* be the dual of H, and assume that the space J of left integrals in H^* is a free k-module (of rank 1). The following statements follow immediately from the results of [12, Section 2].

If $0 \neq \lambda \in J$, then $\Theta : H \to H^*$, $a \mapsto \lambda \leftharpoondown a$ is a right H-module and a right H-comodule isomorphism, and denote $\Lambda = \Theta^{-1}(\epsilon)$. Then Λ is a right integral (that is, $\Lambda a = \epsilon(a)\Lambda$ for all $a \in H$), and $\lambda(\Lambda) = \lambda(S(\Lambda)) = 1$. Let $\sum_i x_i \otimes_B y_i = \beta^{-1}(1 \otimes \Lambda) \in A \otimes_B A$ and define

$$\begin{aligned}
&\eta : A \to B, && \eta(a) = \sum a_0 \lambda(a_1), \\
&\delta : A \to A \otimes_B A, && \delta(a) = a \sum_i x_i \otimes_B y_i.
\end{aligned}$$

Then η is a (B,B)-bimodule map and δ is an (A,A)-bimodule map satisfying (3.3.1). Consequently, B/A is a Frobenius extension. Notice also that $e_A(1) = \sum_i x_i y_i = \epsilon(\Lambda) \cdot 1$.

3.6. *Hopf subalgebras.* Let A be a finite dimensional Hopf k-algebra with Δ, ϵ and S as above. We assume that A has cocomutative coradical and that A is unimodular, so A has an integral γ (that is, a nonzero element satisfying $a\gamma = \gamma a = \epsilon(a)\gamma$ for all $a \in A$). The condition on the coradical implies that A is free left and right module over any Hopf subalgebra.

Let B be a unimodular Hopf subalgebra of A, and let $\eta : A \to B$ be the projection onto B. Clearly, η is a (B,B)-bimodule map. By [13, Lemma 2.9] there is an element α of A whose image in $A \otimes_B k$ is nonzero and A-invariant. Denote $\Delta(\alpha) = \sum \alpha' \otimes \alpha''$ and define

$$\delta : A \to A \otimes_B A, \quad \delta(a) = \sum \alpha' \otimes_B S(\alpha'')a.$$

Then δ is an (A,A)-bimodule map. It can be easily seen that $\sum \alpha' \eta(S(\alpha'')) = \eta(\alpha')S(\alpha'') = 1$, and this implies that that η and δ satisfy (4.3.1), hence A/B is a Frobenius extension.

3.7. *Modules over Hopf-algebras.* In [8], projectivity relative to a module over a group algebra was considered. The next observations show that many of the results of [8, Section 2, 3] can be generalized to Hopf algebras, and follow from Theorem 2.2.

Let k be a commutative ring, H a Hopf k-algebra, and M a (left) H-module, finitely generated and projective as k-module. Let M^* be the k-dual of M and let $\{(m_i, m_i^*),\ i = \overline{1,n}\}$ be a dual basis for ${}_kM$.

Consider also the H-map $\theta : k \to M \otimes_k M^*$, given by $\theta(1) = \sum_{i=1}^n m_i \otimes_k m_i^*$ (this is independent of the choice of dual basis). We also have the evaluation map $\epsilon : M^* \otimes_k M \to k$. For any H-module Y, this induces the H-map $(Y \otimes_k M^*) \otimes_k M \to Y$, $(y \otimes m^*) \otimes m \mapsto (y \otimes m^*)(m) = ym^*(m)$.

Consider the functors $F, G : H\text{-mod} \to H\text{-mod}$, $F = - \otimes_k M$ and $G = - \otimes_k M^*$. (Notice that if M is faithful k-module then F is a faithful functor.) Then (F,G) is a Frobenius pair.

Indeed, it is not difficult to verify that for any H-modules X, Y the map:

$$\alpha_{X,Y} : \mathrm{Hom}_H(X \otimes_k M, Y) \to \mathrm{Hom}_H(X, Y \otimes_k M^*),$$

$$\alpha_{X,Y}(f)(x) = \sum_{i=1}^n f(x \otimes_k m_i) \otimes_k m_i^*$$

is an isomorphism with inverse $\alpha^{-1}_{X,Y}(g)(x\otimes_k m) = g(x)(m)$. The unit and counit of this adjuction are

$$\eta_X : X \to X \otimes_k M \otimes_k M^*, \quad \eta_X(x) = \sum_{i=1}^{n} x \otimes_k m_i \otimes_k m_i^*$$

$$\varepsilon_Y : Y \otimes_k M^* \otimes_k M \to Y, \quad \varepsilon_Y(y \otimes_k m^* \otimes_k m) = m^*(m)y.$$

Similarly, we have the adjunction isomorphism

$$\gamma : \mathrm{Hom}_H(Y \otimes_k M^*, X) \to \mathrm{Hom}_H(Y, X \otimes_k M),$$

$$\gamma(g)(y) = \sum_{i=1}^{n} g(y \otimes_k m_i^*) \otimes_k m_i,$$

$$\gamma^{-1}(f)(y \otimes_k m^*) = (\mathrm{id}_X \otimes_k m^*)(f(y)),$$

with unit and counit

$$\xi_Y : Y \to Y \otimes_k M^* \otimes_k M, \quad \xi_Y(y) = \sum_{i=1}^{n} y \otimes_k m_i^* \otimes_k m_i$$

$$\tau_X : T \otimes_k M \otimes_k M^* \to X, \quad \tau_X(x \otimes_k m \otimes_k m^*) = m^*(m)x.$$

According to 1.7, we can define the corresponding transfer maps:

$$\mathrm{Tr}_M : \mathrm{Hom}_H(X \otimes_k M, Y \otimes_k M) \to \mathrm{Hom}_H(X, Y),$$

$$\mathrm{Tr}_M(f)(x) = \sum_{i=1}^{n} (\mathrm{id}_Y \otimes_k m_i^*) f(x \otimes_k m_i)$$

and

$$\mathrm{Tr}_{M^*} : \mathrm{Hom}_H(X \otimes_k M^*, Y \otimes_k M^*) \to \mathrm{Hom}_H(X, Y),$$

$$\mathrm{Tr}_{M^*}(g)(x) = \sum_{i=1}^{n} g(x \otimes_k m_i^*)(m_i).$$

Finally, for $X, Y \in H$-mod, we have

$$e(M)_X = \tau_X \circ \eta_X : X \to X, \quad e(M)_X(x) = \left(\sum_{i=1}^{n} m_i^*(m_i)\right)x,$$

$$e(M)_Y = \varepsilon_Y \circ \xi_Y : Y \to Y, \qquad e(M^*)_Y(y) = \left(\sum_{i=1}^{n} m_i^*(m_i)\right) y.$$

The general characterization of Frobenius functors between module categories is due to K. Morita (see [10] for a short proof). We add to that another equivalent condition, generalizing (and also giving another proof of) [4, Theorem 2.4].

3.8. Theorem. *Consider the rings A and B and let $F : A$-Mod $\to$ B-Mod and $G : B$-Mod $\to A$-Mod be two functors. The following statements are equivalent:*

a) *(F, G) is a Frobenius pair.*

b) *$F \simeq M \otimes_A -$ and $G \simeq N \otimes_B -$, where ${}_BM_A$ and ${}_AN_B$ are bimodules such that there exist bimodule maps*

$$\delta_A : A \to N \otimes_B M, \quad \eta_B : M \otimes_A N \to B,$$
$$\delta_B : B \to M \otimes_A N, \quad \eta_A : N \otimes_B M \to A$$

satisfying

$$(\eta_B \otimes_B \mathrm{id}_M) \circ (\mathrm{id}_M \otimes_A \delta_A) = \mathrm{id}_M, \tag{3.8.1}$$
$$(\mathrm{id}_N \otimes_B \eta_B) \circ (\delta_A \otimes_A \mathrm{id}_N) = \mathrm{id}_N,$$
$$(\eta_A \otimes_A \mathrm{id}_N) \circ (\mathrm{id}_N \otimes_B \delta_B) = \mathrm{id}_N, \tag{3.8.2}$$
$$(\mathrm{id}_M \otimes_A \eta_A) \circ (\delta_B \otimes_B \mathrm{id}_M) = \mathrm{id}_M .$$

Proof. If (F, G) is a Frobenius pair, then it is well-known that $F \simeq M \otimes_A -$ and $G \simeq N \otimes_B -$, where $M = F({}_AA)$ and $N = G({}_BB)$. For any $X \in A$-Mod and $Y \in B$-Mod we have the natural maps

$$\eta_X : X \to N \otimes_B M \otimes_A X, \quad \varepsilon_Y : M \otimes_A N \otimes_B Y \to Y$$

satisfying 1.3 (a) and (b). Letting $X = {}_AA$ and $Y = {}_BB$ we obtain the maps δ_A and η_B, which are bimodule maps by the naturality of η and ε. Also, in this case, 1.3 (a) and (b) imply (3.8.1).

Similarly, there exist the natural maps

$$\xi_Y : Y \to M \otimes_A N \otimes_B Y, \quad \tau_X : M \otimes_B M \otimes_A X \to X$$

satisfying 1.3 (c) and (d), and letting again $X = {}_AA$ and $Y = {}_BB$ we obtain the bimodule maps δ_B and η_A satisfying (3.8.2).

Conversely, assume the existence of ${}_BM_A$, ${}_AN_B$, δ_A, η_B, δ_B and η_A satisfying (3.8.1) and (3.8.2). For any A-module X let η_X be the composition

$$X \xrightarrow{\simeq} A \otimes_A X \xrightarrow{\delta_A \otimes_A \mathrm{id}_X} N \otimes_B M \otimes_A X,$$

and for any B-module Y let ε_Y be the composition

$$M \otimes_A N \otimes_B Y \xrightarrow{\eta_B \otimes_B \mathrm{id}_Y} B \otimes_B X \xrightarrow{\simeq} Y.$$

Then η and ε are natural transformations, and it is easy to see that (3.8.1) imply that η and τ satisfy 1.3 (a) and (b). It follows that $M \otimes_A -$ is a left adjoint of $N \otimes_B -$. Similarly, using δ_B and η_A we may define the natural transformations ξ and τ satisfying 1.3 (c) and (d), hence $M \otimes_A -$ is also a right adjoint of $N \otimes_B -$. $\square$

Acknowledgements. Parts of this paper were written while the second author was visiting the Mathematical Institutes in Budapest and Jena. He thanks the Domus Hungarica Foundation and the Alexander von Humboldt Foundation for their support. The authors are especially grateful to Professor Stefan Caenepeel whose observations and sugestions improved substantially the presentation of the paper. Proposition 1.6 and the proof of Proposition 1.10 are due to him.

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(Received January 22, 20001; revised April 23, 2001)