Publ. Math. Debrecen
59 / 3-4 (2001), 429–439

The curvature of a curve and variational calculus

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Abstract. The concept of the curvature of a curve is generalized, in order to answer a question about the Euler–Lagrange equations in the calculus of variations raised by Weinstein.

1. Introduction

In a recent paper [7], Weinstein raises the question of how one might formulate the Euler–Lagrange equations for a calculus of variations problem on some manifold M by defining a function S on T^2M in terms of the Lagrangian L, such that the second-order differential equation field on TM corresponding to the equations, considered as a section of the fibration $T^2M \to TM$, is given fibre by fibre by the critical point of the restriction of S to the fibre.

In the case of a regular Lagrangian the answer is clear, with hindsight: we can take

$$S(x,\dot{x},\ddot{x}) = g^{ab}(x,\dot{x})\lambda_a(x,\dot{x},\ddot{x})\lambda_b(x,\dot{x},\ddot{x}),$$

where the λ_a are the coefficients of the Euler–Lagrange form,

$$\lambda_a = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a},$$

and the g^{ab} are the components of the inverse of the Hessian of L with respect to the velocity variables (which is non-singular by assumption).

Mathematics Subject Classification: 53B40, 70H03.
Key words and phrases: curvature, Euler–Lagrange equations.

This expression may be rewritten so as to put in evidence its dependence on the fibre coordinates $\ddot{x}^a$, by noting that λ_a takes the form

$$\lambda_a(x, \dot{x}, \ddot{x}) = g_{ab}(\ddot{x}^b - F^b(x, \dot{x})),$$

where the Euler–Lagrange equations solved for $\ddot{x}^a$ are $\ddot{x}^a = F^a(x, \dot{x})$; then

$$S(x, \dot{x}, \ddot{x}) = g_{ab}(\ddot{x}^a - F^a(x, \dot{x}))(\ddot{x}^b - F^b(x, \dot{x})).$$

The critical points of S, considered as a function of $\ddot{x}^a$, are given exactly by

$$\lambda_a(x, \dot{x}, \ddot{x}) = 0.$$

Though S has been specified in terms of coordinates it is a well-defined function on T^2M: this follows from the fact that the Euler–Lagrange equations of a regular Lagrangian determine a well-defined vector field on TM. Multiplying S by any non-vanishing function on TM gives a new function on T^2M that would serve equally well to specify the Euler–Lagrange equations in the desired manner.

However, this cannot be regarded as a satisfactory answer to Weinstein's question in the absence of any justification for the choice of S, other than that it is well-defined and gives the right answer. In this note I wish to point out that there is a good reason for making this choice, which is geometrically based in the concept of the curvature of a curve. In its simplest version, the result outlined above says that in Euclidean space the shortest curves are also the straightest. I shall show that a natural sequence of generalizations of this observation leads to the formula above for S, up to a scalar factor.

The derivation of the equations of motion of a dynamical system consisting of a number of particles subject to Newtonian external forces, and to possibly non-holonomic perfect constraints, from a principle which does not involve integration, was already a subject of interest in the second half of the nineteenth century. The results are described in Chapter IX of Whittaker's *Analytical Dynamics* [8], articles 105–107. The term "curvature" was introduced by Hertz for the basic function of the theory. It is defined as follows: let the mass of the r th particle be m_r, its Cartesian coordinates (x_r, y_r, z_r), and the components of the total external force acting on it (X_r, Y_r, Z_r); consider a kinematically possible path of the system, so that the values of (x_r, y_r, z_r) and $(\dot{x}_r, \dot{y}_r, \dot{z}_r)$ represent possible positions and velocities of an actual path of the system; then the square of the curvature of the path at any instant is

$$\sum m_r \left\{ \left(\ddot{x}_r - \frac{X_r}{m_r} \right)^2 + \left(\ddot{y}_r - \frac{Y_r}{m_r} \right)^2 + \left(\ddot{z}_r - \frac{Z_r}{m_r} \right)^2 \right\}.$$

The principle of least curvature, to which the names of Gauss and Hertz are attached, is expressed by Whittaker as follows: of all paths consistent with the constraints (which are supposed to do no work), the actual trajectory is that which has the least curvature. For holonomic constraints the square of the curvature can be written in terms of generalized coordinates and their derivatives, as was shown by Lipschitz; for conservative forces the result is a particular case of the formula for S quoted above. In the course of the derivation the definition of curvature is modified by dropping a term which has the same value for all possible paths for which the particles have the same positions and velocities; as a result, the curvature, when re-defined, of the actual path is zero. Finally, Appell showed how to derive from the principle of least curvature a way of writing the equations of motion, applicable whether the constraints are holonomic or non-holonomic, which comes close to that sought by Weinstein.

Some of these results have been put into a more modern framework by Klein [3].

In contrast to the approach described by Whittaker, I shall attack the problem in a more obviously geometrical manner, and I shall obtain a function S for an arbitrary, possibly time-dependent, regular Lagrangian whose Hessian with respect to the velocity variables is non-singular, and not just for a Lagrangian of mechanical type. However, I shall not discuss non-holonomic constraints. My function S is a natural generalization of the square of the curvature of a curve in Euclidean space. It differs from the one given above by an overall scalar factor, independent of $\ddot{x}^a$, which is due to the fact that the curves in question are not parametrized by arc-length, in an appropriate sense. (I conclude that Hertz's use of the term "curvature" was not quite accurate.) Like Klein, I shall make use of the homogeneous formalism.

I start by reminding the reader of some elementary facts about the curvature of a curve.

The curvature of a curve in Euclidean 3-space, parametrized by arc-length, is just the length of the acceleration vector. The justification for this terminology is that it is the integrand of the integral which gives the length of the spherical image of a segment of the curve, and this integral is a measure of the total change in direction of the tangent vector along the segment (see, for example, the discussion in Chapter III of STOKER's *Differential Geometry* [5]).

Consider next a curve σ lying in a surface in Euclidean 3-space. The square of its curvature can be written as the sum of two non-negative terms ${\kappa_\sigma}^2 + (n_{ij}\dot\sigma^i\dot\sigma^j)^2$, where n_{ij}, $i, j = 1, 2$, are the components of the second fundamental form of the surface; on the other hand κ_σ, the geodesic curvature of the curve, is an intrinsic quantity. Now the second term takes the same value for all curves on the surface through a given point, having the same (unit) tangent vector there, so κ_σ is as satisfactory a measure of the curvature of a curve on a surface as the Euclidean length of its acceleration vector; so (just as in Lipschitz's calculation referred to above) it makes sense to change the definition of curvature to geodesic curvature. Moreover, the latter generalizes naturally to Riemannian geometry.

In Riemannian geometry, the (geodesic) curvature vector of a curve σ at the point $\sigma(s)$ on it is $c(s) = \nabla_{\dot\sigma(s)}\dot\sigma$, where $\dot\sigma$ is the tangent vector field along σ, which is parametrized by arc-length. The curvature $\kappa_\sigma(s)$ of σ is given by $\kappa_\sigma(s)^2 = g(c(s), c(s))$. In coordinates,

$${\kappa_\sigma}^2 = g_{ij}\left(\ddot\sigma^i + \Gamma^i_{kl}\dot\sigma^k\dot\sigma^l\right)\left(\ddot\sigma^j + \Gamma^j_{mn}\dot\sigma^m\dot\sigma^n\right),$$

where the Γ^i_{jk} are of course the Christoffel symbols of the Levi–Civita connection.

The curvature vector can be interpreted as the vertical component of the tangent vector to the natural lift of σ to TM, with respect to the horizontal distribution determined by the Levi–Civita connection.

This construction generalizes easily to Finsler geometry. The key point is that there is a horizontal distribution on the slit tangent bundle $\mathcal{T}M$, canonically associated with the Finsler structure. This horizontal distribution is uniquely defined by the following conditions, where X^H and X^V are the horizontal and vertical lifts of a vector field X on M to $\mathcal{T}M$:

1. the horizontal distribution is conservative: $X^H(E) = 0$, where E is the Finslerian energy function;
2. the horizontal distribution is homogeneous: $[\Delta, X^H] = 0$, where Δ is the dilation field;

3. the horizontal distribution is torsion-free: for any vector fields X, Y on M,
$$[X^H, Y^V] - [Y^H, X^V] - [X, Y]^V = 0.$$

(See [6], where this result is called the fundamental lemma of Finsler geometry.) When the Finsler structure is actually Riemannian, these conditions determine, via its horizontal distribution, the Levi–Civita connection. There is a unique horizontal second-order differential equation field: it is the one whose base integral curves are the affinely parametrized geodesics of the Finsler structure. (The horizontal distribution is induced by this second-order differential equation field, and this property is often used to define it, as in [2]; but in the present context to define it in this way would be to beg the question, and it is important to know that it can be defined directly in terms of the Finsler structure.)

Let me briefly indicate how the horizontal distribution may be specified, given the defining conditions above. I shall take a somewhat broad interpretation of the expression "Finsler structure": I shall assume merely that the Finsler function F is defined (and smooth) on an open subset of $\mathcal{T}M$ which is invariant under dilations, and that the induced Finsler metric is non-singular. Take coordinates $(x^i, \dot{x}^i)$ on TM. Recall that the components g_{ij} of the Finsler metric tensor are given in terms of the Finsler function F and the energy $E = \frac{1}{2}F^2$ by

$$g_{ij} = \frac{\partial^2 E}{\partial \dot{x}^i \partial \dot{x}^j} = F \frac{\partial^2 F}{\partial \dot{x}^i \partial \dot{x}^j} + \frac{\partial F}{\partial \dot{x}^i}\frac{\partial F}{\partial \dot{x}^j},$$

and that by homogeneity

$$E = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j.$$

The horizontal distribution may be defined by giving a local basis of horizontal vector fields of the form

$$\frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial \dot{x}^j}.$$

The functions Γ_i^j must be homogeneous of degree 1 in the velocity coordinates $\dot{x}^i$ by the homogeneity condition, and they must satisfy

$$\frac{\partial \Gamma_j^i}{\partial \dot{x}^k} = \frac{\partial \Gamma_k^i}{\partial \dot{x}^j}$$

by the condition of absence of torsion. The condition that the horizontal distribution is conservative is equivalent to

$$\Gamma_i^j g_{jk}\dot{x}^k = \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\dot{x}^j\dot{x}^k,$$

from which the Γ_i^j can be found by repeated differentiation with respect to the velocity coordinates and use of the other conditions. One finds that

$$\Gamma_i^j = \Gamma_{ik}^j\dot{x}^k - C_{im}^j\Gamma_{kl}^m\dot{x}^k\dot{x}^l$$

where

$$\Gamma_{jk}^i = \frac{1}{2}g^{il}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right)$$

is formally the same as a Christoffel symbol, and

$$C_{jk}^i = \frac{1}{2}g^{il}\frac{\partial g_{jk}}{\partial \dot{x}^l} = \frac{1}{2}g^{il}\left(\frac{\partial g_{jl}}{\partial \dot{x}^k} + \frac{\partial g_{kl}}{\partial \dot{x}^j} - \frac{\partial g_{jk}}{\partial \dot{x}^l}\right)$$

is the first Cartan tensor.

The curvature vector of a curve σ, parametrized by arc-length as determined by the Finsler function, can be defined as before as the vertical component of the tangent vector to the natural lift of σ (as in [3]). It follows from the homogeneity of the g_{ij} that the formula for the curvature scalar is formally identical to that for Riemannian geometry, but of course g_{ij} and Γ_{jk}^i now depend on velocity as well as position. Moreover, the equations of geodesics of the Finsler structure, parametrized by arc-length, are formally identical to the corresponding equations in Riemannian geometry. Thus the (rough and ready) dictum that the shortest curves are also the straightest continues to apply.

I now generalize further to a general time-dependent Lagrangian. The base manifold M will now be assumed to be fibred over the real line, which represents the time axis; the Lagrangian is a function on the first jet bundle $J^1\pi$ of this fibration, which can be regarded as the subbundle of $\mathcal{T}M$ given by $\dot{x}^0 = 1$, where x^0 is the time coordinate. A Lagrangian can be used to define a Finsler structure (in the broad sense) by using the homogenization trick. Let $a, b, \ldots$ range and sum from 1 to n, and $i, j, \ldots$ range and sum from 0 to n. Take coordinates $(x^i, \dot{x}^i)$ on $\mathcal{T}M$ and $(t, x^a, \acute{x}^a)$ on $J^1\pi$, such that the imbedding of $J^1\pi$ in $\mathcal{T}M$ is given by $(t, x^a, \acute{x}^a) \to (t, x^a, 1, \acute{x}^a)$. The Lagrangian $L(t, x^a, \acute{x}^a)$ defines a Finsler function $F(x^i, \dot{x}^i)$ by

$$F(x^i, \dot{x}^i) = \dot{x}^0 L(x^0, x^a, \dot{x}^a/\dot{x}^0).$$

I wish to compute the curvature of a curve, as it is determined by the Finsler structure, but expressing the answer in terms of the Lagrangian, and reparametrizing the curve from arc-length s to time t.

The relevant partial derivatives of F are given by

$$\frac{\partial F}{\partial \dot{x}^a} = \frac{\partial L}{\partial \acute{x}^a}, \quad \frac{\partial F}{\partial \dot{x}^0} = L - \acute{x}^a \frac{\partial L}{\partial \acute{x}^a} = -H,$$

where it must be understood that quantities derived from L are evaluated at $t = x^0$, $\acute{x}^a = \dot{x}^a/\dot{x}^0$; here H is the Hamiltonian corresponding to L, but considered as a function on $J^1\pi$. Of course

$$\frac{\partial F}{\partial x^a} = \dot{x}^0 \frac{\partial L}{\partial x^a}, \quad \frac{\partial F}{\partial x^0} = \dot{x}^0 \frac{\partial L}{\partial t}.$$

Furthermore,

$$\frac{\partial^2 F}{\partial \dot{x}^a \partial \dot{x}^b} = \frac{1}{\dot{x}^0} \frac{\partial^2 L}{\partial \acute{x}^a \partial \acute{x}^b}, \quad \frac{\partial^2 F}{\partial \dot{x}^0 \partial \dot{x}^a} = -\frac{\acute{x}^b}{\dot{x}^0} \frac{\partial^2 L}{\partial \acute{x}^a \partial \acute{x}^b},$$
$$\frac{\partial^2 F}{\partial (\dot{x}^0)^2} = \frac{\acute{x}^a \acute{x}^b}{\dot{x}^0} \frac{\partial^2 L}{\partial \acute{x}^a \partial \acute{x}^b}.$$

Set

$$G_{ab} = \frac{\partial^2 L}{\partial \acute{x}^a \partial \acute{x}^b}.$$

Recall that the Finsler metric tensor g_{ij} is given in terms of F by

$$g_{ij} = F \frac{\partial^2 F}{\partial \dot{x}^i \partial \dot{x}^j} + \frac{\partial F}{\partial \dot{x}^i} \frac{\partial F}{\partial \dot{x}^j}.$$

It follows that

$$g_{ab} = LG_{ab} + p_a p_b$$
$$g_{0a} = -LG_{ab}v^b - Hp_a$$
$$g_{00} = LG_{ab}v^a v^b + H^2,$$

where $p_a = \partial L/\partial \acute{x}^a$. In other words,

$$g = LG_{ab}(dx^a - \acute{x}^a dt) \otimes (dx^b - \acute{x}^b dt) + {\theta_L}^2$$

where θ_L is the Cartan 1-form of L,

$$\theta_L = Ldt - p_a(dx^a - \acute{x}^a dt) = p_a dx^a - Hdt.$$

It follows that if G is non-singular and L is non-vanishing, g will be non-singular. Furthermore, this formula expresses g as direct sum of an n-dimensional metric on the space annihilated by the Cartan 1-form and a 1-dimensional metric on the space annihilated by the contact 1-forms $dx^a - \acute{x}^a dt$, these two subspaces being orthogonal. We can take advantage of this when computing the inverse g^{ij} of g_{ij}, considered as a symmetric bilinear form on the covector space spanned by the dx^i: the covector space is the orthogonal direct sum of the n-dimensional subspace spanned by the contact 1-forms and the 1-dimensional subspace spanned by the Cartan 1-form, and the inverse of g will also be a direct sum. In particular, the restriction of the inverse to the contact subspace will have components G^{ab}/L with respect to the basis $dx^a - \acute{x}^a dt$, $a = 1, 2, \ldots, n$, where the G^{ab} are the components of the inverse of G.

Let σ be a curve in M parametrized by Finsler arc-length s, such that $\dot{\sigma}^0 > 0$. Let $\phi = \phi_i dx^i$ be the Euler–Lagrange 1-form defined along σ corresponding to the Finsler function F, so that

$$\phi_i = \frac{d}{ds}\left(\frac{\partial F}{\partial \dot{x}^i}\right) - \frac{\partial F}{\partial x^i}.$$

Then the curvature κ_σ of σ is given by

$${\kappa_\sigma}^2 = g^{ij}\phi_i\phi_j.$$

Since σ is parametrized by arc-length

$$F(\sigma^i, \dot{\sigma}^i) = 1 = \dot{\sigma}^0 L(\sigma^0, \sigma^a, \dot{\sigma}^a/\dot{\sigma}^0).$$

Thus if $\sigma^0 = t$ is used as parameter on σ rather than s, $dt/ds = 1/L$; and so

$$\phi_a = \frac{1}{L}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial v^a}\right) - \frac{\partial L}{\partial x^a}\right),$$

while

$$\phi_0 = -\frac{1}{L}\left(\frac{dH}{dt} + \frac{\partial L}{\partial t}\right).$$

In fact $\phi = \phi_i dx^i$ is a linear combination of contact 1-forms:

$$\phi = \phi_a(dx^a - \acute{x}^a dt) = \phi_a\left(dx^a - \frac{d\sigma^a}{dt}dt\right).$$

To confirm that this is so, I have to show that $\phi_0 = -\phi_a v^a$, or equivalently that

$$\frac{d\sigma^a}{dt}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial v^a}\right) - \frac{\partial L}{\partial x^a}\right) = \frac{dH}{dt} + \frac{\partial L}{\partial t}.$$

But

$$\begin{aligned}\frac{dH}{dt} &= \frac{d}{dt}\left(\frac{d\sigma^a}{dt}\frac{\partial L}{\partial \acute{x}^a} - L\right)\\ &= \frac{d^2\sigma^a}{dt^2}\frac{\partial L}{\partial \acute{x}^a} + \frac{d\sigma^a}{dt}\frac{d}{dt}\left(\frac{\partial L}{\partial \acute{x}^a}\right) - \frac{dL}{dt}\\ &= \frac{d\sigma^a}{dt}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \acute{x}^a}\right) - \frac{\partial L}{\partial x^a}\right) - \frac{\partial L}{\partial t},\end{aligned}$$

as required.

Bearing in mind the remarks about the inverse of g made earlier, it follows that

$$\kappa_\sigma{}^2 = g^{ij}\phi_i\phi_j = g^{ab}\phi_a\phi_b = \frac{1}{L}G^{ab}\phi_a\phi_b.$$

So if we denote by λ_a the components of the Euler–Lagrange 1-form of the Lagrangian L, so that

$$\lambda_a = \frac{d}{dt}\left(\frac{\partial L}{\partial \acute{x}^a}\right) - \frac{\partial L}{\partial x^a} = L\phi_a,$$

then

$$\kappa_\sigma{}^2 = \frac{1}{L^3}G^{ab}\lambda_a\lambda_b.$$

Finally, I must explain how $\kappa_\sigma{}^2$ defines a function on T^2M (or more properly, the restriction of T^2M to an appropriate submanifold of TM, though I shall leave this qualification to be understood). I first consider the Finsler case. It is clear that two curves in M which are second-order tangent at a point have the same value of ${\kappa_\sigma}^2$, assuming that both are parametrized by arc-length. So there is a well-defined function, say κ^2, on the submanifold of T^2M consisting of those points $(x^i, \dot{x}^i, \ddot{x}^i)$ such that

$$g_{ij}\dot{x}^i\dot{x}^j = 1, \quad g_{ij}\dot{x}^i(\ddot{x}^j + \Gamma^j_{kl}\dot{x}^k\dot{x}^l) = 0,$$

such that for any curve σ parametrized by arc-length, $\kappa^2(\sigma(s), \dot{\sigma}(s), \ddot{\sigma}(s)) = {\kappa_\sigma}^2(s)$. To extend the definition of κ^2 to the rest of T^2M one can make use of what might be called the reparametrization group. This group consists of pairs of real numbers (k, l) with $k > 0$ (corresponding to sense-preserving reparametrizations) with the multiplication

$$(k_1, l_1) \cdot (k_2, l_2) = (k_1 k_2, l_1 + {k_1}^2 l_2)$$

(so the group is a semi-direct product). The group acts on T^2M by

$$(x^i, \dot{x}^i, \ddot{x}^i) \to (x^i, k\dot{x}^i, k^2\ddot{x}^i + l\dot{x}^i) = \varphi_{(k,l)}(x^i, \dot{x}^i, \ddot{x}^i).$$

For any point $(x^i, \dot{x}^i, \ddot{x}^i) \in T^2M$ with $(x^i, \dot{x}^i) \in \mathcal{T}M$, there is a unique (k, l), $k > 0$, such that $\varphi_{(k,l)}(x^i, \dot{x}^i, \ddot{x}^i) = (x^i, \dot{y}^i, \ddot{y}^i)$ satisfies

$$g_{ij}\dot{y}^i\dot{y}^j = 1, \quad g_{ij}\dot{y}^i(\ddot{y}^j + \Gamma^j_{kl}\dot{y}^k\dot{y}^l) = 0;$$

we make use here of the fact that g_{ij} is homogeneous of degree zero in the velocity variables. We define κ^2 generally by setting

$$\kappa^2(x^i, \dot{x}^i, \ddot{x}^i) = \kappa^2(x^i, \dot{y}^i, \ddot{y}^i).$$

Thus curvature becomes a property of the curve as point set, not just of the curve with a particular form of parametrization, as it should be. We find that

$$\begin{aligned}\kappa^2(x^i, \dot{x}^i, \ddot{x}^i) &= F(x^i, \dot{x}^i)^{-4}\left(g_{ij}(\ddot{x}^i + \Gamma^i_{kl}\dot{x}^k\dot{x}^l)(\ddot{x}^j + \Gamma^j_{mn}\dot{x}^m\dot{x}^n) - \dot{F}(x^i, \dot{x}^i)^2\right)\\ &= \frac{g_{ij}(\ddot{x}^i + \Gamma^i_{kl}\dot{x}^k\dot{x}^l)(\ddot{x}^j + \Gamma^j_{mn}\dot{x}^m\dot{x}^n)g_{pq}\dot{x}^p\dot{x}^q - (g_{ij}\dot{x}^i(\ddot{x}^j + \Gamma^j_{kl}\dot{x}^k\dot{x}^l))^2}{(g_{ij}\dot{x}^i\dot{x}^j)^3}.\end{aligned}$$

The condition for a critical point with respect to $\ddot{x}^i$ is

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k \propto \dot{x}^i,$$

as expected. The expression for κ^2 in terms of the coefficients ϕ_i of the Euler–Lagrange form is simply

$$\kappa^2 = F^{-2} g^{ij} \phi_i \phi_j.$$

In order to obtain the corresponding formula for a Lagrangian L we restrict to the submanifold of T^2M given by $\dot{x}^0 = 1$, $\ddot{x}^i = 0$; noting that $L(t, x^a, \acute{x}^a) = F(t, x^a, 1, \acute{x}^a)$, we find that

$$\kappa^2(t, x^a, \acute{x}^a) = \frac{1}{L^3} G^{ab} \lambda_a \lambda_b,$$

in agreement with the result already obtained by consideration of curves.

The results obtained here should be relevant to the larger issues raised by Weinstein in [7], which are concerned with relations between variational problems on Lie algebroids and Lie groupoids. Some progress in answering the questions posed by Weinstein is reported on in [4]. Techniques for obtaining results about general Lagrangians from those for Finsler structures, similar to those employed here, have been used previously in a discussion of the second variation formula in [1].

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(Received January 26, 2001; revised June 8, 2001)