Publ. Math. Debrecen
59 / 3-4 (2001), 441–458

The jet prolongations of fibered fibered manifolds and the flow operator

By W. M. MIKULSKI (Kroków)

Abstract. A characterization of the flow operator lifting projectable vector fields to (r,s,q)-jet prolongation functors over fibered fibered manifolds for $r,s,q \in \mathbb{N}$ with $s \geq r \leq q$ as an unique in some sense natural operator is given. Similar result for 2-fibered manifolds is deduced.

0. Introduction

Natural bundles over n-manifolds were introduced by NIJENHUIS [13] as a modern approach to the theory of geometric objects. The concept of natural bundles was extended by KOLÁŘ, MICHOR and SLOVÁK [4] to bundle functors over a local category over manifolds. Bundle functors over some admissible categories were studied, [4].

The well-known example of an admissible category over manifolds is the category of fibered manifolds and (more generally) the category of k-fibered manifolds. Another example is the category of fibered fibered manifolds. The objects of the last category are surjective fibered submersions between fibered manifolds. They appear naturally in differential geometry if we consider transverse natural bundles in the sense of WOLAK [14], see Subsection 2.1.

The fibre product preserving bundle functors over fibered manifolds play improtrant role in differential geometry because to them one can lifts many geometric structures. Such functors are classified, [5]. The most improtrant examples of such bundle functors are the r-jet prolongation

Mathematics Subject Classification: 58A20, 53A55.
Key words and phrases: bundle functors, natural operators, fibered manifolds, jets.

functors. They play essential roles in the theory of higher order connections and Lagrangians, [4]. They have the well-known universal property with respect to operators of order r on sections of fibered manifolds, [4]. In the case of fibered fibered manifolds, using the concept of (r, s, q)-jets on fibered manifolds, [1], [4], we extend the notion of r-jet prolongation functors to the (r, s, q)-jet prolongation functors for $r, s, q \in \mathbb{N}$, $s \geq r \leq q$. These prolongation functors have the similar universal property with respect to operators of order (r, s, q) on fibered sections of fibered fibered manifolds. In the case of 2-fibered manifolds, using the concept of (r, s)-jets, [4], we can define the (r, s)-jet prolongation functors for $r, s \in \mathbb{N}$, $s \geq r$. The respective universal property is satisfied.

Natural operators lifting functions, vector fields, etc. are used practically in all papers in which problem of prolongations of geometric structures was studied, [2], [4], [12]. That is why such natural operators are classified by many authors, [3], [4], [6]–[11], [15], etc.

Let $\mathcal{K}$ be an admissible category (manifolds, fibered manifolds, k-fibered manifolds, fibered fibered manifolds, etc.) with objects of fixed (generalized) dimension and local isomorphisms. The most known natural operator lifting projectable vector fields to bundle functors over $\mathcal{K}$ is the flow operator. (A vector fields is called projectable if its flow is formed by $\mathcal{K}$-morphisms.) In the case of some bundle functors over manifolds and fibered manifolds we have several characterizations of this operator, [4]. The purpose of this paper is to characterize the flow operator lifting projectable vector fields to the (r, s, q)-jet prolongation functor without using constructions.

We say that a bundle functor F over $\mathcal{K}$ is poor if every natural operator lifting projectable vector fields to F is a constant multiple of the flow operator. The flow operator to poor bundle functors can be characterized as the unique natural operator lifting every projectable vector field X into a projectable vector field covering X.

The theorem of Kolář and Slovák [7] says that the r-jet prolongation functors over fibered manifolds are poor. Another examples of poor bundle functors are the ones of contact elements of order (r, s, q) and dimension (k, l) over fib. manifolds for respective r, s, q, k, l, [6].

The main result of this paper says that for $r, s, q \in \mathbb{N}$ with $s \geq r \leq q$ the (r, s, q)-jet prolongation bundle functors over fibered fibered manifolds (with fixed their dimension and local isomorphisms) are poor. Formally, the result is a non-trivial extension of [7]. Since fibered foliated manifolds (i.e. surjrective foliated submersions between foliated manifolds) are locally fibered fibered manifolds, from the main result of the paper it follows that the r-jet prolongation functors over fibered foliated manifolds (defined by r-jets of foliated sections) are poor.

As a first application of the main result we dedduce that for $r, s \in \mathbb{N}$ with $s \geq r$ the (r, s)-jet prolongation bundle functors over 2-fibered manifolds are poor.

As a second application of the main result we reobtain (in a simple way) the cited above result from [6].

All manifolds and maps are assumed to be of class C^∞.

1. Functor J^r and the flow operator. The theorem of Kolář and Slovák

For a comfort let us cite the result of [7].

1.1. A fibered manifold is a surjective submersion $\pi_0 : Y_1 \to Y_0$ between manifolds. If $\overline{\pi}_0 = \overline{Y}_1 \to \overline{Y}_0$ is another fibered manifold, a fibered map $\pi_0 \to \overline{\pi}_0$ is a map $f : Y_1 \to \overline{Y}_1$ such that there is a map $f_0 : Y_0 \to \overline{Y}_0$ with $\overline{\pi}_0 \circ f = f_0 \circ \pi_0$. Thus all fibered manifolds form a category which we will denote by $\mathcal{FM}$. This category is over manifolds, local and admissible, [4].

1.2. A fibered manifold $Y_1 \to Y_0$ has dimension (m_1, n_1) if Y_1 has dimension $m_1 + n_1$ and Y_0 has dimension m_1. All fibered manifolds of dimension (m_1, n_1) and their local isomorphisms form a subcategory $\mathcal{FM}_{m_1,n_1} \subset \mathcal{FM}$. Every $\mathcal{FM}_{m_1,n_1}$-object is locally isomorphic to $\mathbb{R}^{m_1} \times \mathbb{R}^{n_1} \to \mathbb{R}^{m_1}$, the usual projection.

1.3. Let $\pi_0 : Y = Y_1 \to Y_0$ be an $\mathcal{FM}_{m_1,n_1}$-object. Denote the set of (local) sections of π_0 by ΓY. $J^r Y = \{j^r_{y_0}\sigma \mid \sigma \in \Gamma Y,\ y_0 \in Y_0\}$ is a fibered manifold over Y_1 with respect to the target projection. If $\overline{\pi}_0 : \overline{Y} = \overline{Y}_1 \to \overline{Y}_0$ is another $\mathcal{FM}_{m_1,n_1}$-object and $f : Y \to \overline{Y}$ is an $\mathcal{FM}_{m_1,n_1}$-isomorphism with the underlying map f_0, we have $J^r f : J^r Y \to J^r \overline{Y}$, $J^r f(j^r_{y_0}\sigma) = j^r_{f_0(y_0)}(f \circ \sigma \circ f_0^{-1})$, $\sigma \in \Gamma Y$, $y_0 \in Y_0$. Then $J^r : \mathcal{FM}_{m_1,n_1} \to \mathcal{FM}$ is a bundle functor, [4].

1.4. Let $\pi_0 : Y = Y_1 \to Y_0$ be an $\mathcal{FM}_{m_1,n_1}$-object. A vector field Z on Y_1 is projectable on Y if there exist a vector field Z_0 on Y_0 such that Z is π_0-related with Z_0. If Z is projectable on Y, then its flow is formed by local $\mathcal{FM}$-isomorphisms. If $F : \mathcal{FM}_{m_1,n_1} \to \mathcal{FM}$ is a bundle functor, we have a vector field $\mathcal{F}Z = \frac{\partial}{\partial t}_{|t=0} F \operatorname{Exp} tZ$ on FY. The flow operator $\mathcal{F} : T_{\mathrm{proj}\,|\mathcal{FM}_{m_1,n_1}} \rightsquigarrow TF$ is $\mathcal{FM}_{m_1,n_1}$-natural, [4].

1.5. Theorem (Kolář, Slovák, [7]). *Every natural operator $T_{\mathrm{proj}|\mathcal{FM}_{m_1,n_1}} \rightsquigarrow TJ^r$ is a constant multiple of the flow operator $\mathcal{J}^r$.*

The rest of the paper is devoted to obtain extensions of the above theorem.

2. The category of fibered fibered manifolds. Functor $J^{r,s,q}$ and its flow operator. The main result

2.1. A fibered fibered manifold is a fibered surjective submersion $\pi : Y \to X$ between fibered manifolds, i.e. a surjective submersion which sends fibers into fibers such that the restricted and corestricted maps are submersions. (We will write Y instead of π if π is clear.) If $\overline{\pi} : \overline{Y} \to \overline{X}$ is another fibered fibered manifold, a morphism $\pi \to \overline{\pi}$ is a fibered map $f : Y \to \overline{Y}$ such that there is a fibered map $f_0 : X \to \overline{X}$ with $\overline{\pi} \circ f = f_0 \circ \pi$. Thus all fibered fibered manifolds form a category which we will denote by $\mathcal{F}^2\mathcal{M}$. This category is over manifolds, local and admissible.

Fibered fibered manifolds appear naturally in differential geometry. To see this, we consider a fibered manifold $p : X \to M$. Then X has the foliated structure $\mathcal{F}$ by fibres. Its normal bundle $Y = \mathcal{N}(X, \mathcal{F}) = TX/T\mathcal{F}$ has the induced foliation, [14]. This foliation is by the fibered manifold $[Tp] : Y \to TM$, the quotient map of the differential $Tp : TX \to TM$. Clearly, the projection $\pi : Y \to X$ of the normal bundle is a fibered fibered manifold. Considering other transverse natural bundles in the sense of [14] instead of $\mathcal{N}(X, \mathcal{F})$ we can produce many fibered fibered manifolds.

2.2. A fibered fibered manifold $\pi : Y \to X$ has dimension (m_1, m_2, n_1, n_2) if Y has dimension $(m_1{+}n_1, m_2{+}n_2)$ and X has dimension (m_1, m_2). All fibered fibered manifolds of dimension (m_1, m_2, n_1, n_2) and their local isomorphisms form a subcategory $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2} \subset \mathcal{F}^2\mathcal{M}$. Every $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object is locally isomorphic to $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times$

$\mathbb{R}^{n_2} \to \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, the projection, where $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ (or $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$) is over $\mathbb{R}^{m_1} \times \mathbb{R}^{n_1}$ (or $\mathbb{R}^{m_1}$).

2.3. Let $\pi : Y \to X$ be an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object. Denote the set of (local) fibered maps $\sigma : X \to Y$ with $\pi \circ \sigma = \mathrm{id}_{\mathrm{dom}(\sigma)}$ by $\Gamma_{\mathrm{fib}} Y$. Let $r, s, q \in \mathbb{N}$, $s \geq r \leq q$. By 12.19 in [4], $\sigma, \rho \in \Gamma_{\mathrm{fib}} Y$ represent the same (r,s,q)-jet $j^{r,s,q}_x \sigma = j^{r,s,q}_x \rho$ at a point $x \in X$ if $j^r_x \sigma = j^r_x \rho$, $j^s_x(\sigma | X_{\overline{x}}) = j^s_x(\rho | X_{\overline{x}})$ and $j^q_{\overline{x}} \overline{\sigma} = j^q_{\overline{x}} \overline{\rho}$, where B_X and B_Y are the bases of X and Y, $\overline{x} \in B_X$ is the element under x, $X_{\overline{x}}$ is the fibre of X over $\overline{x}$ and $\overline{\sigma}, \overline{\rho} : B_X \to B_Y$ are the underlying maps of ρ, σ. $J^{r,s,q} Y = \{ j^{r,s,q}_x \sigma \mid \sigma \in \Gamma_{\mathrm{fib}} Y, x \in X \}$ is a fibered manifold over Y with the target projection. If $\overline{\pi} : \overline{Y} \to \overline{X}$ is another $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object and $f : Y \to \overline{Y}$ is an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-isomorphism with the underlying map f_0, we define $J^{r,s,q} f : J^{r,s,q} Y \to J^{r,s,q} \overline{Y}$, $J^{r,s,q} f (j^{r,s,q}_x \sigma) = j^{r,s,q}_{f_0(x)} (f \circ \sigma \circ f_0^{-1})$, $\sigma \in \Gamma_{\mathrm{fib}} Y$, $x \in X$. Then $J^{r,s,q} : \mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2} \to \mathcal{FM}$ is a bundle functor in the sense of [4].

2.4. Let $\pi : Y \to X$ be an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object. A projectable vector field Z on Y is projectable on π if there exists a π-related (with Z) projectable vector field Z_0 on X. If Z is projectable on π, then its flow is formed by local $\mathcal{F}^2\mathcal{M}$-isomorphisms. If $F : \mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2} \to \mathcal{FM}$ is a bundle functor, we have a vector field $FZ = \frac{\partial}{\partial t}_{|t=0} F \,\mathrm{Exp}\, tZ$ on FY. The flow operator $F : T_{\mathrm{proj}\,|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \rightsquigarrow TF$ is natural in the sense of [4] with respect to $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-morphisms.

The main result of this paper is the following theorem.

2.5. Theorem. *Let $m_1, m_2, n_1, n_2 \in \mathbb{N} \cup \{0\}$, $m_1 \geq 1$, $r, s, q \in \mathbb{N}$, $s \geq r \leq q$. Every natural operator $T_{\mathrm{proj}|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \rightsquigarrow TJ^{r,s,q}$ is a constant multiple of the flow operator $J^{r,s,q}$.*

The proof of Theorem 2.5 will occupy Sections 3–9.

From now on let m_1, m_2, n_1, n_2, r, s, q be as in Theorem 2.5.

3. The notations

3.1. We shall use the indices running as follows: $i, \tilde{i}, \tilde{\tilde{i}}, \overline{i}, \overline{\overline{i}} \in \langle 1, m_1\rangle \cap \mathbb{N}$, $I, \tilde{I}, \overline{I}, \overline{\overline{I}} \in \langle m_1+1, m_1+m_2\rangle \cap \mathbb{N}$, $p, \overline{p}, \overline{\overline{p}} \in \langle 1, n_1\rangle \cap \mathbb{N}$, $P, \overline{P}, \overline{\overline{P}} \in \langle n_1+1, n_1+n_2\rangle \cap \mathbb{N}$, $\alpha, \tilde{\tilde{\alpha}}, \overline{\alpha} \in \mathbb{N}^q_{m_1}$, $(\beta,\gamma), (\tilde{\tilde{\beta}}, \tilde{\tilde{\gamma}}), (\overline{\beta}, \overline{\gamma}) \in \mathbb{N}^r_{m_1,m_2}$ and $\delta, \tilde{\tilde{\delta}}, \overline{\delta} \in \mathbb{N}^s_{m_2}$, where $\mathbb{N}^q_{m_1} = \{\alpha \in (\mathbb{N}\cup\{0\})^{m_1} \mid 1 \leq |\alpha| \leq q\}$, $\mathbb{N}^r_{m_1,m_2} = \{(\beta,\gamma) \in (\mathbb{N}\cup\{0\})^{m_1} \times (\mathbb{N}\cup\{0\})^{m_2} \mid |\beta| \geq 1,\ |\beta| + |\gamma| \leq r\}$ and $\mathbb{N}^s_{m_2} = \{\delta \in (\mathbb{N}\cup\{0\})^{m_2} \mid 1 \leq |\delta| \leq s\}$. To these indices we shall apply the Einstein sumation convention.

3.2. Let $S^{r,s,q}$ or $Z^{s,r,q}$ denote the fiber of $J^{r,s,q}(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ or $TJ^{r,s,q}(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ over $0 \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, respectively, where $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is the standard $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object.

3.3. Let V^d denotes the space of all d-jets of projectable vector fields on the fibered fibered manifold $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with source $0 \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Let $V_0 \subset V^d$ denotes the subset of all constant vector fields with zero component in $\mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

3.4. Let $G^{d+1}_{m_1,m_2,n_1,n_2}$ denotes the Lie group of $(d+1)$-jets at $0 \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ of 0-preserving local $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-isomorphisms $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. If $d = \max(s,q)$, then $G^{d+1}_{m_1,m_2,n_1,n_2}$ acts on V^d, $S^{r,s,q}$ and $Z^{r,s,q}$ in obvious way. In this case, let $\tilde{G}^{d+1}_{m_1,m_2,n_1,n_2}$ denotes the subgroup in $G^{d+1}_{m_1,m_2,n_1,n_2}$ of all g with $g.V_0 \subset V_0$. Then $\tilde{G}^{d+1}_{m_1,m_2,n_1,n_2}$ acts on V_0, $S^{r,s,q}$, $Z^{r,s,q}$.

3.5. Let x^i, x^I, y^p, y^P be the usual coordinates on $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Let X^i be the induced coordinates on V_0. Let $y^p_\alpha = \overline{y}^p_{\alpha|S^{r,s,q}}$, $y^P_{(\beta,\gamma)} = \overline{y}^P_{(\beta,\gamma)|S^{r,s,q}}$, $y^P_\delta = \overline{y}^P_{\delta\,|S^{r,s,q}}$ be the induced coordinates on $S^{r,s,q}$ and $W^i = \overline{W}^i_{|Z^{r,s,q}}$, $W^I = \overline{W}^I_{|Z^{r,s,q}}$, $Z^p = \overline{Z}^p_{|Z^{r,s,q}}$, $Z^P = \overline{Z}^P_{|Z^{r,s,q}}$, $Z^p_\alpha = \overline{Z}^p_{\alpha|Z^{r,s,q}}$, $Z^P_{(\beta,\gamma)} = \overline{Z}^P_{(\beta,\gamma)|Z^{r,s,q}}$, $Z^P_\delta = \overline{Z}^P_{\delta\,|Z^{r,s,q}}$ the additional coordinates on $Z^{r,s,q}$ (see 3.1 for domains of the indices), where functions $\overline{x}^i, \overline{x}^I, \overline{y}^p, \overline{y}^P, \overline{y}^p_\alpha, \overline{y}^P_{(\beta,\gamma)}, \overline{y}^P_\delta : J^{r,s,q}(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \to \mathbb{R}$ are given by $\overline{x}^i(j^{r,s,q}_a\sigma) = x^i(a)$, $\overline{x}^I(j^{r,s,q}_a\sigma) = x^I(a)$, $\overline{y}^p(j^{r,s,q}_a\sigma) = y^p(\sigma(a))$, $\overline{y}^P(j^{r,s,q}_a\sigma) = y^P(\sigma(a))$, $\overline{y}^p_\alpha(j^{r,s,q}_a\sigma) = (\frac{\partial^{\alpha_1}}{\partial(x^1)^{\alpha_1}} \cdots \frac{\partial^{\alpha_{m_1}}}{\partial(x^{m_1})^{\alpha_{m_1}}}(y^p \circ \sigma))(a)$,

$\overline{y}^P_{(\beta,\gamma)}(j^{r,s,q}_a\sigma) = \left(\frac{\partial^{\beta_1}}{\partial(x^1)^{\beta_1}} \dots \frac{\partial^{\gamma_{m_2}}}{\partial(x^{m_1+m_2})^{\gamma_{m_2}}}(y^P\circ\sigma)\right)(a)$, $\overline{y}^P_\delta(j^{r,s,q}_a\sigma) = \left(\frac{\partial^{\delta_1}}{\partial(x^{m_1+1})^{\delta_1}} \dots \frac{\partial^{\delta_{m_2}}}{\partial(x^{m_1+m_2})^{\delta_{m_2}}}(y^P\circ\sigma)\right)(a)$, $j^{r,s,q}_a\sigma \in J^{r,s,q}(\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ and where $\overline{W}^i = d\overline{x}^i$, $\overline{W}^I = d\overline{x}^I$, $\overline{Z}^p = d\overline{y}^p$, $\overline{Z}^P = d\overline{y}^P$, $\overline{Z}^p_\alpha = d\overline{y}^p_\alpha$, $\overline{Z}^P_{(\beta,\gamma)} = d\overline{y}^P_{(\beta,\gamma)}$, $\overline{Z}^P_\delta = d\overline{y}^P_\delta$.

4. The order estimation

4.1. Lemma. *If Z is a projectable vector field on an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object $\pi : Y \to X$ and $y \in Y$ is a point such that the projection of Z_y onto the basis of X is non-zero, then $f_*Z = \frac{\partial}{\partial x^1}$ near y for some local $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-map f.*

Proof. The proof is a simple modification of the proof of the similar well-known fact for non-vanishing vector fields on manifolds. In the oryginal proof we apply the fact that the flow of Z is formed by local $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-isomorphisms. □

4.2. Lemma. *If Z_1 and Z_2 are projectable vector fields on an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object $\pi : Y \to X$ and $y \in Y$ is a point such that the projection of $Z_{1|y}$ onto the basis of X is non-zero and $j^d_yZ_1 = j^d_yZ_2$, then there exists a local $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-isomorphism $f : Y \to Y$ with $j^{d+1}_y f = \mathrm{id}$ and $f_*Z_1 = Z_2$ near y.*

Proof. The proof is a simple modification of the proof of Lemma 42.4 in [4]. It is sufficient to observe that f from the proof of Lemma 42.4 in [4] (for Z_1 and Z_2 instead of X and Y) is a local $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-isomorphisms. □

4.3. Proposition. *Let $F : \mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2} \to \mathcal{FM}$ be a bundle functor of order $\le d$. Every natural operator $T_{\mathrm{proj}|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \rightsquigarrow TF$ is of order $\le d$.*

Proof. We use Lemmas 4.1 and 4.2 and modify 42.5 in [4]. □

5. The reduction of the problem

From now on $d := \max(s,q)$.

5.1. Proposition. *The natural operators $T_{\mathrm{proj}|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \rightsquigarrow TJ^{r,s,q}$ are in bijection with the $G^{d+1}_{m_1,m_2,n_1,n_2}$-maps $V^d \times S^{r,s,q} \to Z^{r,s,q}$ over the identity of $S^{r,s,q}$.*

Proof. It is a consequence of the general theory (see [4]) and Proposition 4.3. □

5.2. Lemma. *Let $\mathcal{A} : V^d \times S^{r,s,q} \to Z^{r,s,q}$ be a $G^{d+1}_{m_1,m_2,n_1,n_2}$-map. Then $\mathcal{A}$ is determined by its restriction $\mathcal{A}_0$ to $V_0 \times S^{r,s,q}$. Moreover, $\mathcal{A}_0$ is a $\tilde{G}^{d+1}_{m_1,m_2,n_1,n_2}$-map and $\mathcal{A}_0(0,.)$ ($0 \in V_0$) is a $G^{d+1}_{m_1,m_2,n_1,n_2}$-map.*

Proof. It is a simple consequence of Lemma 4.1. □

5.3. Lemma. *Every $\mathcal{A}_0 : V_0 \times S^{r,s,q} \to Z^{r,s,q}$ covering the identity of $S^{r,s,q}$ can be written in the form $W^i = f^i(X^{\bar{i}}, y^{\bar{p}}_{\bar{\alpha}}, y^{\bar{P}}_{(\bar{\beta},\bar{\gamma})}, y^{\bar{P}}_{\bar{\delta}})$, $W^I = f^I(X^{\bar{i}}, y^{\bar{p}}_{\bar{\alpha}}, y^{\bar{P}}_{(\bar{\beta},\bar{\gamma})}, y^{\bar{P}}_{\bar{\delta}})$, $Z^p = g^p(X^{\bar{i}}, y^{\bar{p}}_{\bar{\alpha}}, y^{\bar{P}}_{(\bar{\beta},\bar{\gamma})}, y^{\bar{P}}_{\bar{\delta}})$, $Z^P = g^P(X^{\bar{i}}, y^{\bar{p}}_{\bar{\alpha}}, y^{\bar{P}}_{(\bar{\beta},\bar{\gamma})}, y^{\bar{P}}_{\bar{\delta}})$, $Z^p_\alpha = g^p_\alpha(X^{\bar{i}}, y^{\bar{p}}_{\bar{\alpha}}, y^{\bar{P}}_{(\bar{\beta},\bar{\gamma})}, y^{\bar{P}}_{\bar{\delta}})$, $Z^P_{(\beta,\gamma)} = g^P_{(\beta,\gamma)}(X^{\bar{i}}, y^{\bar{p}}_{\bar{\alpha}}, y^{\bar{P}}_{(\bar{\beta},\bar{\gamma})}, y^{\bar{P}}_{\bar{\delta}})$, $Z^P_\delta = g^P_\delta(X^{\bar{i}}, y^{\bar{p}}_{\bar{\alpha}}, y^{\bar{P}}_{(\bar{\beta},\bar{\gamma})}, y^{\bar{P}}_{\bar{\delta}})$.*

Proof. The lemma is obvious. □

Now, Theorem 2.5 is a simple consequence of the following proposition.

5.4. Proposition. *Let $\mathcal{A}_0 : V_0 \times S^{r,s,q} \to Z^{r,s,q}$ be a $\tilde{G}^{d+1}_{m_1,m_2,n_1,n_2}$-map over the identity of $S^{r,s,q}$ such that $\mathcal{A}_0(0,.) : S^{r,s,q} \to Z^{r,s,q}$ is a $G^{d+1}_{m_1,m_2,n_1,n_2}$-map, $d = \max(s,q)$. Let f^i, f^I, g^p, g^P, g^p_α, $g^P_{(\beta,\gamma)}$ and g^P_δ be as in Lemma 5.3. Then there is $k \in \mathbb{R}$ such that $f^i = kX^i$ and $f^I = g^p = g^P = g^p_\alpha = g^P_{(\beta,\gamma)} = g^P_\delta = 0$.*

The proof of this proposition will occupy Sections 6–9.

6. The beginning of the proof

6.1. Lemma. *$f^i = kX^i$ for some $k \in \mathbb{R}$.*

Proof. By the equivariance of f^i with respect to the homotheties in $G^1_{n_1} \times G^1_{n_2} \subset \tilde{G}^{d+1}_{m_1,m_2,n_1,n_2}$ we obtain $f^i(X^{\bar{i}}, y^{\overline{p}}_{\overline{\alpha}}, y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}, y^{\overline{P}}_{\overline{\delta}}) =$ $f^i(X^{\bar{i}}, ty^{\overline{p}}_{\overline{\alpha}}, ty^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}, ty^{\overline{P}}_{\overline{\delta}})$ for every $t \in \mathbb{R}_+$, so that f^i depends on $X^{\bar{i}}$ only. Then the equivariance of f^i with respect to $G^1_{m_1}$ yelds $f^i = kX^i$ for some k, see 44.3 in [4]. □

6.2. Lemma. *$f^I = 0$.*

Proof. By the similar argument as in the proof of Lemma 6.1, f^I depends only on $X^{\bar{i}}$. Then using the equivariance of f^I with respect to the homotheties in $G^1_{m_2} \subset \tilde{G}^{d+1}_{m_1,m_2,n_1,n_2}$ we obtain $tf^I(X^{\bar{i}}) = f^I(X^{\bar{i}})$ for $t \in \mathbb{R}_+$, i.e. $f^I = 0$. □

6.3. Lemma. *$g^p = g^p(X^{\bar{i}}, y^{\overline{p}}_{\overline{\alpha}})$, $g^p_\alpha = g^p_\alpha(X^{\bar{i}}, y^{\overline{p}}_{\overline{\alpha}})$, $g^P = g^P(X^{\bar{i}}, y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}, y^{\overline{P}}_{\overline{\delta}})$, $g^P_{(\beta,\gamma)} = g^P_{(\beta,\gamma)}(X^{\bar{i}}, y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}, y^{\overline{P}}_{\overline{\delta}})$ and $g^P_\delta = g^P_\delta(X^{\bar{i}}, y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}, y^{\overline{P}}_{\overline{\delta}})$.*

Proof. By the equivariance of g^p with respect to the homotheties in $G^1_{n_2}$ we get $g^p(X^{\bar{i}}, y^{\overline{p}}_{\overline{\alpha}}, ty^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}, ty^{\overline{P}}_{\overline{\delta}}) = g^p(X^{\bar{i}}, y^{\overline{p}}_{\overline{\alpha}}, y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}, y^{\overline{P}}_{\overline{\delta}})$ for $t \in \mathbb{R}_+$, so g^p depends on $X^{\bar{i}}$ and $y^{\overline{p}}_{\overline{\alpha}}$ only. By the same argument $g^p_\alpha = g^p_\alpha(X^{\bar{i}}, y^{\overline{p}}_{\overline{\alpha}})$. Similarly, by the equivariance of g^P, $g^P_{(\beta,\gamma)}$ and g^P_δ with respect to the homotheties in $G^1_{n_1}$, $g^P = g^P(X^{\bar{i}}, y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}, y^{\overline{P}}_{\overline{\delta}})$, $g^P_{(\beta,\gamma)} = g^P_{(\beta,\gamma)}(X^{\bar{i}}, y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}, y^{\overline{P}}_{\overline{\delta}})$ and $g^P_\delta = g^P_\delta(X^{\bar{i}}, y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}, y^{\overline{P}}_{\overline{\delta}})$. □

6.4. Lemma. *$g^p = 0$.*

Proof. The element from $\tilde{G}^{d+1}_{m_1,m_2,n_1,n_2}$ given by $(x^{\bar{\bar{i}}}, x^{\bar{\bar{I}}}, y^{\bar{\bar{p}}} + (y^{\bar{\bar{p}}})^{q+1}, y^{\bar{\bar{P}}})$ preserves $X^{\bar{i}}$ and $y^{\overline{p}}_{\overline{\alpha}}$, and if $\alpha = (\alpha_1, \dots, \alpha_{m_1})$ is such that $|\alpha| = q$ then it sends Z^p_α into $Z^p_\alpha + (q+1)!Z^p(y^p_{e_1})^{\alpha_1} \dots (y^p_{e_{m_1}})^{\alpha_{m_1}}$, where $e_i = (0, \dots, 1, \dots, 0) \in \mathbb{N}^q_{m_1}$, 1 in i-th position. Now, by the equivariance of g^p_α with respect to this element we get $g^p_\alpha = g^p_\alpha + (q+1)!g^p(y^p_{e_1})^{\alpha_1} \dots (y^p_{e_{m_1}})^{\alpha_{m_1}}$ if $|\alpha| = q$. So, $g^p = 0$. □

6.5. Lemma. $g^P = 0$.

PROOF. The element from $\tilde{G}^{d+1}_{m_1,m_2,n_1,n_2}$ corresponding to $(x^{\bar{\bar{i}}}, x^{\bar{\bar{I}}}, y^{\bar{\bar{p}}}, y^{\bar{\bar{P}}} + (y^{\bar{\bar{P}}})^{s+1})$ preserves $X^{\bar{i}}$, $y^{\overline{P}}_{(\bar{\beta},\bar{\gamma})}$ and $y^{\overline{P}}_{\bar{\delta}}$, and if $\delta = (\delta_1, \dots, \delta_{m_2})$ is such that $|\delta| = s$ then it sends Z^P_δ into $Z^P_\delta + (s+1)! Z^P (y^P_{e_{m_1+1}})^{\delta_1} \dots \dots (y^P_{e_{m_1+m_2}})^{\delta_{m_2}}$, where $e_I = (0, \dots, 1, \dots, 0) \in \mathbb{N}^s_{m_2}$, 1 in $(I - m_1)$-th position. This gives the equivariant condition $g^P_\delta = g^P_\delta + (s+1)! g^P (y^P_{e_{m_1+1}})^{\delta_1} \dots \dots (y^P_{e_{m_1+m_2}})^{\delta_{m_2}}$ if $|\delta| = s$. Hence $g^P = 0$. □

To prove $g^p_\alpha = g^P_{(\beta,\gamma)} = g^P_\delta = 0$ we shall proced by induction on $d = \max(s, q)$.

7. The first inductive step

Now, $d = r = s = q = 1$. Let $e_i \in \mathbb{N}^1_{m_1}$ and $e_I \in \mathbb{N}^1_{m_2}$ be as in 6.4 and 6.5.

7.1. Lemma. $g^p_{e_i} = 0$.

PROOF. By Lemma 6.3, $g^p_{e_i} = g^p_{e_i}(X^{\bar{i}}, y^{\bar{p}}_{e_{\bar{i}}})$. The equivariance of $g^p_{e_i}$ with respect to the homotheties in $G^1_{n_1}$ yields the homogeneity condition $t g^p_{e_i}(X^{\bar{i}}, y^{\bar{p}}_{e_{\bar{i}}}) = g^p_{e_i}(X^{\bar{i}}, t y^{\bar{p}}_{e_{\bar{i}}})$ for $t \in \mathbb{R}_+$. This type of homogeneity implies that $g^p_{e_i} = h^{pe_{\bar{i}}}_{e_i \bar{p}}(X^{\tilde{i}}) y^{\bar{p}}_{e_{\bar{i}}}$. (We use the Einstein sumation convention, see 3.1 for the domain of the indices.) Now, using the equivariance of $g^p_{e_i}$ with respect to the homotheties in $G^1_{m_1}$ we get that $h^{pe_{\bar{i}}}_{e_i \bar{p}}(tX^{\tilde{i}}) = h^{pe_{\bar{i}}}_{e_i \bar{p}}(X^{\tilde{i}})$, i.e. $h^{pe_{\bar{i}}}_{e_i \bar{p}} = \text{const}$. So, $g^p_{e_i} = h^{pe_{\bar{i}}}_{e_i \bar{p}} y^{\bar{p}}_{e_{\bar{i}}}$. Next, we shall apply the equivariance of $\mathcal{A}_0(0, .)$. Consider $\xi(a^{\bar{\bar{p}}}_{\tilde{\tilde{i}}}) \in G^2_{m_1,m_2,n_1,n_2}$ corresponding to $(x^{\bar{\bar{i}}}, x^{\bar{\bar{I}}}, y^{\bar{\bar{p}}} + a^{\bar{\bar{p}}}_{\tilde{\tilde{i}}} x^{\tilde{\tilde{i}}}, y^{\bar{\bar{P}}})$ for $a^{\bar{\bar{p}}}_{\tilde{\tilde{i}}} \in \mathbb{R}$. Every $\xi(a^{\bar{\bar{p}}}_{\tilde{\tilde{i}}})$ sends $y^{\bar{p}}_{e_{\tilde{i}}}$ into $y^{\bar{p}}_{e_{\tilde{i}}} + a^{\bar{p}}_{\tilde{i}}$ and preserves $Z^p_{e_i}$. Then we obtain the equivariant condition $h^{pe_{\bar{i}}}_{e_i \bar{p}} y^{\bar{p}}_{e_{\bar{i}}} = h^{pe_{\bar{i}}}_{e_i \bar{p}} (y^{\bar{p}}_{e_{\bar{i}}} + a^{\bar{p}}_{\bar{i}})$. Then $h^{pe_{\bar{i}}}_{e_i \bar{p}} = 0$. So, $g^p_{e_i} = 0$. □

7.2. Lemma. $g^P_{e_i} = 0$.

PROOF. By Lemma 6.3, $g^P_{e_i} = g^P_{e_i}(X^{\bar{i}} y^{\overline{P}}_{e_{\bar{i}}}, y^{\overline{P}}_{e_{\bar{I}}})$. Similarly as in the proof of Lemma 7.1, the equivariance of $g^P_{e_i}$ with respect to the homotheties in $G^1_{n_2}$ yields $g^P_{e_i} = h^{Pe_{\bar{i}}}_{e_i \overline{P}}(X^{\tilde{i}}) y^{\overline{P}}_{e_{\bar{i}}} + \tilde{h}^{Pe_{\bar{I}}}_{e_i \overline{P}}(X^{\tilde{i}}) y^{\overline{P}}_{e_{\bar{I}}}$. If we use the equivariance of $g^P_{e_i}$ with respect to the homotheties in $G^1_{m_2}$ we get the homogeneity condition $\tilde{h}^{Pe_{\overline{I}}}_{e_i\overline{P}}(X^{\tilde{i}}) = \frac{1}{t}\tilde{h}^{Pe_{\overline{I}}}_{e_i\overline{P}}(X^{\tilde{i}})$, i.e. $\tilde{h}^{Pe_{\overline{I}}}_{e_i\overline{P}} = 0$. Then if we use the equivariance of $g^P_{e_i}$ with respect to the homotheties in $G^1_{m_1}$ we get $h^{Pe_{\overline{i}}}_{e_i\overline{P}} =$ const. So, $g^P_{e_i} = h^{Pe_{\overline{i}}}_{e_i\overline{P}} y^{\overline{P}}_{e_{\overline{i}}}$. Next, we shall apply the equivariance of $\mathcal{A}_0(0,.)$. Consider $\xi(a^{\overline{\overline{P}}}_{\tilde{\tilde{i}}}) \in G^2_{m_1,m_2,n_1,n_2}$ corresponding to $(x^{\overline{\overline{i}}}, x^{\overline{\overline{I}}}, y^{\overline{\overline{p}}}, y^{\overline{\overline{P}}} + a^{\overline{\overline{P}}}_{\tilde{\tilde{i}}} x^{\tilde{\tilde{i}}})$ for $a^{\overline{\overline{P}}}_{\tilde{\tilde{i}}} \in \mathbb{R}$. Every $\xi(a^{\overline{\overline{P}}}_{\tilde{\tilde{i}}})$ sends $y^{\overline{P}}_{e_{\overline{i}}}$ into $y^{\overline{P}}_{e_{\overline{i}}} + a^{\overline{P}}_{\overline{i}}$ and preserves $Z^P_{e_i}$. Then the equivariance of $g^P_{e_i}$ gives $h^{Pe_{\overline{i}}}_{e_i\overline{P}} y^{\overline{P}}_{e_{\overline{i}}} = h^{Pe_{\overline{i}}}_{e_i\overline{P}}(y^{\overline{P}}_{e_{\overline{i}}} + a^{\overline{P}}_{\overline{i}})$. Then $h^{Pe_{\overline{i}}}_{e_i\overline{P}} = 0$. So, $g^P_{e_i} = 0$. $\square$

7.3. Lemma. $g^P_{e_I} = 0$.

Proof. By Lemma 6.3, $g^P_{e_I} = g^P_{e_I}(X^{\overline{i}}, y^{\overline{P}}_{e_{\overline{i}}}, y^{\overline{P}}_{e_{\overline{I}}}) = 0$. Similarly as in the proof of Lemma 7.2, the equivariance of $g^P_{e_I}$ with respect to the homotheties in $G^1_{n_2}$ yields $g^P_{e_I} = h^{Pe_{\overline{i}}}_{e_I\overline{P}}(X^{\tilde{i}}) y^{\overline{P}}_{e_{\overline{i}}} + \tilde{h}^{Pe_{\overline{I}}}_{e_I\overline{P}}(X^{\tilde{i}}) y^{\overline{P}}_{e_{\overline{I}}}$. If we use the equivariance of $g^P_{e_I}$ with respect to the homotheties in $G^1_{m_2}$ and next the equivariance of $g^P_{e_I}$ with respect to the homotheties in $G^1_{m_1}$ we get $h^{Pe_{\overline{i}}}_{e_I\overline{P}} = 0$ and $\tilde{h}^{Pe_{\overline{I}}}_{e_I\overline{P}} =$ const. So, $g^P_{e_I} = \tilde{h}^{Pe_{\overline{I}}}_{e_I\overline{P}} y^{\overline{P}}_{e_{\overline{I}}}$. Next, we shall apply the equivariance of $\mathcal{A}_0(0,.)$. Consider $\xi(a^{\overline{\overline{P}}}_{\tilde{\tilde{I}}}) \in G^2_{m_1,m_2,n_1,n_2}$ corresponding to $(x^{\overline{\overline{i}}}, x^{\overline{\overline{I}}}, y^{\overline{\overline{p}}}, y^{\overline{\overline{P}}} + a^{\overline{\overline{P}}}_{\tilde{\tilde{I}}} x^{\tilde{\tilde{I}}})$ for $a^{\overline{\overline{P}}}_{\tilde{\tilde{I}}} \in \mathbb{R}$. Every $\xi(a^{\overline{\overline{P}}}_{\tilde{\tilde{I}}})$ sends $y^{\overline{P}}_{e_{\overline{I}}}$ into $y^{\overline{P}}_{e_{\overline{I}}} + a^{\overline{P}}_{\overline{I}}$ and preserves $Z^P_{e_I}$. Then, by equivariance of $g^P_{e_I}$, $\tilde{h}^{Pe_{\overline{I}}}_{e_I\overline{P}} y^{\overline{P}}_{e_{\overline{I}}} = \tilde{h}^{Pe_{\overline{I}}}_{e_I\overline{P}}(y^{\overline{P}}_{e_{\overline{I}}} + a^{\overline{P}}_{\overline{I}})$, i.e. $\tilde{h}^{Pe_{\overline{I}}}_{e_I\overline{P}} = 0$. So, $g^P_{e_I} = 0$. $\square$

We have finished the proof for $d = 1$.

8. Some preparations

We assume $d = \max(s, q) \geq 2$. $e_i \in \mathbb{N}^q_{m_1}$, $e_I \in \mathbb{N}^s_{m_2}$ are as in 6.4 and 6.5.

8.1. Lemma. *If $|\alpha| \leq \min(d-1, q)$, g^p_α is independent of $y^{\overline{p}}_{\overline{\alpha}}$ for $|\overline{\alpha}| = d$.*

Proof. We can assume that $d = q$ as if $q < d$ the $y^{\overline{p}}_{\overline{\alpha}}$ for $|\overline{\alpha}| = d$ do not occur. Let $|\alpha| \leq d - 1$. Using the equivariance of $g^p_\alpha = g^p_\alpha(X^{\overline{i}}, y^{\overline{p}}_{\overline{\alpha}})$ with respect to the homotheties in $G^1_{n_1}$ we get $g^p_\alpha = h^{p\overline{\alpha}}_{\alpha\overline{p}}(X^{\tilde{i}}) y^{\overline{p}}_{\overline{\alpha}}$. Consider

$\eta \in \tilde{G}^{d+1}_{m_1,m_2,n_1,n_2}$ corresponding to $(x^{\bar{\bar{i}}}, x^{\bar{\bar{I}}}, y^{\bar{\bar{p}}} + (y^{\bar{\bar{p}}})^d, y^{\bar{\bar{P}}})$. It preserves $X^{\bar{i}}$ and $y^{\bar{p}}_{\bar{\alpha}}$ with $|\bar{\alpha}| < d$, and it sends $y^{\bar{p}}_{\bar{\alpha}}$ into $y^{\bar{p}}_{\bar{\alpha}} + d!(y^{\bar{p}}_{e_1})^{\bar{\alpha}_1} \dots (y^{\bar{p}}_{e_{m_1}})^{\bar{\alpha}_{m_1}}$ for $\bar{\alpha} = (\bar{\alpha}_1, \dots, \bar{\alpha}_{m_1})$ with $|\bar{\alpha}| = d$. Moreover it sends Z^p_α into $Z^p_\alpha + Z^p \cdot (\dots)$ because of $|\alpha| \le d-1$.

Now, since $g^p = 0$, the equivariance of g^p_α with respect to η yields $h^{p\bar{\alpha}}_{\alpha\bar{p}}(X^{\bar{i}}) y^{\bar{p}}_{\bar{\alpha}} = h^{p\bar{\alpha}}_{\alpha\bar{p}}(X^{\bar{i}}) y^{\bar{p}}_{\bar{\alpha}} + d! \sum_{|\bar{\alpha}|=d} h^{p\bar{\alpha}}_{\alpha\bar{p}}(X^{\bar{i}})(y^{\bar{p}}_{e_1})^{\bar{\alpha}_1} \dots (y^{\bar{p}}_{e_{m_1}})^{\bar{\alpha}_{m_1}}$. Then $\sum_{|\bar{\alpha}|=d} h^{p\bar{\alpha}}_{\alpha\bar{p}}(X^{\bar{i}})(y^{\bar{p}}_{e_1})^{\bar{\alpha}_1} \dots (y^{\bar{p}}_{e_{m_1}})^{\bar{\alpha}_{m_1}} = 0$. Then $h^{p\bar{\alpha}}_{\alpha\bar{p}}(X^{\bar{i}}) = 0$ if $|\bar{\alpha}| = d$, so g^p_α is independent of the $y^{\bar{p}}_{\bar{\alpha}}$ for $|\bar{\alpha}| = d$. □

8.2. Lemma. *$g^P_{(\beta,\gamma)}$ is independent of the $y^{\bar{P}}_{\bar{\delta}}$ with $|\bar{\delta}| = d$.*

Proof. Using the equivariance of $g^P_{(\beta,\gamma)} = g^P_{(\beta,\gamma)}(X^{\bar{i}}, y^{\bar{P}}_{(\bar{\beta},\bar{\gamma})}, y^{\bar{P}}_{\bar{\delta}})$ with respect to the homotheties in $G^1_{n_2}$ we can write $g^P_{(\beta,\gamma)} = h^{P,(\bar{\beta},\bar{\gamma})}_{(\beta,\gamma),\bar{P}}(X^{\bar{i}}) y^{\bar{P}}_{(\bar{\beta},\bar{\gamma})} + \tilde{h}^{P\bar{\delta}}_{(\beta,\gamma)\bar{P}}(X^{\bar{i}}) y^{\bar{P}}_{\bar{\delta}}$. Then by the equivariance of $g^P_{(\beta,\gamma)}$ with respect to the homotheties in $G^1_{m_2}$ we deduce that the second sum is over $\bar{\delta}$ with $|\bar{\delta}| = |\gamma|$ (if $|\bar{\delta}| \ne |\gamma|$, then $\tilde{h}^{P\bar{\delta}}_{(\beta,\gamma)\bar{P}}(X^{\bar{i}}) = 0$). Cleary, $|\gamma| \le r - |\beta| \le r - 1 < d$. Then $g^P_{(\beta,\gamma)}$ is independent of the $y^{\bar{P}}_{\bar{\delta}}$ with $|\bar{\delta}| = d$. □

8.3. Lemma. *If $|\beta| + |\gamma| \le \min(d-1, r)$ then $g^P_{(\beta,\gamma)}$ is independent of the $y^{\bar{P}}_{(\bar{\beta},\bar{\gamma})}$ with $|\bar{\beta}| + |\bar{\gamma}| = d$.*

Proof. We can assume that $d = r$ as if $r < d$ then the $y^{\bar{P}}_{(\bar{\beta},\bar{\gamma})}$ with $|\bar{\beta}| + |\bar{\gamma}| = d$ do not occur. Let $|\beta| + |\gamma| \le d - 1$. We can write $g^P_{(\beta,\gamma)} = h^{P,(\bar{\beta},\bar{\gamma})}_{(\beta,\gamma),\bar{P}}(X^{\bar{i}}) y^{\bar{P}}_{(\bar{\beta},\bar{\gamma})} + \tilde{h}^{P\bar{\delta}}_{(\beta,\gamma)\bar{P}}(X^{\bar{i}}) y^{\bar{P}}_{\bar{\delta}}$, where in the second sum we have $|\bar{\delta}| < d$, see Proof of Lemma 8.2. The element $\xi \in \tilde{G}^{d+1}_{m_1,m_2,n_1,n_2}$ corresponding to $(x^{\bar{\bar{i}}}, x^{\bar{\bar{I}}}, y^{\bar{\bar{p}}}, y^{\bar{\bar{P}}} + (y^{\bar{\bar{P}}})^d)$ preserves $X^{\bar{i}}$ and $y^{\bar{P}}_{\bar{\delta}}$ with $|\bar{\delta}| < d$ and $y^{\bar{P}}_{(\bar{\beta},\bar{\gamma})}$ with $|\bar{\beta}| + |\bar{\gamma}| < d$, sends $y^{\bar{P}}_{(\bar{\beta},\bar{\gamma})}$ with $|\bar{\beta}| + |\bar{\gamma}| = d$ into $y^{\bar{P}}_{(\bar{\beta},\bar{\gamma})} + d!(y^{\bar{P}}_{e_1})^{\bar{\beta}_1} \dots (y^{\bar{P}}_{e_{m_1}})^{\bar{\beta}_{m_1}} (y^{\bar{P}}_{e_{m_1+1}})^{\bar{\gamma}_1} \dots (y^{\bar{P}}_{e_{m_1+m_2}})^{\bar{\gamma}_{m_2}}$, where $\bar{\beta} = (\bar{\beta}_1, \dots, \bar{\beta}_{m_1})$ and $\bar{\gamma} = (\bar{\gamma}_1, \dots, \bar{\gamma}_{m_2})$, and sends $Z^P_{(\beta,\gamma)}$ into $Z^P_{(\beta,\gamma)} + Z^P \cdot (\dots)$ because of $|\beta| + |\gamma| \le d - 1$. Now, since $g^P = 0$, the equivariance of $g^P_{(\beta,\gamma)}$ yields $\sum_{|\bar{\beta}|+|\bar{\gamma}|=d} h^{P,(\bar{\beta},\bar{\gamma})}_{(\beta,\gamma),\bar{P}}(X^{\bar{i}})(y^{\bar{P}}_{e_1})^{\bar{\beta}_1} \dots (y^{\bar{P}}_{e_{m_1}})^{\bar{\beta}_{m_1}} (y^{\bar{P}}_{e_{m_1+1}})^{\bar{\gamma}_1} \dots (y^{\bar{P}}_{e_{m_1+m_2}})^{\bar{\gamma}_{m_2}} = 0$. Consequently $h^{P,(\bar{\beta},\bar{\gamma})}_{(\beta,\gamma),\bar{P}}(X^{\bar{i}}) = 0$ if $|\bar{\beta}| + |\bar{\gamma}| = d$. Therefore $g^P_{(\beta,\gamma)}$ is independent of the $y^{\bar{P}}_{(\bar{\beta},\bar{\gamma})}$ with $|\bar{\beta}| + |\bar{\gamma}| = d$. □

8.4. Lemma. *If $|\delta| \le \min(d-1,s)$ then g^P_δ is independent of the $y^{\overline{P}}_{\overline{\delta}}$ with $|\overline{\delta}| = d$ and of the $y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}$ with $|\overline{\beta}| + |\overline{\gamma}| = d$.*

Proof. We can assume that $d = s$ as if $s < d$ then the $y^{\overline{P}}_{\overline{\delta}}$ with $|\overline{\delta}| = d$ and the $y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}$ with $|\overline{\beta}| + |\overline{\gamma}| = d$ do not occur. Let $|\delta| \le d-1$. Using the equivariance of g^P_δ with respect to the homotheties in $G^1_{n_2}$ we can write $g^P_\delta = h^{P,(\overline{\beta},\overline{\gamma})}_{\delta\overline{P}}(X^{\overline{i}})y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})} + \tilde{h}^{P\overline{\delta}}_{\delta\overline{P}}(X^{\overline{i}})y^{\overline{P}}_{\overline{\delta}}$. The element $\xi \in \tilde{G}^{d+1}_{m_1,m_2,n_1,n_2}$ as in the proof of Lemma 8.3 has additionally the following properties. It preserves $X^{\overline{i}}$ and $y^{\overline{P}}_{\overline{\delta}}$ with $|\overline{\delta}| \le d-1$ and $y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}$ with $|\overline{\beta}| + |\overline{\gamma}| \le \min(d-1,r)$. If $d = r$, it sends $y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}$ with $|\overline{\beta}| + |\overline{\gamma}| = d$ into $y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})} + d!(y^{\overline{P}}_{e_1})^{\overline{\beta}_1} \dots (y^{\overline{P}}_{e_{m_1}})^{\overline{\beta}_{m_1}}(y^{\overline{P}}_{e_{m_1+1}})^{\overline{\gamma}_1} \dots (y^{\overline{P}}_{e_{m_1+m_2}})^{\overline{\gamma}_{m_2}}$, where $\overline{\beta} = (\overline{\beta}_1,\dots,\overline{\beta}_{m_1})$ and $\overline{\gamma} = (\overline{\gamma}_1,\dots,\overline{\gamma}_{m_2})$. It sends $y^{\overline{P}}_{\overline{\delta}}$ with $|\overline{\delta}| = d$ into $y^{\overline{P}}_{\overline{\delta}} + d!(y^{\overline{P}}_{e_{m_1+1}})^{\overline{\delta}_1} \dots (y^{\overline{P}}_{e_{m_1+m_2}})^{\overline{\delta}_{m_2}}$, where $\overline{\delta} = (\overline{\delta}_1,\dots,\overline{\delta}_{m_2})$. It sends Z^P_δ into $Z^P_\delta + Z^P \cdot (\dots)$ because of $|\delta| \le d-1$. Hence (similarly as in 8.3) the equivariance of g^P_δ yields $\sum_{|\overline{\beta}|+|\overline{\gamma}|=d} h^{P,(\overline{\beta},\overline{\gamma})}_{\delta\overline{P}}(X^{\overline{i}})(y^{\overline{P}}_{e_1})^{\overline{\beta}_1} \dots (y^{\overline{P}}_{e_{m_1}})^{\overline{\beta}_{m_1}}(y^{\overline{P}}_{e_{m_1+1}})^{\overline{\gamma}_1} \dots (y^{\overline{P}}_{e_{m_1+m_2}})^{\overline{\gamma}_{m_2}} + \sum_{|\overline{\delta}|=d} \tilde{h}^{P\overline{\delta}}_{\delta\overline{P}}(X^{\overline{i}})(y^{\overline{P}}_{e_{m_1+1}})^{\overline{\delta}_1} \dots \dots (y^{\overline{P}}_{e_{m_1+m_2}})^{\overline{\delta}_{m_2}} = 0$ (if $r < d$ the first sum do not occur). This implies g^P_δ is independent of the $y^{\overline{P}}_{\overline{\delta}}$ with $|\overline{\delta}| = d$ and of the $y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}$ with $|\overline{\beta}| + |\overline{\gamma}| = d$. □

9. The second inductive step

9.1. Lemma. *By the inductive assumption, $g^p_\alpha = 0$ if $|\alpha| \le \min(d-1,q)$, $g^P_{(\beta,\gamma)} = 0$ if $|\beta| + |\gamma| \le \min(d-1,r)$, and $g^P_\delta = 0$ if $|\delta| \le \min(d-1,q)$.*

Proof. Section 8 shows that by the projectability the system $f^i = kX^i$, $f^I = 0$, $g^p = 0$, $g^P = 0$, g^p_α with $|\alpha| \le \min(d-1,q)$, $g^P_{(\beta,\gamma)}$ with $|\beta|+|\gamma| \le \min(d-1,r)$ and g^P_δ with $|\delta| \le \min(d-1,q)$ corresponds to some $\tilde{G}^d_{m_1,m_2,n_1,n_2}$-map $\overline{\mathcal{A}}_0 : V_0 \times S^{\overline{r},\overline{s},\overline{q}} \to Z^{\overline{r},\overline{s},\overline{q}}$ such that $\overline{\mathcal{A}}_0(0,.) : S^{\overline{r},\overline{s},\overline{q}} \to Z^{\overline{r},\overline{s},\overline{q}}$ is a $G^d_{m_1,m_2,n_1,n_2}$-map, where $\overline{r} = \min(d-1,r)$, $\overline{s} = \min(d-1,s)$ and $\overline{q} = \min(d-1,q)$. □

So, it remains to show that $g^p_\alpha = 0$ if $|\alpha| = d = q$, $g^P_{(\beta,\gamma)} = 0$ if $|\beta| + |\gamma| = d = r$, and $g^P_\delta = 0$ if $|\delta| = d = s$.

9.2. Lemma. *If $|\alpha| = d = q$, then $g^p_\alpha = 0$.*

Proof. Let $d = q$, $|\alpha| = d$. By the equivariance of $g^p_\alpha = g^p_\alpha(X^{\bar{i}}, y^{\overline{p}}_{\overline{\alpha}})$ with respect to the homotheties in $G^1_{n_1}$ we deduce that $g^p_\alpha = h^{p\overline{\alpha}}_{\alpha\overline{p}}(X^{\bar{i}}) y^{\overline{p}}_{\overline{\alpha}}$. Now, using the equivariance of g^p_α with respect to the homotheties in $G^1_{m_1}$ we deduce that $h^{p\overline{\alpha}}_{\alpha\overline{p}} = 0$ if $|\overline{\alpha}| < d$, and $h^{p\overline{\alpha}}_{\alpha\overline{p}} = \mathrm{const}$ if $|\overline{\alpha}| = d$. Next, we shall apply the equivariance of $\mathcal{A}_0(0, .)$. Consider $\xi(a^{\overline{\overline{p}}}_{\tilde{\tilde{\alpha}}}) \in G^{d+1}_{m_1,m_2,n_1,n_2}$ corresponding to $(x^{\bar{\bar{i}}}, x^{\overline{\overline{I}}}, y^{\overline{\overline{p}}} + \sum_{|\tilde{\tilde{\alpha}}|=d} a^{\overline{\overline{p}}}_{\tilde{\tilde{\alpha}}} x^{\tilde{\tilde{\alpha}}}, y^{\overline{\overline{P}}})$ for $a^{\overline{\overline{p}}}_{\tilde{\tilde{\alpha}}} \in \mathbb{R}$, where $x^{\tilde{\tilde{\alpha}}} = (x^1)^{\tilde{\tilde{\alpha}}_1} \dots (x^{m_1})^{\tilde{\tilde{\alpha}}_{m_1}}$. Every $\xi(a^{\overline{\overline{p}}}_{\tilde{\tilde{\alpha}}})$ preserves Z^p_α (because of $|\alpha| = d$) and sends $y^{\overline{p}}_{\overline{\alpha}}$ with $|\overline{\alpha}| = d$ into $y^{\overline{p}}_{\overline{\alpha}} + \overline{\alpha}! a^{\overline{p}}_{\overline{\alpha}}$. Then using the equivariance of g^p_α we get $h^{p\overline{\alpha}}_{\alpha\overline{p}} y^{\overline{p}}_{\overline{\alpha}} = h^{p\overline{\alpha}}_{\alpha\overline{p}}(y^{\overline{p}}_{\overline{\alpha}} + \overline{\alpha}! a^{\overline{p}}_{\overline{\alpha}})$ where the sums are over $\overline{\alpha}$ with $|\overline{\alpha}| = d$ and over $\overline{p}$. Hence $h^{p\overline{\alpha}}_{\alpha\overline{p}} = 0$. Therefore $g^p_\alpha = 0$. □

9.3. Lemma. *If $|\beta| + |\gamma| = d = r$, then $g^P_{(\beta,\gamma)} = 0$.*

Proof. Let $d = r$, $|\beta| + |\gamma| = d$. Then $s = r = d$. By the equivariance of $g^P_{(\beta,\gamma)} = g^P_{(\beta,\gamma)}(X^{\bar{i}}, y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}, y^{\overline{P}}_{\overline{\delta}})$ with respect to the homotheties in $G^1_{n_2}$ we can write $g^P_{(\beta,\gamma)} = h^{P,(\overline{\beta},\overline{\gamma})}_{(\beta,\gamma),\overline{P}}(X^{\bar{i}}) y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})} + \tilde{h}^{P\overline{\delta}}_{(\beta,\gamma)\overline{P}}(X^{\bar{i}}) y^{\overline{P}}_{\overline{\delta}}$. Now, by the equivariance of $g^P_{(\beta,\gamma)}$ with respect to the homotheties in $G^1_{m_1} \times G^1_{m_2}$ we deduce that $h^{P,(\overline{\beta},\overline{\gamma})}_{(\beta,\gamma),\overline{P}} = 0$ if $|\overline{\beta}| + |\overline{\gamma}| < d$, $h^{P,(\overline{\beta},\overline{\gamma})}_{(\beta,\gamma),\overline{P}} = \mathrm{const}$ if $|\overline{\beta}| + |\overline{\gamma}| = d$, and $\tilde{h}^{P\overline{\delta}}_{(\beta,\gamma)\overline{P}} = 0$ if $|\overline{\delta}| < d$. Similarly, by the equivariance of $g^P_{(\beta,\gamma)}$ with respect to the homotheties in $G^1_{m_2}$ we deduce that $\tilde{h}^{P\overline{\delta}}_{(\beta,\gamma)\overline{P}} = 0$ if $|\overline{\delta}| = d$ (for $|\gamma| = d - |\beta| \le d - 1$). So, $g^P_{(\beta,\gamma)} = h^{P,(\overline{\beta},\overline{\gamma})}_{(\beta,\gamma),\overline{P}} y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}$, where the sum is over $(\overline{\beta}, \overline{\gamma})$ with $|\overline{\beta}| + |\overline{\gamma}| = d$ and over $\overline{P}$. Next, we shall apply the equivariance of $\mathcal{A}_0(0, .)$. Consider $\xi(a^{\overline{\overline{P}}}_{(\tilde{\tilde{\beta}},\tilde{\tilde{\gamma}})}) \in G^{d+1}_{m_1,m_2,n_1,n_2}$ corresponding to $(x^{\bar{\bar{i}}}, x^{\overline{\overline{I}}}, y^{\overline{\overline{p}}}, y^{\overline{\overline{P}}} + \sum_{|\tilde{\tilde{\beta}}|+|\tilde{\tilde{\gamma}}|=d} a^{\overline{\overline{P}}}_{(\tilde{\tilde{\beta}},\tilde{\tilde{\gamma}})} x^{(\tilde{\tilde{\beta}},\tilde{\tilde{\gamma}})})$ for $a^{\overline{\overline{P}}}_{(\tilde{\tilde{\beta}},\tilde{\tilde{\gamma}})} \in \mathbb{R}$, where $x^{(\tilde{\tilde{\beta}},\tilde{\tilde{\gamma}})} = (x^1)^{\tilde{\tilde{\beta}}_1} \dots (x^{m_1})^{\tilde{\tilde{\beta}}_{m_1}} (x^{m_1+1})^{\tilde{\tilde{\gamma}}_1} \dots (x^{m_1+m_2})^{\tilde{\tilde{\gamma}}_{m_2}}$. Every $\xi(a^{\overline{\overline{P}}}_{(\tilde{\tilde{\beta}},\tilde{\tilde{\gamma}})})$ preserves $Z^P_{(\beta,\gamma)}$ (because of $|\beta| + |\gamma| = d$) and sends $y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}$ with $|\overline{\beta}| + |\overline{\gamma}| = d$ into $y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})} + \overline{\beta}!\overline{\gamma}! a^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}$. Then by the equivariance of $g^P_{(\beta,\gamma)}$ we get $h^{P,(\overline{\beta},\overline{\gamma})}_{(\beta,\gamma),\overline{P}} y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})} = h^{P,(\overline{\beta},\overline{\gamma})}_{(\beta,\gamma),\overline{P}}(y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})} + \overline{\beta}!\overline{\gamma}! a^{\overline{P}}_{(\overline{\beta},\overline{\gamma})})$, where the sums are over $(\overline{\beta}, \overline{\gamma})$ with $|\overline{\beta}| + |\overline{\gamma}| = d$ and over $\overline{P}$. Hence $h^{P,(\overline{\beta},\overline{\gamma})}_{(\beta,\gamma),\overline{P}} = 0$. So, $g^P_{(\beta,\gamma)} = 0$. □

9.4. Lemma. *If* $|\delta| = d = s$, *then* $g^P_\delta = 0$.

Proof. Let $d = s$, $|\delta| = d$. If we use the equivariance of $g^P_\delta = g^P_\delta(X^{\overline{i}}, y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})}, y^{\overline{P}}_{\overline{\delta}})$ with respect to the homotheties in $G^1_{n_2}$ we can write $g^P_\delta = h^{P,(\overline{\beta},\overline{\gamma})}_{\delta\overline{P}}(X^{\overline{i}})y^{\overline{P}}_{(\overline{\beta},\overline{\gamma})} + \tilde{h}^{P\overline{\delta}}_{\delta\overline{P}}(X^{\overline{i}})y^{\overline{P}}_{\overline{\delta}}$. Now, if we apply the equivariance of g^P_δ with respect to the homotheties in $G^1_{m_2}$ we deduce that $h^{P,(\overline{\beta},\overline{\gamma})}_{\delta\overline{P}} = 0$ (as $|\overline{\gamma}| \leq r-1 < d = |\delta|$), and $\tilde{h}^{P\overline{\delta}}_{\delta\overline{P}}(X^{\overline{i}}) = 0$ if $|\overline{\delta}| < d$. Similarly, if we apply the equivariance of g^P_δ with respect to the homotheties in $G^1_{m_1}$ we deduce that $\tilde{h}^{P\overline{\delta}}_{\delta\overline{P}}(X^{\overline{i}}) = \text{const}$. So, $g^P_\delta = \tilde{h}^{P\overline{\delta}}_{\delta\overline{P}}y^{\overline{P}}_{\overline{\delta}}$, where the sum is over $\overline{\delta}$ with $|\overline{\delta}| = d$ and over $\overline{P}$. Next, we shall apply the equivariance of $\mathcal{A}_0(0,.)$. Consider $\xi(a^{\overline{\overline{P}}}_{\tilde{\overline{\delta}}}) \in G^{d+1}_{m_1,m_2,n_1,n_2}$ corresponding to $(x^{\overline{\overline{i}}}, x^{\overline{\overline{I}}}, y^{\overline{\overline{p}}}, y^{\overline{\overline{P}}} + \sum_{|\tilde{\overline{\delta}}|=d} a^{\overline{\overline{P}}}_{\tilde{\overline{\delta}}} x^{\tilde{\overline{\delta}}})$ for $a^{\overline{\overline{P}}}_{\tilde{\overline{\delta}}} \in \mathbb{R}$ where $x^{\tilde{\overline{\delta}}} = (x^{m_1+1})^{\tilde{\overline{\delta}}_1} \dots (x^{m_1+m_2})^{\tilde{\overline{\delta}}_{m_2}}$. Every $\xi(a^{\overline{\overline{P}}}_{\tilde{\overline{\delta}}})$ preserves Z^P_δ (because of $|\delta| = d$) and sends $y^{\overline{P}}_{\overline{\delta}}$ with $|\overline{\delta}| = d$ into $y^{\overline{P}}_{\overline{\delta}} + \overline{\delta}! a^{\overline{P}}_{\overline{\delta}}$. Then using the equivariance of g^P_δ we get $\tilde{h}^{P\overline{\delta}}_{\delta\overline{P}}y^{\overline{P}}_{\overline{\delta}} = \tilde{h}^{P\overline{\delta}}_{\delta\overline{P}}(y^{\overline{P}}_{\overline{\delta}} + \overline{\delta}! a^{\overline{P}}_{\overline{\delta}})$, where the sums are over $\overline{\delta}$ with $|\overline{\delta}| = d$ and over $\overline{P}$. Hence $\tilde{h}^{P\overline{\delta}}_{\delta\overline{P}} = 0$. So, $g^P_\delta = 0$.

The inductive proof of Proposition 5.4 is complete.

The proof of Theorem 2.5 is complete. □

10. The category of 2-fibered manifolds. Functor $J^{r,s}$ and its flow operator. The second main result

10.1. A 2-fibered manifold is a sequence $Y = Y_2 \xrightarrow{\pi_1} Y_1 \xrightarrow{\pi_0} Y_0$ of surjective submersions between manifolds. If $\overline{Y} = \overline{Y}_2 \xrightarrow{\overline{\pi}_1} \overline{Y}_1 \xrightarrow{\overline{\pi}_0} \overline{Y}_0$ is another 2-fibered manifold, a morphism $Y \to \overline{Y}$ is a map $f : Y_2 \to \overline{Y}_2$ such that there are maps $f_0 : Y_0 \to \overline{Y}_0$ and $f_1 : Y_1 \to \overline{Y}_1$ with $\overline{\pi}_0 \circ f_1 = f_0 \circ \pi_0$ and $\overline{\pi}_1 \circ f = f_1 \circ \pi_1$. Thus all 2-fibered manifolds form a category which we will denote by 2-$\mathcal{FM}$. This category is over manifolds, local and admissible.

2-fibered manifolds appear naturally in differential geometry. If E is a bundle functor on manifolds (for example, $E = T$) and $Y_1 \to Y_0$ is a fibered manifold, then $EY_1 \to Y_1 \to Y_0$ is a 2-fibered manifold.

10.2. A 2-fibered manifold $Y = Y_2 \to Y_1 \to Y_0$ has dimension (m_1, n_1, n_2) if Y_2 has dimension $m_1 + n_1 + n_2$, Y_1 has dimension $m_1 + n_1$

and Y_0 has dimension m_1. All 2-fibered manifolds of dimension (m_1, n_1, n_2) and their local isomorphisms form a subcategory $2\text{-}\mathcal{FM}_{m_1,n_1,n_2} \subset 2\text{-}\mathcal{FM}$. Every $2\text{-}\mathcal{FM}_{m_1,n_1,n_2}$-object is locally isomorphic to $\mathbb{R}^{m_1} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{m_1} \times \mathbb{R}^{n_1} \to \mathbb{R}^{m_1}$, the usual projections.

10.3. Let $Y = Y_2 \xrightarrow{\pi_1} Y_1 \xrightarrow{\pi_0} Y_0$ be an $2-\mathcal{FM}_{m_1,n_1,n_2}$-object. Denote the set of (local) sections of $\pi_1 : Y_2 \to Y_1$ by ΓY. Let $r, s, \in \mathbb{N}$, $s \geq r$. By 12.19 in [12], $\sigma, \rho \in \Gamma Y$ represent the same (r, s)-jet $j^{r,s}_x \sigma = j^{r,s}_x \rho$ at a point $y_1 \in Y_1$ if $j^r_{y_1} \sigma = j^r_{y_1} \rho$ and $j^s_{y_1}(\sigma|Y_{1y_0}) = j^s_{y_1}(\rho|Y_{1y_0})$, where $y_0 = \pi_1(y_1) \in Y_0$ and Y_{1y_0} is the fibre of $\pi_0 : Y_1 \to Y_0$ over y_0. $J^{r,s}Y = \{j^{r,s}_{y_1}\sigma \mid \sigma \in \Gamma Y, y_1 \in Y_1\}$ is a fibered manifold over Y_2 with respect to the target projection. If $\overline{Y} = \overline{Y}_2 \to \overline{Y}_1 \to \overline{Y}_0$ is another $2\text{-}\mathcal{FM}_{m_1,n_1,n_2}$-object and $f : Y \to \overline{Y}$ is an $2\text{-}\mathcal{FM}_{m_1,n_1,n_2}$-isomorphism with the underlying maps f_1, f_0, we define $J^{r,s}f : J^{r,s}Y \to J^{r,s}\overline{Y}$, $J^{r,s}f(j^{r,s}_{y_1}\sigma) = j^{r,s}_{f_1(y_1)}(f \circ \sigma \circ f_1^{-1})$, $\sigma \in \Gamma Y$, $y_1 \in Y_1$. Then $J^{r,s} : 2\text{-}\mathcal{FM}_{m_1,n_1,n_2} \to \mathcal{FM}$ is a bundle functor.

10.4. Let $Y = Y_2 \xrightarrow{\pi_1} Y_1 \xrightarrow{\pi_0} Y_0$ be an $2\text{-}\mathcal{FM}_{m_1,n_1,n_2}$-object. A vector field Z on Y_2 is projectable on Y if there exist a vector field Z_1 on Y_1 and a vector field Z_0 on Y_0 such that Z is π_1-related with Z_1 and Z_1 is π_0-related with Z_0. If Z is projectable on Y, then its flow is formed by local $2\text{-}\mathcal{FM}$-isomorphisms. If $F : 2\text{-}\mathcal{FM}_{m_1,n_1,n_2} \to \mathcal{FM}$ is a bundle functor, we have a vector field $\mathcal{F}Z = \frac{\partial}{\partial t}_{|t=0} F \operatorname{Exp} tZ$ on FY. The flow operator $\mathcal{F} : T_{\mathrm{proj}\,|2-\mathcal{FM}_{m_1,n_1,n_2}} \rightsquigarrow TF$ is $2\text{-}\mathcal{FM}_{m_1,n_1,n_2}$-natural.

10.5. Theorem. *Let $m_1, n_1, n_2 \in \mathbb{N} \cup \{0\}$, $m_1 \geq 1$, $r, s \in \mathbb{N}$, $s \geq r$. Every natural operator $T_{\mathrm{proj}|2-\mathcal{FM}_{m_1,n_1,n_2}} \rightsquigarrow TJ^{r,s}$ is a constant multiple of the flow operator $\mathcal{J}^{r,s}$.*

Proof. $2\text{-}\mathcal{FM}_{m_1,n_1,n_2}$ is a full subcategory in $\mathcal{F}^2\mathcal{M}_{m_1,n_1,0,n_2}$. (A 2-fibered manifold $Y = Y_2 \xrightarrow{\pi_1} Y_1 \xrightarrow{\pi_0} Y_0$ is a fibered fibered manifold $\pi_1 : Y_2 \to Y_1$, where Y_2 and Y_1 are the fibered manifolds $\pi_0 \circ \pi_1$ and π_0, respectively.) Every natural operator $T_{\mathrm{proj}|\mathcal{F}^2\mathcal{M}_{m_1,n_1,0,n_2}} \rightsquigarrow TJ^{r,s,q}$ is determined by its restriction to the $\mathcal{F}^2\mathcal{M}_{m_1,n_1,0,n_2}$-object $\mathbb{R}^{m_1} \times \mathbb{R}^{n_1} \times \{0\} \times \mathbb{R}^{n_2} = \mathbb{R}^{m_1} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Every natural operator $T_{\mathrm{proj}|2-\mathcal{FM}_{m_1,n_1,n_2}} \rightsquigarrow TJ^{r,s}$ is also determined by its restriction to the $2\text{-}\mathcal{FM}_{m_1,n_1,n_2}$-object $\mathbb{R}^{m_1} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. So, applying Theorem 2.5 we end the proof. $\square$

11. Theorem 1.5 as a consequence of Theorem 2.5

Proof of Theorem 1.5. $\mathcal{FM}_{m_1,n_1}$ is a subcategory in $\mathcal{F}^2_{m_1,0,n_1,0}$. (A fibered manifold $\pi_0 : Y_1 \to Y_0$ can be considered as a fibered fibered manifold $\pi_0 : Y_1 \to Y_0$, where Y_1 and Y_0 are the fibered manifolds id : $Y_1 \to Y_1$ and id : $Y_0 \to Y_0$, respectively.) Now, we use the arguments similar to the one of Theorem 10.5. □

12. An application

In [6], we defined the bundle functor $K^{r,s,q}_{k,l}$ of contact elements of dimension (k,l) and of order (r,s,q) over fibered manifolds, and we proved the following theorem.

12.1. Theorem ([6]). *Let $r,s,q,k,l,m,n \in \mathbb{N}$, $s \geq r \leq q$, $k \geq m$, $l \geq n$. Every natural operator $A : T_{\mathrm{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow TK^{r,s,q}_{k,l}$ is a constant multiple of the flow operator.*

Proof. We can deduce this theorem from Theorem 2.5 by the (obviously adapted) proof of Proposition 44.4 in [4]. □

References

[1] M. Doupovec and I. Kolář, *On the jets of fibered manifold morphisms*, *Cahiers Topo. Geom. Diff. Categoriques* **XL** (1999), 21–30.

[2] J. Gancarzewicz, W. M. Mikulski and Z. Pogoda, Lifts of some tensor fields and connections to product preserving functors, *Nagoya Math. J.* **135** (1994), 1–41.

[3] P. Kobak, Natural liftings of vector fields to tangent bundles of bundles of 1-forms, *Math. Bohemica* **116** (1991), 319–326.

[4] I. Kolář, P. Michor and J. Slovák, Natural Operations In Differential Geometry, *Springer Verlag*, *Berlin*, 1993.

[5] I. Kolář and W. M. Mikulski, On the fiber product preserving bundle functors, *Diff. Geom. and Its Appl.* **11** (1999), 105–115.

[6] I. Kolář and W. M. Mikulski, Contact elements on fibered manifolds, *Czech. Math. J.* (*to appear*).

[7] I. Kolář and J. Slovák, Prolongation of vector fields to jet bundles, *Supl. Rend. Circolo Math. Palermo* **22** (II) (1990), 103–111.

[8] J. Kurek, On the first order natural operators transforming 1-forms on manifolds to the tangent bundle, *Ann. UMCS* XLIII **8** (1989), 79–83.

[9] W. M. Mikulski, Some natural constructions on vector fields and higher order cotangent bundles, *Mh. Math.* **117** (1994), 107–119.

[10] W. M. Mikulski, The natural operators lifting vector fields to $(J^rT^*)^*$, *Arch. Math. Brno* **36**(4) (2000), 255–260.

[11] W. M. Mikulski, The natural operators lifting vector fields to generalized higher order tangent bundles, *Arch. Math. Brno* **36**(3) (2000), 207–212.

[12] A. Morimoto, Prolongations of connections to bundles of infinitely near points, *J. Diff. Geom.* **40** (1970), 85–97.

[13] A. Nijenhuis, Natural bundles and their general properties, Diff. Geom. in honor of K. Yano, Tokyo 1972, 317–334.

[14] R. Wolak, On transverse structures of foliations, *Suppl. Rend. Circ. Mat. Palermo* **9** (1985), 227–243.

[15] J. Tomáš, Natural operators on vector fields on the cotangent bundles of (k, r)-velocities, *Rend. Circ. Mat. Palermo* (1998), 113–124, 239–249.

W. M. MIKULSKI
INSTITUTE OF MATHEMATICS
JAGIELLONIAN UNIVERSITY
REYMONTA 4, KRAKÓW
POLAND

E-mail: mikulski@im.uj.edu.pl

(Received January 31, 2001; revised June 1, 2001)