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A characterization of polynomials through Flett's MVT

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Abstract. We consider the following functional equation

$$f(a)=\sum_{k=0}^{n} g_k(a-tc)(tc)^k+(\Phi(a)-\Phi(a-c))(tc)^n, \quad a,c\in\mathbb{R},\ t\in\mathbb{Q}\cap(0,1)$$

related to a generalization of the Flett's Mean Value Theorem [4]. We present the solutions of the above equation without any regularity assumptions on the functions $f, g_0, \dots, g_n, \Phi:\mathbb{R}\to\mathbb{R}$.

1. Introduction

The Lagrange Mean Value Theorem says that for every function $f:[a,b]\to\mathbb{R}$, continuous on $[a,b]$ and differentiable on (a,b), there exists a point $c\in(a,b)$ such that

$$f(b)-f(a)=(b-a)f'(c).$$

It is well known that taking $c=\frac{a+b}{2}$ the above equation characterizes quadratic polynomials. J. Aczél in [1] and Sh. Haruki in [3] considered a more general equation as follows

$$f(x)-g(y)=h(x+y)(x-y).$$

They proved that f, g, h satisfy this equation if and only if $f(x)=g(x)=ax^2+bx+c$ and $h(x)=ax+b$, for some $a,b,c\in\mathbb{R}$. Analogously, T. Riedel

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and M. Sablik in [5] solved the functional equation

$$f(c)-f(a)=(c-a)h(c)-\frac{1}{2}\frac{h(b)-h(a)}{b-a}(c-a)^2. \tag{1}$$

Their work was motivated by a generalization of Flett's Mean Value Theorem due to T. Riedel and P. K. Sahoo [6].

Theorem. *If $f:[a,b]\to\mathbb{R}$ is differentiable on $[a,b]$, then there exists a point $c\in(a,b)$ such that*

$$f(c)-f(a)=(c-a)f'(c)-\frac{1}{2}\frac{f'(b)-f'(a)}{b-a}(c-a)^2.$$

T. Riedel and M. Sablik proved that with $c=\frac{a+3b}{4}$ (1) characterizes cubic polynomials.

The following theorem (see [4]) is a generalization of Flett's Mean Value Theorem.

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be an n times differentiable function. Then there exists a $t=t(a,b)\in(0,1)$ such that*

$$\begin{aligned} f(a)=&\sum_{k=0}^{n}\frac{t^k f^{(k)}(a+t(b-a))}{k!}(a-b)^k \\ &+\frac{t^{n+1}}{(n+1)!}\big(f^{(n)}(a)-f^{(n)}(b)\big)(a-b)^n. \end{aligned} \tag{2}$$

If we define functions $g_0,\dots,g_n,\Phi:[a,b]\to\mathbb{R}$ by

$$g_i(x)=\frac{f^{(i)}(x)}{i!},\qquad i\in\{0,\dots,n\}, \tag{3}$$

$$\Phi(x)=\frac{tf^{(n)}(x)}{(n+1)!}, \tag{4}$$

then (2) implies that $(f,g_0,\dots,g_n,\Phi)$ satisfies the equation

$$f(a)=\sum_{k=0}^{n}g_k\big(a+t(b-a)\big)\big(t(a-b)\big)^k+\big(\Phi(a)-\Phi(b)\big)\big(t(a-b)\big)^n. \tag{5}$$

Motivated by [5], [7] we pose two questions about the equation (5).

(i) Find the unknown functions $f, g_0, \dots, g_n, \Phi : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(a) = \sum_{k=0}^{n} g_k(a + t(b-a))(t(a-b))^k + (\Phi(a) - \Phi(b))(t(a-b))^n. \tag{5}$$

for all $a, b \in \mathbb{R}$ and for a *fixed* $t \in (0,1)$.

(ii) Determine if the relations (3), (4) hold for any solution.

The aim of the paper is to answer these questions.

The formula (2) is asymmetric in a, b. One may ask whether Theorem 1.1 is still true if we interchange a and b. The answer is positive.

Theorem 1.1'. *Let $f : [a,b] \to \mathbb{R}$ be an n times differentiable function. Then there exists an $s = s(a,b) \in (0,1)$ such that*

$$\begin{aligned} f(b) = \sum_{k=0}^{n} & \frac{s^k f^{(k)}(b + s(a-b))}{k!}(b-a)^k \\ & + \frac{s^{n+1}}{(n+1)!}\big(f^{(n)}(b) - f^{(n)}(a)\big)(b-a)^n. \end{aligned} \tag{2'}$$

Proof. Let us define a transformation $T : [a,b] \to [a,b]$ by

$$T(x) = a + b - x.$$

Applying Theorem 1.1 to the composition $f \circ T$ completes the proof.

2. Main results

Now we present answers to the forementioned questions. Firstly, we prove a theorem which concerns the connection between polynomials and the set of t's for which (2) is satisfied.

We begin with some lemmas. The first one easily follows from the binomial formula.

Lemma 2.1. *Let $n \in \mathbb{N}$, $a_i, x, y \in \mathbb{R}$ for $i \in \{0, \dots, n\}$. Then the following formula holds*

$$\sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} \sum_{k=i}^{n} \binom{k}{i} a_k (x-y)^{k-i} y^i.$$

We need also the following lemma (cf. [8]).

Lemma 2.2. *Let $N \in \mathbb{N}$ and $C_i : \mathbb{R} \to \mathbb{R}$, $i \in \{0, \dots, n\}$ be functions homogeneous of i-th order with respect to $k \in \mathbb{N}$, i.e. $C_i(kz) = k^i C_i(z)$. If*

$$\sum_{i=0}^{N} C_i(z) = 0, \quad \textit{then} \quad C_i(z) = 0$$

for every $i \in \{0, \dots, N\}$.

Let us denote by p_N a polynomial of N-th degree, i.e. $p_N = \sum_{i=0}^{N} d_i x^i$, d_i, $x \in \mathbb{R}$, $i \in \{0, \dots, N\}$, $d_N \neq 0$. We have the following

Theorem 2.3. *Let $n \in \mathbb{N}$ and $f = p_N$ for some $N \in \{0, \dots, n+2\}$. If $N \leq n+1$ then (2) (resp. (2')) holds for every $a, b \in \mathbb{R}$, $a < b$ with any $t \in (0,1)$ (resp. $s \in (0,1)$). If $N = n+2$ then (2) (resp. (2')) holds for every $a, b \in \mathbb{R}$, $a < b$, if and only if $t = \frac{n+2}{2n+2}$ (resp. $s = \frac{n+2}{2n+2}$).*

Proof. Putting $c := a - b$, fix $n \in \mathbb{N}$ and $N \in \{0, \dots, n+2\}$. It is obvious that the k-th derivative of p_N has the following form

$$p_N^{(i)}(x) = \sum_{k=i}^{N} k(k-1) \cdot \ldots \cdot (k-i+1) d_k x^{k-i}. \tag{6}$$

For $f = p_N$ and $N \leq n$ using (6) we get from (2) the following

$$\sum_{i=0}^{N} d_i a^i = \sum_{i=0}^{N} \sum_{k=i}^{N} \binom{k}{i} d_k (a - tc)^{k-i} (tc)^i.$$

The application of Lemma 2.1 ends the proof in this case.

Applying Lemma 2.1 and (6) for $N = n+1$ in (2) we have for every $t \in (0,1)$

$$\sum_{i=0}^{N-1} \sum_{k=i}^{N} \binom{k}{i} d_k (a-tc)^{k-i} (tc)^i + \frac{t^N}{N!} (N! d_N a - N! d_N (a-c)) c^{N-1}$$
$$= \sum_{i=0}^{N-1} \sum_{k=i}^{N-1} \binom{k}{i} d_k (a-tc)^{k-i} (tc)^i$$

$$+\sum_{i=0}^{N-1}\binom{N}{i}d_N(a-tc)^{N-i}(tc)^i+d_N(tc)^N$$

$$=\sum_{i=0}^{N-1}d_ia^i+d_Na^N=\sum_{i=0}^{N}d_ia^i,$$

which means that (2) holds with any $t\in(0,1)$ if $f=p_{n+1}$.

Finally, if $N=n+2$ the equality (2) yields

$$\sum_{i=0}^{N}d_ia^i=\sum_{i=0}^{N-2}\sum_{k=i}^{N}\binom{k}{i}d_k(a-tc)^{k-i}(tc)^i+\frac{t^{N-1}}{(N-1)!}$$
$$\times\big((N-1)!d_{N-1}c+3\cdot\ \ldots\ \cdot Nd_N(a^2-(a-c)^2)\big)c^{N-2}$$
$$=\sum_{i=0}^{N-2}\sum_{k=i}^{N}\binom{k}{i}d_k(a-tc)^{k-i}(tc)^i+d_{N-1}(tc)^{N-1}+Nd_N(tc)^{N-1}\left(a-\frac{c}{2}\right)$$
$$=\sum_{i=0}^{N-2}\sum_{k=i}^{N-2}\binom{k}{i}d_k(a-tc)^{k-i}(tc)^i+\sum_{i=0}^{N-2}\binom{N-1}{i}d_{N-1}(a-tc)^{N-1-i}(tc)^i$$
$$+\sum_{i=0}^{N-2}\binom{N}{i}d_N(a-tc)^{N-i}(tc)^i+d_{N-1}(tc)^{N-1}+Nd_N(tc)^{N-1}\left(a-\frac{c}{2}\right).$$

Using Lemma 2.1 and the binomial formula we get therefore

$$\sum_{i=0}^{N}d_ia^i=\sum_{i=0}^{N-2}d_ia^i+d_{N-1}a^{N-1}$$
$$+d_Na^N-Nd_N(a-tc)(tc)^{N-1}-d_N(tc)^N+Nd_N(tc)^{N-1}\left(a-\frac{c}{2}\right)$$
$$=\sum_{i=0}^{N}d_ia^i-Nd_N(a-tc)(tc)^{N-1}-d_N(tc)^N+Nd_N(tc)^{N-1}\left(a-\frac{c}{2}\right),$$

whence

$$N(a-tc)+tc=N\left(a-\frac{c}{2}\right)$$

and finally

$$t=\frac{N}{2N-2}=\frac{n+2}{2n+2},$$

which ends the proof, since the case of (2') may be treated similarly.

So we have got that polynomials of degree $n+2$ satisfy (2) with a unique $t = \frac{n+2}{2n+2}$ while for polynomials of lower degree (2) holds for any t from $(0,1)$. In Theorems 1.1 and 1.1' $t = t(a,b)$ and $s = s(a,b)$ usually are not equal. The above result shows that for polynomials of degree at most $n+2$, t and s may (or have to, if $N = n+2$) be equal and do not depend on a and b. Now the question is whether (2) is satisfied only by polynomials if we fix a $t \in (0,1)$. Observe that in view of the above remarks we admit that (2) holds for any $a, b \in \mathbb{R}$, not necessarily $a < b$.

Putting $c := a - b$ in (5) we obtain the related functional equation in the following form

$$f(a) = \sum_{k=0}^{n} g_k(a-tc)(tc)^k + (\Phi(a) - \Phi(a-c))(tc)^n, \quad a, c \in \mathbb{R}. \tag{7}$$

Let us introduce a notation. For any $n \in \mathbb{N}$ denote by $SA^n(\mathbb{R},\mathbb{R})$ the family of all n-additive and symmetric mappings from $\mathbb{R}^n$ into $\mathbb{R}$. If $B \in SA^n(\mathbb{R},\mathbb{R})$ then B^d will stand for the diagonalization of B. By $B(y^l, z^{l-j})$ we mean a value of B at any l-tuple in which l entries are equal to y and the remaining ones to z.

We are going now to apply Lemma 2.3 from [8] to infer that Φ is a polynomial function. First, observe that any function $\gamma : \mathbb{R} \to \mathbb{R}$ generates a mapping from $\mathbb{R}$ into $SA^k(\mathbb{R},\mathbb{R})$, ($k \in \mathbb{N}$) given by

$$\gamma(x)(y_1, \ldots, y_k) = \gamma(x) y_1 \cdot \ldots \cdot y_k,$$

for every $x, y_1, \ldots, y_k \in \mathbb{R}$. Therefore any $\gamma : \mathbb{R} \to \mathbb{R}$ can be identified with a mapping from $\mathbb{R}$ into $SA^k(\mathbb{R},\mathbb{R})$ which will be denoted by γ as well. Taking this into account, let us put $x = a$, $y = tc$ and rewrite (7) in the following, equivalent form

$$\Phi(x)(y^n) - f(x) = \sum_{k=0}^{n} -g_k(x-y)(y^k) + \Phi\left(x - \frac{y}{t}\right)(y^n). \tag{7'}$$

Now (7') is a particular case of the equation (4) from Lemma 2.3 in [8]. To see it let us admit (cf. the notation in [8]) $N = n$, $\varphi_N = \Phi$, $\varphi_{N-1} = \cdots = \varphi_1 = 0$, $\varphi_0 = -f$, $I_0 = \cdots = I_{N-1} = \{(id, -id)\}$, $I_N = \{(id, -id), (id, -\frac{1}{t} id)\}$, $\psi_{k,(id,-id)} = -g_k$, $k \in \{0, \ldots, N\}$ and $\psi_{N,(id,-\frac{1}{t}id)} = \Phi$.

Because $t \neq 0$ we have $-\frac{1}{t} id(\mathbb{R}) = \mathbb{R}$, and thus all the assumptions of Lemma 2.3 from [8] are satisfied. In view of the Lemma we see therefore that $\Phi = \varphi_N$ is a polynomial function. It is a well known fact [9] that Φ can be represented in form

$$\Phi = B_0^d + \cdots + B_s^d, \tag{8}$$

where $B_i \in SA^i(\mathbb{R}, \mathbb{R})$, $i \in \{0, \ldots, s\}$. Then assuming some regularity on function Φ (e.g. continuity, boundedness on a nonvoid open set, measurability) we conclude that Φ is a polynomial, i.e. Φ has the form $\Phi(x) = b_0 + b_1 x + \cdots + b_s x^s$ for some $b_i \in \mathbb{R}$, $i \in \{0, \ldots, s\}$. Our aim is to show that this is true even with no regularity assumption. However, we will assume that the fixed t with which (7) holds, is rational.

Let us observe that (7) may be written equivalently with new variables $z := tc$, $y := a - z$ and $u := 1 - \frac{1}{t}$ in the form

$$f(y+z) = \sum_{k=0}^{n} g_k(y) z^k + (\Phi(y+z) - \Phi(y+uz)) z^n. \tag{9}$$

Putting $y := 0$ in (9) we infer

$$f(z) = \sum_{k=0}^{n} g_k(0) z^k + (\Phi(z) - \Phi(uz)) z^n. \tag{10}$$

Taking (8) into account we see that

$$f(z) = \sum_{k=0}^{n} a_k z^k + (A_1^d(z) + \cdots + A_s^d(z)) z^n, \tag{11}$$

where $A_i \in SA^i(\mathbb{R}, \mathbb{R})$, $i \in \{1, \ldots, s\}$, while $a_k \in \mathbb{R}$, $k \in \{0, \ldots, n\}$ are some constants. Inserting (8) and (11) into (9) and using rational homogeneity of Φ (note that t hence u are rational) we get

$$\sum_{l=0}^{n} \sum_{i=0}^{l} \binom{l}{i} a_l \, y^{l-i} z^i + \sum_{l=1}^{s} \sum_{j=0}^{l} \sum_{i=0}^{n} \binom{l}{j} \binom{n}{i} A_l(y^{l-j}, z^j) y^{n-i} z^i$$

$$= \sum_{k=0}^{n} g_k(y) z^k + \left[\sum_{l=0}^{s} \sum_{j=0}^{l} \binom{l}{j} (1 - u^j) B_l(y^{l-j}, z^j) \right] z^n. \tag{12}$$

Suitably arranging terms on both sides of (12) we get the following

$$L_0(y) + \sum_{i=1}^{n+s} L_i(y,z) = R_0(y) + \sum_{i=1}^{n+s} R_i(y,z),$$

where L_i and R_i are those terms on the left and right hand side of (12), respectively, which are homogeneous of degree i in the variable z for every $y, z \in \mathbb{R}$ and $i \in \{0, \dots, n+s\}$. Lemma 2.2 yields $L_i(y,z) = R_i(y,z)$ for $y, z \in \mathbb{R}$ and $i \in \{0, \dots, n+s\}$. In particular we have

$$L_{n+s}(y,z) = A_s(z^s)z^n$$

and

$$R_{n+s}(y,z) = (1-u^s)B_s(z^s)z^n,$$

whence

$$A_s^d(z)z^n = (1-u^s)B_s^d(z)z^n. \tag{13_0}$$

It is easy to check that for every $k \in \{1, \dots, s-1\}$

$$L_{n+s-k}(y,z) = \sum_{l=1}^{s} \sum_{(i,j)\in D_{k,l}} \binom{l}{j}\binom{n}{i} A_l(y^{l-j}, z^j)y^{n-i}z^i,$$

$$R_{n+s-k}(y,z) = \sum_{l=s-k}^{s} \binom{l}{s-k} B_l(y^{l-s+k}, z^{s-k})(1-u^{s-k})z^n,$$

where $D_{k,l} = \{(i,j) \in \{0, \dots, n\} \times \{0, \dots, l\} : i + j = n + s - k\}$. Obviously the righthand sides of the above formulas are sums of functions $\mathbb{N}$-homogeneous with respect to y. Applying Lemma 2.2 gives in particular the equality of terms which are homogeneous of k-th order with respect to y, $k \in \{1, \dots, s-1\}$. An easy computation shows that

$$\sum_{(i,j)\in D_{k,s}} \binom{s}{j}\binom{n}{i} A_s(y^{s-j}, z^j)y^{n-i}z^i = \binom{s}{s-k}(1-u^{s-k})B_s(y^k, z^{s-k})z^n.$$

Putting $y := z$ in the above we get

$$\sum_{(i,j)\in D_{k,s}} \binom{s}{j}\binom{n}{i} A_s^d(z)z^n = \binom{s}{s-k}(1-u^{s-k})B_s^d(z)z^n.$$

The sum in the above equation may be written equivalently

$$\sum_{(i,j)\in D_{k,s}} \binom{s}{j}\binom{n}{i} = \sum_{i=n-k}^{n} \binom{s}{n+s-k-i}\binom{n}{i}$$

$$= \sum_{i=n-k}^{n} \binom{s}{k-(n-i)}\binom{n}{n-i} = \sum_{j=0}^{k} \binom{s}{k-j}\binom{n}{j} = \binom{n+s}{k}$$

where the last equality follows from the combinatorial meaning of the binomial coefficients. So we have

$$(13_k) \qquad A_s^d(z)z^n = \frac{s(s-1)\cdot\ \dots\ \cdot(s-k+1)}{(n+s)(n+s-1)\cdot\ \dots\ \cdot(n+s-k+1)} \times (1-u^{s-k})B_s^d(z)z^n$$

for $k \in \{1,\dots,s-1\}$. Let us note that the above formula holds also for $k=0$, if we admit the convention that $\prod_{i=p}^{q} i = 1$ for $q<p$ (cf. (13_0) above). This convention will be used also in the proof of the following lemma.

Lemma 2.4. *Let $f, g_0, \dots, g_n, \Phi : \mathbb{R} \to \mathbb{R}$ and $t \in \mathbb{Q} \cap (0,1)$ be such that $(f, g_0, \dots, g_n, \Phi)$ is a solution of (7). Then Φ and f are polynomial functions of degrees at most 2 and $n+2$, respectively.*

Proof. Suppose that contrary to our claim $s \geq 3$ and consider the equalities (13_{s-3}), (13_{s-2}) and (13_{s-1}). Assuming $B_s \neq 0$ we get

$$\frac{s(s-1)\cdot\ \dots\ \cdot 2}{(n+s)\cdot\ \dots\ \cdot(n+2)}(1-u) = \frac{s(s-1)\cdot\ \dots\ \cdot 3}{(n+s)\cdot\ \dots\ \cdot(n+3)}(1-u^2)$$

$$= \frac{s(s-1)\cdot\ \dots\ \cdot 4}{(n+s)\cdot\ \dots\ \cdot(n+4)}(1-u^3)$$

or

$$(14) \qquad \frac{6}{(n+2)(n+3)} = \frac{3}{(n+3)}(1+u) = 1+u+u^2.$$

A simple calculation shows that (14) implies $u^2 < 0$ which contradicts our assumption on s and ends the proof. □

We have proved therefore that (cf. (8) and (10))

$$\Phi(z) = B_0 + B_1(z) + B_2^d(z)$$

and

$$f(z) = \sum_{k=0}^{n} a_k z^k + (1-u)B_1(z)z^n + (1-u^2)B_2^d(z)z^n \tag{15}$$

for $z \in \mathbb{R}$. In view of Lemma 2.4 and (13_0) we can rewrite (12) in the following way

$$\begin{aligned} &\sum_{l=0}^{n}\sum_{i=0}^{l}\binom{l}{i} a_l y^{l-i} z^i + (1-u)\left(B_1(y) + B_1(z)\right)\sum_{i=0}^{n}\binom{n}{i} y^{n-i} z^i \\ &\qquad + (1-u^2)B_2^d(y+z)\sum_{i=0}^{n}\binom{n}{i} y^{n-i} z^i \\ &= \sum_{k=0}^{n} g_k(y) z^k + (1-u)B_1(z)z^n + 2(1-u)B_2(y,z)z^n + (1-u^2)B_2^d(z)z^n \end{aligned} \tag{16}$$

for $y, z \in \mathbb{R}$. We can arrange terms on both sides of (16) to get sums of $\mathbb{N}$-homogeneous functions with respect to z. Taking into account summands which are homogeneous of $(n+1)$-th order in (16) and applying Lemma 2.2 yields

$$\begin{aligned} &(1-u)B_1(z)z^n + 2(1-u^2)B_2(y,z)z^n + (1-u^2)B_2^d(z)nyz^{n-1} \\ &\qquad = (1-u)B_1(z)z^n + 2(1-u)B_2(y,z)z^n \end{aligned}$$

for all $y, z \in \mathbb{R}$. Hence we get

$$n(1+u)B_2^d(z)y = -2uB_2(y,z)z \tag{17}$$

for $z \neq 0$. Replacing y by z we get

$$n(1+u)B_2^d(z)z = -2uB_2^d(z)z$$

for $z \neq 0$. Taking $z \in \mathbb{R}$ such that $B_2^d(z) \neq 0$ we have

$$n(1+u) = -2u$$

hence

$$u = -\frac{n}{n+2}$$

or

$$t = \frac{n+2}{2(n+1)}.$$

For $t = \frac{n+2}{2(n+1)}$ we get from (17) for $z \neq 0$

$$B_2^d(z)y = B_2(y,z)z.$$

Dividing by yz^2 and using the symmetry of $B_2(y,z)$ yields

$$\frac{B_2^d(z)}{z^2} = \frac{B_2(y,z)}{yz} = \frac{B_2(z,y)}{zy} = \frac{B_2^d(y)}{y^2}$$

for $y, z \neq 0$. It follows that $\frac{B_2^d(z)}{z^2}$ is a constant, which we denote by b_2. Thus we get

$$B_2^d(z) = b_2 z^2 \tag{18}$$

and this holds for all $z \in \mathbb{R}$ because $B_2^d(0) = 0$. Now we consider terms which are homogeneous of n-th order with respect to z in (16). By Lemma 2.2 and (18) we get

$$\begin{aligned} a_n z^n + (1-u)B_1(y)z^n + n(1-u)B_1(z)yz^{n-1} + (1-u^2)b_2y^2z^n \\ + 2n(1-u^2)b_2y^2z^n + \binom{n}{n-2}(1-u^2)b_2y^2z^n = g_n(y)z^n \end{aligned}$$

for all $z \in \mathbb{R}$, whence for $z \neq 0$

$$\begin{aligned} g_n(y) = a_n + (1-u)B_1(y) + n(1-u)\frac{B_1(z)y}{z} + (1-u^2)b_2y^2 \\ + 2n(1-u^2)b_2y^2 + \binom{n}{n-2}(1-u^2)b_2y^2. \end{aligned}$$

Hence for $y, z \neq 0$ we have

$$\begin{aligned} n(1-u)\frac{B_1(z)}{z} = \frac{1}{y}\Big[g_n(y) - a_n - (1-u)B_1(y) - (1-u^2)b_2y^2 \\ - 2n(1-u^2)b_2y^2 - \binom{n}{n-2}(1-u^2)b_2y^2\Big]. \end{aligned}$$

It is clear that the righthand side does not depend on z. Thus $B_1(z)$ is linear, say

$$B_1(z) := b_1 z \tag{19}$$

for all $z \in \mathbb{R}$. Inserting (18) and (19) in (15) we get

$$f(z) = \begin{cases} \sum_{k=0}^{n} a_k z^k + \frac{1}{t} b_1 z^{n+1} + \frac{2t-1}{t^2} b_2 z^{n+2}, & \text{if } t = \frac{n+2}{2(n+1)}, \\ \sum_{k=0}^{n} a_k z^k + \frac{1}{t} b_1 z^{n+1}, & \text{if } t \neq \frac{n+2}{2(n+1)}. \end{cases} \tag{20}$$

Theorem 2.5. *Let $(f, g_0, \ldots, g_n, \Phi)$ be a solution of (7), where $t \in \mathbb{Q} \cap (0,1)$ is fixed. If $t \neq \frac{n+2}{2n+2}$ then Φ and f are polynomials of degrees at most 1 and $n+1$, respectively. If $t = \frac{n+2}{2n+2}$ then the respective degrees are less than or equal to 2 and to $n+2$. Moreover, if $f(x) = \sum_{l=0}^{n+2} a_l x^l$ and $\Phi(x) = b_0 + b_1 x + b_2 x^2$ then*

$$b_1 = t a_{n+1} \tag{21}$$

and

$$b_2 = \begin{cases} 0, & \text{if } t \neq \frac{n+2}{2n+2}, \\ \frac{(n+2)^2}{4(n+1)} a_{n+2}, & \text{if } t = \frac{n+2}{2n+2}. \end{cases} \tag{22}$$

In particular

$$\Phi(x) = c_0 + \frac{t}{(n+1)!} f^{(n)}(x) \tag{23}$$

for some constant c_0.

Proof. It remains to prove the last part of the assertion. Defining $a_{n+1} := \frac{1}{t} b_1$ and $a_{n+2} := \frac{2t-1}{t^2} b_2$ and by (20) we get (22). It is now matter of a straightforward calculation to show that (23) holds. □

Now we can present the solution of the equation (7). First however let us prove the following

Lemma 2.6. *Let* $f, g_0, \dots, g_n, \Phi : \mathbb{R} \to \mathbb{R}$ *and* $t \in \mathbb{Q} \cap (0,1)$ *be such that* $(f, g_0, \dots, g_n, \Phi)$ *is a solution of* (7). *Then*

$$g_i(x) = \frac{f^{(i)}(x)}{i!},$$

for $i \in \{0, \dots, n\}$.

Proof. From the equation (7) and Theorem 2.5 we infer that g is also a polynomial. Indeed, if $f(x) = \sum_{i=0}^{n+2} a_i x^i$ then comparing terms with $(tc)^i$ for $i \in \{0, \dots, n\}$ on both sides of (7) we get

$$g_i(y) = \binom{i}{i} a_i + \binom{i+1}{i} a_{i+1} y + \binom{i+2}{i} a_{i+2} y^2 + \dots$$
$$+ \binom{n}{i} a_n y^{n-i} + \binom{n+1}{i} a_{n+1} y^{n+1-i} + \binom{n+2}{i} a_{n+2} y^{n+2-i},$$

whence we have the following

$$g_0(y) = f(y)$$

and (cf. (6))

$$i! g_i(y) = f^{(i)}(y)$$

which completes the proof. □

Summarizing, our results can be collected in the following

Theorem 2.7. *Let* $t \in \mathbb{Q} \cap (0,1)$ *be fixed. The functions* $f, g_0, \dots$ $\dots, g_n, \Phi : \mathbb{R} \to \mathbb{R}$ *satisfy the equation*

$$\text{(7)} \qquad f(a) = \sum_{k=0}^{n} g_k(a - tc)(tc)^k + (\Phi(a) - \Phi(a-c))(tc)^n, \quad a, c \in \mathbb{R}$$

if and only if

$$f(x) = \begin{cases} \sum_{i=0}^{n+2} a_i x^i, & \text{if } t = \dfrac{n+2}{2n+2}, \\ \sum_{i=0}^{n+1} a_i x^i, & \text{if } t \neq \dfrac{n+2}{2n+2}, \end{cases}, \qquad g_i(x) = \frac{f^{(i)}(x)}{i!}$$

for $i \in \{0, \dots, n\}$, *and*

$$\Phi(x) = c_0 + \frac{t}{(n+1)!} f^{(n)}(x),$$

where a_i, $i \in \{0, \dots, n+2\}$ *and* c_0 *are arbitrary constants.*

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References

[1] J. Aczél, A mean value propery of the derivative of quadratic polynomials – without mean values and derivatives, *Math. Magazine* **58** (1985), 42–45.

[2] T. M. Flett, A mean value theorem, *Math. Gazette* **42** (1958), 38–39.

[3] Sh. Haruki, A property of quadratic polynomials, *Amer. Math. Monthly* **86** (1979), 577–579.

[4] I. Pawlikowska, An extension of a theorem of Flett, *Dem. Math.* **32** (1999), 281–286.

[5] T. Riedel and M. Sablik, Characterizing polynomial functions by a mean value property, *Publ. Math. Debrecen* **52** (1998), 597–610.

[6] T. Riedel and P. K. Sahoo, Mean value theorems and functional equations, *World Scientific*, *Singapore – New Jersey – London – Hong Kong*, 1998.

[7] W. Rudin, Problem E3338, *Amer. Math. Monthly* **96** (1989), 641.

[8] M. Sablik, Taylor's theorem and functional equations, *Aequationes Math.* **60** (2000), 258–267.

[9] L. Székelyhidi, Convolution type functional equations on topological Abelian groups, *World Scientific*, *Singapore – New Jersey – London – Hong Kong*, 1991.

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