Publ. Math. Debrecen
60 / 1-2 (2002), 15–22

An optimal control problem on the Lie group $\mathbf{SE}(2,\mathbb{R})$

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Abstract. An optimal control problem on the Lie group $\mathrm{SE}(2,\mathbb{R})$ is discussed and some of its properties are pointed out.

1. Introduction

Recent work in nonlinear control has drawn attention to drift-free systems with fewer controls than state variables. These arise in problems of motion planning for wheeled robots subject to nonholonomic controls [6], [7], models of kinematic drift effects in space systems subject to appendage vibrations or articulations [3], [4], models of self-propulsion of paramecia at low Reynolds number [12], kinematic model of an automobile [9] and kinematic model of an automobile with $(n-3)$-trailers [11].

The goal of our paper is to discuss a similar problem for the Lie group $\mathrm{SE}(2,\mathbb{R})$ which is in fact the phase space of the laser-matter dynamics and which appears naturally in the study of the 3-dimensional real valued Maxwell–Bloch equations.

1. An optimal problem for the Lie group $\mathrm{SE}(2,\mathbb{R})$

Let $\mathrm{SE}(2,\mathbb{R})$ be the special Euclidean group of the plane, i.e.

$$\mathrm{SE}(2,\mathbb{R}) = \mathrm{SO}(2,\mathbb{R}) \times \mathbb{R}^2,$$

Mathematics Subject Classification: 70Q05.
Key words and phrases: control problem, Lie group, dynamics, integration.

with the group operation given by:

$$(A,a)\cdot(B,b) = (AB, Ab+a),$$

for each $(A,a),(B,b) \in \mathrm{SO}(2,\mathbb{R})\times\mathbb{R}^2$. It is easy to see that via the map ϕ given by

$$\phi : \mathrm{SE}(2,\mathbb{R}) \to \mathrm{GL}(3,\mathbb{R})$$
$$\phi(A,a) = \begin{bmatrix} A & a \\ 0 & 1 \end{bmatrix},$$

it is a closed subgroup of $\mathrm{GL}(3,\mathbb{R})$ and so it is a Lie group. Its Lie algebra $\mathcal{L}\,\mathrm{SE}(2,\mathbb{R})$ can be canonically identified with $\mathrm{se}(2,\mathbb{R})$, where

$$\mathrm{se}(2,\mathbb{R}) = \left\{ \begin{bmatrix} 0 & -a & v_1 \\ a & 0 & v_2 \\ 0 & 0 & 0 \end{bmatrix} \;\middle|\; a\in\mathbb{R},\ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2 \right\}.$$

Let

$$A_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};\quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};\quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

be the canonical basis of $\mathrm{se}(2,\mathbb{R})$ with the bracket operation $[\cdot,\cdot]$ given by:

$[\cdot,\cdot]$	A_1	A_2	A_3
A_1	0	A_3	$-A_2$
A_2	$-A_3$	0	0
A_3	A_2	0	0

and let us consider the following left-invariant controlled system on the matrix Lie group $\mathrm{SE}(2,\mathbb{R})$:

$$\dot{X} = X(A_1u_1 + A_2u_2). \tag{2.1}$$

Then an easy computation leads us to

Theorem 2.1. *The system* (2.1) *is controllable.*

Let J be the cost function given as usual by

$$J(u_1,u_2)=\frac{1}{2}\int_0^{t_f}\left[c_1u_1^2(t)+c_2u_2^2(t)\right]dt,\quad c_1>0,\ c_2>0.$$

Theorem 2.2. *The controls which minimize J and steer the system* (2.1) *from $X=X_0$ at $t=0$ to $X=X_f$ at $t=t_f$ are given by*

$$u_1=\frac{1}{c_1}P_1,\quad u_2=\frac{1}{c_2}P_2, \tag{2.2}$$

where P_i's are solutions of

$$\begin{cases}\dot P_1=-\dfrac{1}{c_2}P_2P_3\\[2mm] \dot P_2=\dfrac{1}{c_1}P_1P_3\\[2mm] \dot P_3=-\dfrac{1}{c_1}P_1P_2.\end{cases} \tag{2.3}$$

Proof. Let us apply Krishnaprasad's theorem [2] to this special case. Then the extremal controls are given by (2.2), where P_i's are solutions of the reduced Hamilton's equations from $T^*\,\mathrm{SE}(2,\mathbb{R})$ to $((\mathrm{se}(2,\mathbb{R}))^*_-\simeq\mathbb{R}^3$. Here $(\mathrm{se}(2,\mathbb{R}))^*_-$ means $(\mathrm{se}(2,\mathbb{R}))^*$ together with the minus-Lie–Poisson structure $\{\,\cdot\,,\,\cdot\,\}_-$, i.e. the Poisson structure generated by the matrix

$$\Pi_-=\begin{bmatrix}0&-P_3&P_2\\P_3&0&0\\-P_2&0&0\end{bmatrix}.$$

Therefore,

$$\begin{bmatrix}\dot P_1\\\dot P_2\\\dot P_3\end{bmatrix}=\begin{bmatrix}0&-P_3&P_2\\P_3&0&0\\-P_2&0&0\end{bmatrix}\cdot\nabla H,$$

where H is given by

$$H(P_1,P_2,P_3)=\frac{1}{2c_1}P_1^2+\frac{1}{2c_2}P_2^2 \tag{2.4}$$

or equivalently,

$$\begin{cases} \dot{P}_1 = -\frac{1}{c_2} P_2 P_3 \\ \dot{P}_2 = \frac{1}{c_1} P_1 P_3 \\ \dot{P}_3 = -\frac{1}{c_1} P_1 P_2, \end{cases}$$

as required. □

Remark 2.1. The same result can be obtained using Lagrangian reduction [5].

Remark 2.2. The function C given by

$$C(P_1, P_2, P_3) = \frac{1}{2}(P_2^2 + P_3^2),$$

is a Casimir of our configuration $((\mathrm{se}(2,\mathbb{R}))^*, \{\cdot\,,\cdot\}_-) \simeq (\mathbb{R}^3, \{\cdot\,,\cdot\}_-)$, i.e.

$$\{C, f\}_- = 0,$$

for each $f \in C^\infty(\mathbb{R}^3, \mathbb{R})$.

Remark 2.3. The integral curves of the system (2.3) are intersections of the cylinders:

$$\frac{P_1^2}{c_1} + \frac{P_2^2}{c_2} = 2H$$

and

$$P_2^2 + P_3^2 = 2C.$$

3. Dynamical and geometrical properties of the equations (2.3)

In this section we want to point out some geometrical and dynamical properties of the equations (2.3).

Theorem 3.1. *The dynamics* (2.3) *is equivalent to the pendulum dynamics.*

Proof. Indeed, C is a constant of motion, so

$$P_2^2 + P_3^2 = 2C = \text{constant}.$$

Let us take now

$$\begin{cases} P_2 = \sqrt{2C}\cos\theta \\ P_3 = \sqrt{2C}\sin\theta. \end{cases}$$

Then

$$\dot{P}_2 = -P_3\dot{\theta},$$

or equivalently,

$$\dot{\theta} = -\frac{\dot{P}_2}{P_3} = -\frac{1}{c_1}\frac{P_1 P_3}{P_3} = -\frac{1}{c_1}P_1.$$

Differentiating again, we get

$$\ddot{\theta} = -\frac{1}{c_1}\dot{P}_1 = -\frac{1}{c_1 c_2}P_2 P_3 = -\frac{C}{c_1 c_2}\sin 2\theta,$$

hence pendulum dynamics. □

Remark 3.1. A similar result is proved in [1] for the free rigid body.

Theorem 3.2. *The system* (2.3) *may be realized as an Hamilton–Poisson system in an infinite number of different ways, i.e. there exists infinitely many different (in general nonisomorphic) Poisson structures on* $\mathbb{R}^3$ *such that the system* (2.3) *is induced by an appropriate Hamiltonian.*

Proof. An easy computation shows us that the triples:

$$(\mathbb{R}^3, \{\,\cdot\,,\,\cdot\,\}_{ab}, H_{cd}),$$

where

$$\{f,g\}_{ab} = -\nabla C_{ab}\cdot(\nabla f \times \nabla g), \quad (\forall) f,g \in C^\infty(R^3,R);$$

$$C_{a,b} = aC + bH;$$

$$H_{cd} = cC + dH;$$

$$C = \frac{1}{2}(P_2^2 + P_3^2);$$

$$H = \frac{1}{2c_1}P_1^2 + \frac{1}{2c_2}P_2^2;$$

$$a,b,c,d \in \mathbb{R}, \quad ad - bc = 1$$

are Hamilton–Poisson realizations of the system (2.3). □

An easy computation leads us to:

Theorem 3.3. *The equations* (2.3) *may be explicitely integrated by elliptic functions.*

It is easy to see that the equilibrium states of our system (2.2) are:

$$e_1 = (M, 0, 0);\ e_2 = (0, M, 0);\ e_3 = (0, 0, M), \quad M \in \mathbb{R}.$$

Now we shall discuss their nonlinear stability. Recall that an equilibrium state P_e is nonlinear stable if trajectories starting close to P_e stay close to P_e, or in other words, a neighborhood of P_e must be flow invariant. We have the following result:

Theorem 3.4. *The equilibrium states e_1, e_2, e_3 have the following behaviour:*

(i) *The equilibrium states* $(M, 0, 0)$, $M \in \mathbb{R}$, $M \neq 0$ *are nonlinear stable.*

(ii) *The equilibrium states* $(0, M, 0)$, $M \in \mathbb{R}$, $M \neq 0$ *are unstable.*

(iii) *The equilibrium states* $(0, 0, M)$, $M \in \mathbb{R}$, $M \neq 0$ *are nonlinear stable.*

(iv) *The equilibrium state* $(0, 0, 0)$ *is nonlinear stable.*

4. Numerical integration of the equations (2.3)

In this section we shall discuss the numerical integration of the system (2.3) via the Lie–Trotter integrator and we shall point out some of its properties.

To begin with, let us observe that the Hamiltonian vector field X_H splits as follows:

$$X_H = X_{H_1} + X_{H_2},$$

where

$$H_1(P_1, P_2, P_3) = \frac{1}{2c_1} P_1^2$$

and

$$H_2(P_1, P_2, P_3) = \frac{1}{2c_2} P_2^2.$$

The integral curves of X_{H_1} and X_{H_2} are given by:

$$P(t) = \exp(tX_{H_1}) \cdot P(0) = \phi_1(t, P(0))$$

and respectively,

$$P(t) = \exp(tX_{H_1}) \cdot P(0) = \phi_2(t, P(0)).$$

Now following [8], [10], [13] the Lie–Trotter formula gives rise to an explicit integrator of the equations (2.3), namely:

$$\tag{4.1} \begin{cases} P_1^{n+1} = P_1^n - \frac{P_2(0)}{c_2} P_3^n t \\ P_2^{n+1} = P_2^n \cos \frac{P_1(0)}{c_1} t + P_3^n \sin \frac{P_1(0)}{c_1} t \\ P_3^{n+1} = -P_2^n \sin \frac{P_1(0)}{c_1} t + P_3^n \cos \frac{P_1(0)}{c_1} t. \end{cases}$$

Some of its properties are sketched in the following theorem:

Theorem 4.1. *The numerical integrator* (4.1) *has the following properties:*

(i) *It preserves the Poisson structure* $\{\,\cdot\,,\,\cdot\,\}_-$.

(ii) *Its restriction to the coadjoint orbits* $(\mathcal{O}_k, \omega_k)$, *where*

$$\mathcal{O}_k = \{(P_1, P_2, P_3) \in \mathbb{R}^3 \mid P_2^2 + P_3^2 = k^2\}$$

and

$$\omega_k = \frac{1}{k}(P_3 dP_1 \wedge dP_2 - P_2 dP_1 \wedge dP_3),$$

gives rise to a symplectic integrator.

(iii) *It does not preserve the Hamiltonian* (2.4).

Proof. The items (i) and (ii) hold because ϕ_1 and ϕ_2 are flows of some Hamiltonian vector fields, hence they are Poisson.

The item (iii) is a consequence of the fact that:

$$\frac{1}{2c_1}(P_1^{n+1})^2 + \frac{1}{2c_2}(P_2^{n+1})^2 \neq \frac{1}{2c_1}(P_1^n)^2 + \frac{1}{2c_2}(P_2^n)^2. \qquad \square$$

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(Received July 26, 2000; revised May 2, 2001)