Generalized norms and convexity

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Abstract. Given an ε-convex subset V of an n-dimensional normed space, that is such that

$$\alpha x + (1 - \alpha)y \in V + B(0, \varepsilon)$$

for all $x, y \in V$, $\alpha \in [0, 1]$, we prove that

$$\operatorname{conv}(V) \subset V + K_{n+1}B(0, \varepsilon).$$

Using the notion of generalized norm we obtain as a corollary an improvement of the Theorem of Hyers–Ulam on the stability of convex functions.

1. Generalized norms

In this section we investigate the notion of a generalized norm. To our opinion its main advantage is that it helps us better understand the properties of the epigraph of a given real-valued function. This enables us in consequence to translate the stability of convex sets to the stability of convex functions. This translation results from the fact that in certain extended norm the epigraph of a function is an ε-convex set iff the function is ε-convex (in usual norms this does not hold, which has been shown in Example 1). The obtained results are similar to [2] and [5], but we obtain a better constant of approximation.

The notion of generalized metric was introduced in [3]. In the following definition we simply adapt this notion for norms.

Definition 1. Let E be a vector space. We say that $\| \ \| : E \to \mathbb{R}_+ \cup \{\infty\}$ is a *generalized norm* (g-norm for short) iff

- $\|x\| = 0$ iff $x = 0$,
- $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in E$,
- $\|\alpha x\| = |\alpha| \|x\|$ for $\alpha \in \mathbb{R}$, $x \in E$, where $0 \cdot \infty$ denotes 0.

If E is a g-normed space by $B(0, r)$ we denote the closed ball at the centre at zero and radius r. It should be noted that although a g-normed space is not a topological vector space (as the multiplication by scalars is not continuous in general), it is a Hausdorff topological group with respect to addition. Defining

$$\|x\|_d := \begin{cases} 0 & \text{if } x = 0, \\ \infty & \text{otherwise} \end{cases}$$

we obtain a g-norm which generates the discrete topology on an arbitrary vector space. Let $(E, \| \ \|_E)$, $(F, \| \ \|_F)$ be two g-normed spaces. We say that a g-norm $\| \ \|$ in $E \times F$ is a product g-norm of the g-norms from E and F iff

$$\|(e, 0)\| = \|e\|_E, \quad \|(0, f)\| = \|f\|_F \quad \text{for } (e, f) \in E \times F.$$

Let E be a vector space. We define the following g-norm in $E \times \mathbb{R}$

$$\|(x, r)\|_\infty := \begin{cases} |r| & \text{if } x = 0, \\ +\infty & \text{if } x \neq 0 \end{cases}$$

for $(x, r) \in E \times \mathbb{R}$. One can easily notice that $\| \ \|_\infty$ is a product norm of $\| \ \|_d$ (discrete g-norm) and the common norm on $\mathbb{R}$.

For a given function $f : D \to \mathbb{R}$ we will consider its *epigraph*, that is the set

$$\text{epi}(f) := \{(x, y) \in D \times \mathbb{R} \mid y \geq f(x)\}.$$

Theorem 1. *Let E be a vector space, let $D \subset E$ and let $f, g : D \to \mathbb{R}$. Then*

$$d_H(\text{epi}(f), \text{epi}(g)) = \|f - g\|_{\sup},$$

where the Hausdorff distance d_H is generated by the g-norm $\| \ \|_\infty$.

PROOF. The proof follows directly from the definition of $\| \ \|_\infty$ and the Hausdorff distance. $\qquad \square$

Definition 2. Let $\varepsilon \geq 0$ and let E be a g-normed space. We say that a set $V \subset E$ is ε-*convex*, if

$$\alpha x + (1 - \alpha)y \in V + B(0, \varepsilon)$$

for all $x, y \in V$, $\alpha \in [0, 1]$.

Notice that a given set is convex iff it is 0-convex. Let E be a vector space. We say that the function $f : D \to \mathbb{R}$, where $D \subset E$ is a convex set, is ε-*convex* (cf. [5], p. 430) iff

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) + \varepsilon$$

for every $x, y \in D$ and $\alpha \in [0, 1]$.

The following proposition shows the connection between the ε-convexity of the function and the ε-convexity of its epigraph.

Proposition 1. *Let E be a g-normed space and let $D \subset E$ be a convex set. In $E \times \mathbb{R}$ we take any product g-norm $\| \ \|$. If $f : D \to \mathbb{R}$ is an ε-convex function then* epi(f) *is an ε-convex set in $(E \times \mathbb{R}, \| \ \|)$.*

PROOF. Let $f : D \to \mathbb{R}$ be an ε-convex function. We show that epi(f) is an ε-convex set. Choose arbitrary $(x_0, r_0), (x_1, r_1) \in$ epi(f) and $\alpha \in [0, 1]$. Then $r_0 \geq f(x_0)$ and $r_1 \geq f(x_1)$. As f is ε-convex we have $\alpha r_0 + (1 - \alpha)r_1 + \varepsilon \geq f(\alpha x_0 + (1 - \alpha)x_1)$. In other words

$$\alpha r_0 + (1 - \alpha)r_1 + \varepsilon \in [f(\alpha x_0 + (1 - \alpha)x_1), \infty).$$

This means that

$$\alpha(x_0, r_0) + (1 - \alpha)(x_1, r_1) \in \text{epi}(f) + B(0, \varepsilon). \qquad \square$$

One may ask if an opposite implication is also true, that is if epi(f) is an ε-convex set, is f an ε-convex function then? As the following example shows, in general the answer is negative. The reason is that ε-convexity of a function does not depend on the g-norm in the domain space.

Example 1. Let E be a normed space. In $E \times \mathbb{R}$ we introduce any product norm. Let $D = \{x \in E \mid \|x\| < 1\}$ and let $f : D \to \mathbb{R}$ be given by the formula

$$f(x) := -\frac{1}{1 - \|x\|}.$$

Then, one can easily notice that epi(f) is a 1-convex set in $E \times \mathbb{R}$. However, f is not ε-convex function with any $\varepsilon \in [0, \infty)$.

However, as the following proposition shows, if we take in $E \times \mathbb{R}$ the $\| \ \|_\infty$ as g-norm, the opposite implication holds.

Proposition 2. *Let E be a vector space and let $D \subset E$ be a convex set. A function $f : D \to \mathbb{R}$ is an ε-convex function iff $\mathrm{epi}(f)$ is an ε-convex set in $(E \times \mathbb{R}, \| \ \|_\infty)$.*

PROOF. Let us assume that $\mathrm{epi}(f)$ is an ε-convex set in $(E \times \mathbb{R}, \| \ \|_\infty)$. Let $x_0, x_1 \in D$ and $\alpha \in [0, 1]$ be arbitrary. As $\mathrm{epi}(f)$ is ε-convex there exist $(x, r) \in \mathrm{epi}(f)$ such that

$$(x, r) \in \alpha(x_0, f(x_0)) + (1 - \alpha)(x_1, f(x_1)) + B(0, \varepsilon).$$

According to the definition of the g-norm $\| \ \|_\infty$ this implies that x has to be equal to $\alpha x_0 + (1 - \alpha)x_1$ and that

$$|r - (\alpha f(x_0) + (1 - \alpha)f(x_1))| \leq \varepsilon.$$

As $(\alpha x_0 + (1 - \alpha)x_1, r) \in \mathrm{epi}(f)$, $r \geq f(\alpha x_0 + (1 - \alpha)x_1)$. Joining this with the previous inequality we obtain

$$\alpha f(x_0) + (1 - \alpha)f(x_1) + \varepsilon \geq f(\alpha x_0 + (1 - \alpha)x_1). \qquad \square$$

2. Stability of Convex sets

The following proposition plays an essential role in the proof of our main theorem in this section. We will need a technical Lemma which can be easily proved by elementary induction. Its main use is that it will provide us with a better constant of approximation than the constant of approximation of ε-convex functions.

For $x \in \mathbb{R}$ by $[x]$ we denote the integer part of x. $\mathrm{conv}(V)$ means the convex hull of V.

At first, for the reader's convenience, we quote the Carathéodory theorem (cf. [6], Theorem 17.1 or [5], Lemma 17.4.3):

Carathéodory Theorem. *Let E be an n-dimensional vector space and let $V \subset E$. Then, for every $x \in \mathrm{conv}(V)$ there exist $\alpha_i \in [0, 1]$, $\sum_{i=1}^{n+1} \alpha_i = 1$ such that*

$$x \in \alpha_1 V + \cdots + \alpha_{n+1} V.$$

The proof of the following technical lemma is left for the reader.

Lemma 1. *Let the sequence* $(K_n) \in \mathbb{R}_+^{\mathbb{N}}$ *be defined by the formula*

$$K_n := l - \frac{k}{n}$$

where k, l *are the unique non-negative integers such that* $k < 2^{l-1}$ *and* $n = 2^l - k$. *Then,* K_n *satisfies the following difference equation*

$$K_1 = 0, \quad K_n = \frac{\left[\frac{n}{2}\right]}{n} K_{\left[\frac{n}{2}\right]} + \frac{n - \left[\frac{n}{2}\right]}{n} K_{n-\left[\frac{n}{2}\right]} + 1$$

for $n > 1$.

Proposition 3. *Let* E *be a g-normed space and let* $V \subset E$ *be an* ε-convex set. *Let* $n \in \mathbb{N}$. *Then, for all* $\alpha_1, \ldots, \alpha_n \in [0,1]$ *such that* $\sum_{i=1}^n \alpha_i = 1$

(1)
$$\sum_{i=1}^n \alpha_i V \subset V + K_n B(0, \varepsilon),$$

where K_n *is defined in Lemma 1.*

PROOF. The proof goes by induction over n. For $n = 1$ it is trivial. Suppose that (1) holds for $1 \leq m < n$. Let $\alpha_1, \ldots, \alpha_n \in [0,1]$ be arbitrary such that $\sum_{i=1}^n \alpha_i = 1$. Without any loss of generality we may assume that $\alpha_i \neq 0$ for $i = 1, \ldots, n$. We order α_i so that

(2)
$$\alpha_1 \geq \cdots \geq \alpha_n.$$

One can easily notice that

(3)
$$\sum_{i=1}^{\left[\frac{n}{2}\right]} \alpha_i \geq \frac{\left[\frac{n}{2}\right]}{n}.$$

By the inductive hypothesis we get

$$\sum_{i=1}^{\left[\frac{n}{2}\right]} (\alpha_i V) \subset \left(\sum_{i=1}^{\left[\frac{n}{2}\right]} \alpha_i \right) V + \sum_{i=1}^{\left[\frac{n}{2}\right]} \alpha_i K_{\left[\frac{n}{2}\right]} B(0, \varepsilon),$$

$$\sum_{i=\left[\frac{n}{2}\right]+1}^{n} (\alpha_i V) \subset \left(\sum_{i=\left[\frac{n}{2}\right]+1}^{n} \alpha_i \right) V + \sum_{i=\left[\frac{n}{2}\right]+1}^{n} \alpha_i K_{n-\left[\frac{n}{2}\right]} B(0, \varepsilon),$$

$$\left(\sum_{i=1}^{[\frac{n}{2}]} \alpha_i\right) V + \left(\sum_{i=[\frac{n}{2}]+1}^{n} \alpha_i\right) V \subset V + B(0, \varepsilon).$$

Summing these inequalities up and applying (3) along with the fact that K_n is an increasing sequence we obtain

$$\sum_{i=1}^{n} (\alpha_i V) = \sum_{i=1}^{[\frac{n}{2}]} (\alpha_i V) + \sum_{i=[\frac{n}{2}]+1}^{n} (\alpha_i V)$$

$$\subset V + \left(\frac{[\frac{n}{2}]}{n} K_{[\frac{n}{2}]} + \frac{n - [\frac{n}{2}]}{n} K_{n-[\frac{n}{2}]} + 1\right) B(0, \varepsilon)$$

$$= V + K_n B(0, \varepsilon). \qquad \square$$

Now we prove the main theorem in our paper, namely the theorem on the stability of the convex sets in finite dimensional spaces.

Theorem 2. *Let E be an n-dimensional g-normed space, and let $V \subset E$ be an ε-convex set. Then*

$$\operatorname{conv}(V) \subset V + K_{n+1} B(0, \varepsilon),$$

where K_n is defined by Lemma 1.

PROOF. Let $x \in \operatorname{conv}(V)$ be arbitrary. By the Carathéodory Theorem there exist $\alpha_i \in [0,1]$, $\sum_{i=1}^{n+1} \alpha_i = 1$ such that

$$x \in \sum_{i=1}^{n+1} \alpha_i V,$$

and therefore, by Proposition 3, we get

$$x \in \sum_{i=1}^{n+1} \alpha_i V \subset V + K_{n+1} B(0, \varepsilon). \qquad \square$$

Corollary 1. *Let E be an n-dimensional g-normed space, and let $V \subset E$ be an ε-convex set. Then*

$$d_H(\operatorname{conv}(V); V) \le K_{n+1} \varepsilon,$$

where K_n is defined by Lemma 1.

3. Approximately convex functions

The approach to the stability of convex functions by the stability of convex sets will enable us to improve, in a very simple and geometrical way, the well-known result on the stability of the convex functions obtained by D. H. HYERS and S. M. ULAM in [4] and further improved by P. W. CHOLEWA in [2]:

Theorem (cf. [2], Theorem 2). *Let $D \subset \mathbb{R}^n$ be a convex and open set and let $f : D \to \mathbb{R}$ be an ε-convex function. Then, there exists a continuous convex function $g : D \to \mathbb{R}$ such that*

$$g(x) \leq f(x) \leq g(x) + J_{n+1}\varepsilon$$

for all $x \in D$, where $J_{n+1} = \min\{k_n, l_n\}$, $k_n = \frac{n^2+3n}{2n+2}$ for all $n \in \mathbb{N}$, $l_n = m$ for $2^{m-1} \leq n < 2^m$.

The following Proposition is an analogue of Proposition 3 for approximately convex functions.

Proposition 4. *Let E be a vector space and $D \subset E$ be convex. If $f : D \to \mathbb{R}$ be an ε-convex function, then for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in D$, $t_1, \ldots, t_n \in [0, 1]$, $t_1 + \cdots + t_n = 1$ we have*

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} t_i f(x_i) + K_n \varepsilon.$$

PROOF. One can easily notice that

$$\left(\sum_{i=1}^{n} t_i x_i, \sum_{i=1}^{n} t_i f(x_i)\right) \in \sum_{i=1}^{n} \left(t_i \operatorname{epi}(f)\right).$$

By Propositions 2 and 3 we obtain

$$\left(\sum_{i=1}^{n} t_i x_i, \sum_{i=1}^{n} t_i f(x_i)\right) \in \operatorname{epi}(f) + K_n B(0, \varepsilon).$$

By the definition of $\| \ \|_\infty$ the above implies the assertion of the Proposition 4. $\qquad\square$

Now, we will be able to prove the main Theorem of this section, which improves the previous constant of approximation of approximately convex functions.

Theorem 3. *Let $D \subset \mathbb{R}^n$ be a convex set, and let $f : D \to \mathbb{R}$ be an ε-convex function. Then, there exists a convex function $g : D \to \mathbb{R}$ such that*

$$g(x) \le f(x) \le g(x) + K_{n+1}\varepsilon$$

for all $x \in D$.

PROOF. The proof of the existence of the convex function g follows from Proposition 4 and the proof of Theorem 2 from [2] (instead of Lemma 2 from [2] we apply Proposition 4). $\square$

In the following part of this section we will prove that the constant of approximation in Theorem 3 is nearly sharp. In particular, we will show that there is no stability if E is infinite dimensional (in some special case it was proved by E. CASINI and P. L. PAPINI in [1]). We would like to mention that in Example 3 we have been inspired by an example for approximately Jensen convex functions constructed by P. CHOLEWA in [2].

Example 2. Let $f : [0, 2] \to \mathbb{R}$ be defined by

$$f(x) = 1 - |1 - x|.$$

One can easily check that f is a 1-convex function. However, clearly for every convex function $g : [0, 2] \to \mathbb{R}$ such that $g \le f$

$$\sup_{x \in [0,2]} \{f(x) - g(x)\} \ge 1.$$

This implies that the constant $K_2 = 1$, which we obtain in Theorem 3, is sharp (however, it is equal to the constant obtained by P. CHOLEWA in this case).

Example 3. Let E be a real vector space of dimension $k \in \mathbb{N} \cup \{\infty\}$, and let H be a base of E. Let

$$D := \{x \in E \mid x = t_1 h_1 + \cdots + t_n h_n,$$
$$n \ge 1, \; h_1, \ldots, h_n \in H, \; t_1, \ldots, t_n \in (0, \infty)\}.$$

D is clearly a convex subset of E. Let $x \in D$. There exist a unique $n \in \mathbb{N}$ and distinct points $h_1, \ldots, h_n \in H$ and $t_1, \ldots, t_n \in (0, \infty)$ such that

$$x = t_1 h_1 + \cdots + t_n h_n.$$

We put
$$m(x) := \max\{t_i : i = 1, \ldots, n\}.$$

The definition of D and m imply the inequalities

(4)
$$0 < m(x) \le m(x+y) \quad \text{for } x, y \in D$$

and

(5)
$$m(\alpha x) = \alpha m(x) \quad \text{for } x \in D, \ \alpha \in (0, \infty).$$

We define
$$f(x) := -\log_2(m(x)) \quad \text{for } x \in D.$$

From (4) and (5) one easily gets
$$f(x+y) \le f(x) \quad \text{for } x, y \in D$$

and
$$f(\alpha x) = f(x) - \log_2(\alpha) \quad \text{for } x \in D, \ \alpha \in (0, \infty).$$

Now, we obtain
$$f(\alpha x + (1-\alpha)y) \le f(\alpha x) = f(x) - \log_2(\alpha),$$
$$f(\alpha x + (1-\alpha)y) \le f((1-\alpha)y) = f(y) - \log_2(1-\alpha),$$

for $x, y \in D$, $\alpha \in (0,1)$. Multiplying the first of the above inequalities by α and the second by $(1-\alpha)$ and summing them up we obtain

$$f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha)f(y) - \log_2(\alpha^\alpha(1-\alpha)^{1-\alpha})$$

for $x, y \in D$, $\alpha \in (0,1)$. One can easily verify that $-\log_2(\alpha^\alpha(1-\alpha)^{1-\alpha}) \le 1$. Thus, f is a 1-convex function.

We define function $g : D \to \mathbb{R} \cup \{-\infty\}$ by the formula

$$g(x) := \inf\left\{ \sum_{i=1}^m \alpha_i f(x_i) \;\Big|\; \alpha_i \ge 0, \ \sum_{i=1}^m \alpha_i = 1, \right.$$
$$\left. \sum_{i=1}^m \alpha_i x_i = x, \ x_i \in D, \ m \in \mathbb{N} \right\}.$$

One can easily observe that g is the maximum of all convex functions $\psi : D \to \mathbb{R} \cup \{-\infty\}$ such that $\psi \leq f$.

Now, let $n \in \mathbb{N}$, $n \leq \dim(E)$ be fixed. Then, there exist n distinct points $h_1, \ldots, h_n \in H$. Let $x = \frac{1}{n}h_1 + \cdots + \frac{1}{n}h_n$. Then

$$f(x) - g(x) \geq f\left(\frac{1}{n}h_1 + \cdots + \frac{1}{n}h_n\right) - \sum_{i=1}^{n} \frac{1}{n}f(h_i)$$

$$= \log_2(n) - \sum_{i=1}^{n} \frac{1}{n} \cdot 0 = \log_2(n).$$

Comparing our constant K_{n+1} with J_{n+1} obtained by P. Cholewa one can easily notice that $K_{n+1} \leq J_{n+1}$.

Moreover, if we compare them with $\log_2(n)$ in infinity, we obtain

$$\limsup_{n \to \infty}(K_{n+1} - \log_2(n)) = 1 - \log_2(e \cdot \ln(2)) \simeq 0.086$$

$$\limsup_{n \to \infty}(J_{n+1} - \log_2(n)) = 1.$$

Thus, we see that the constant we obtain is much closer to the sharpness then the one obtained by P. Cholewa. Moreover, it is smaller and defined by one formula instead of two different ones as J_{n+1}.

There arises a natural question if the constant K_{n+1} in Theorem 3 is sharp.

References

[1] E. Casini and P. Papini, A counterexample to the infinity version of Hyers and Ulam stability theorem, *Proc. Amer. Math. Soc.* **118** no. 3 (July 1993), 885–890.

[2] P. W. Cholewa, Remarks on the stability of functional equations, *Aequationes Math.* **27** (1984), 76–86.

[3] H. Covitz and S. D. Nadler, Multivalued contractious mappings in generalized metric spaces, *Israel J. Math.* **8** (1970), 5–11.

[4] D. H. Hyers and S. M. Ulam, Approximately convex functions, *Proc. Amer. Math. Soc.* **3** (1952), 821–828.

[5] M. Kuczma, An Introduction to the Theory of Functional Equations and Inequalities, *PWN, Warszawa–Kraków–Katowice, Silesian University*, 1985.

[6] R. T. ROCKAFELLAR, Convex Analysis, *Princeton University Press*, *Princeton, New Jersey*, 1970.

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