Publ. Math. Debrecen
42 / 3–4 (1993), 247–251

On a basic property of Lagrangians

By J. SZENTHE (Budapest)

Dedicated to Professor Lajos Tamássy on his 70th birthday

Let M be a C^∞ manifold and $\mathcal{O}_{TM}$ the zero section of its tangent bundle. A function $L : TM \to \mathbb{R}$ which is continuous on TM and C^∞ on $\breve{T}M = TM - \mathcal{O}_{TM}$ is said to be *Lagrangian.* The Euler–Lagrange equation for L can be given as follows

$$\iota_X dd_v L = d(L - AL)$$

where $d_v : \wedge(\breve{T}M) \to \wedge(\breve{T}M)$ the *vertical differential* is an antiderivation of degree 1 and $A : TM \to TTM$ is the *Liouville field* of the tangent bundle ([Go] pp. 159–164, 169–175). If a vector field $X : \breve{T}M \to TTM$ satisfies the above equation then it is called a *Lagrangian field* associated with the Lagrangian function L. The Lagrangian function L is said to be *regular* if 2-form $dd_v L$ is nondegenerate. In case of a nondegenerate Lagrangian L the above equation has obviously a unique solution X which is a second-order differential equation ([Go] pp. 170–171). The integral curves of X when projected to M yield those curves which are stationary for L.

If a second-order equation $X : \breve{T}M \to TTM$ is given then the set $\mathbf{L}(X)$ of those Lagrangian functions L with which X is the associated as a Lagrangian field has a canonical $\mathbb{R}$ vector space structure. The study of $\mathbf{L}(X)$ includes also the inverse problem which concerns those conditions on X under which $\mathbf{L}(X)$ is not empty. In the study of $L(X)$ recently differential geometric methods have been successfully applied. Namely, a second-order differential equation X induces the field of endomorphisms $\mathcal{L}_X v$ of the tangent spaces $T_v TM$, $v \in \breve{T}M$ where $v : TTM \to \mathcal{V}TM$ is

This work was supported by Hungarian Nat. Found. for Sci. Research Grant No. I/3 2505.

the vertical endomorphism of the second tangent bundle ([Go] pp. 159–161). Since the field of endomorphisms $\gamma = -\mathcal{L}_X v$ is involutive and its -1 eigenspaces are exactly the vertical subspaces of TTM, a connection is defined by γ on TM which, in general, is not homogeneous, and, in turn, yields a horizontal distribution $\mathcal{H}$ on $\breve{T}M$ [K]. The corresponding field of horizontal projections

$$\chi_v : T_v TM \to \mathcal{H}_v,$$

as a field of endomorphisms of the tangent spaces $v \in \breve{T}M$ defines the derivation

$$\iota_\chi : \wedge(\breve{T}M) \to \wedge(\breve{T}M)$$

of degree 0 and a differential $d_\chi = \iota_\chi d - d\iota_\chi$ which is an antiderivation of degree 1. In that particular case, when X is a *spray*, i.e., a second-order differential equation which is homogeneous of degree 2, the elements of $\mathcal{L}(X)$ are exactly those Lagrangian functions L for which $d_\chi L = 0$ holds according to a result of J. Klein [K]. It is shown below that the elements of $\mathbf{L}(X)$ are characterized by the above property in case of any second-order differential equation X too. The result is based on an extension of the identity

$$\mathcal{L}_X = \iota_X d + d\iota_X$$

to the case when instead of d the vertical differential d_v appears. The extension of the above identity will be presented first.

In fact, if $X \in \mathcal{T}(\breve{T}M)$ then ι_X is an antiderivation of degree -1 of $\wedge(\breve{T}M)$; on the other hand, d_v is an antiderivation of degree 1 of $\wedge(\breve{T}M)$ ([Go] pp. 87–89, 161–164). Consequently,

$$\iota_X d_v + d_v \iota_X$$

is a derivation of degree 0 by a fundamental result concerning composition of antiderivations ([Go] pp. 88–89). Moreover, $\mathcal{L}_{vX}$ is a derivation of degree 0 too. Furthermore, the field of endomorphisms $\gamma = -\mathcal{L}_X v$ induces an endomorphism γ^* of the algebra $\wedge(\breve{T}M)$ which in turn defines a derivation ι_γ of degree 0 of $\wedge(\breve{T}M)$ in the canonical way; similarly, as the vertical endomorphism v defines the vertical derivation ι_v ([Go] pp. 161–162). The extension of the above mentioned identity is given now by the following Proposition.

Proposition. *Let be a C^∞ manifold and $X \in \mathcal{T}(\breve{T}M)$ an arbitrary vector field then the following holds*

$$\iota_X d_v + d_v \iota_X = \mathcal{L}_{vX} + \iota_\gamma$$

where ι_γ is the derivation of degree 0 of $\wedge(\breve{T}M)$ defined by the homomorphism γ^* where $\gamma = -\mathcal{L}_X v$.

Proof. Since two derivations of $\wedge(\breve{T}M)$ are equal if they coincide on those elements $\xi \in \wedge(\breve{T}M)$ which are obtainable as $\xi = f$, *df* where

$f \in \mathcal{F}(\breve{T}M)$ ([Go] pp. 87–89), the validity of the identity has to be verified only in these two cases.

1st case: $\xi = f$. Obvious calculations yield the following

$$\iota_X d_v f = \iota_X(\iota_v d - d\iota_v) f = \iota_X \iota_v df = (vX)f,$$
$$d_v \iota_X f = 0,$$
$$\mathcal{L}_{vX} f = (vX)f,$$
$$\iota_\gamma f = 0.$$

Now the above equalities obviously yield the validity of the identity in the case $\xi = f$.

2nd case: $\xi = df$. Let now $V \in \mathcal{T}(\breve{T}M)$ be an arbitrary vector field, then the following equalities hold:

$$\iota_X d_v df(V) = \iota_X(\iota_v d - d\iota_v) df(V) = -\iota_X d\iota_v df(V) =$$
$$= -d(\iota_v df)(X, V) = -(X(vV)f - V(vX)f - v[X, V]f),$$
$$d_v \iota_X df(V) = (\iota_v d - d\iota_v)\iota_X df(V) =$$
$$= \iota_v d(Xf)(V) - d\iota_v(Xf)(V) = (vV)(Xf),$$
$$\mathcal{L}_{vX} df(V) = (vX)(Vf) - [vX, V]f,$$
$$\iota_\gamma df(V) = -df(\mathcal{L}_X v(V)) = -df([X, vV] - v[X, V]) =$$
$$= v[X, V]f - [X, vV]f.$$

But now the validity of the identity in the case $\xi = df$ follows directly from the above equalities.

The following lemma yields a simple observation which is essential for the main result.

Lemma. *Let $X : \breve{T}M \to TTM$ be a second-order differential equation, χ the horizontal projection field of the connection which is defined by the endomorphism field $\gamma = -\mathcal{L}_X v$, and $d_\chi : \wedge(\breve{T}M) \to \wedge(\breve{T}M)$ the differentiel defined by χ. Then $d_\chi L = 0$ for any Lagrangian function L if and only if*

$$dL(V) = dL(v[X, V]) - dL([X, vV])$$

holds in case of any vector field $V \in \mathcal{T}(\breve{T}M)$.

Proof. Let I be the field of identity endomorphisms of the tangent spaces of $\breve{T}M$ and χ, ϕ the fields of horizontal and vertical projections of these spaces corresponding to the connection which is defined by $\gamma = -\mathcal{L}_X v$. Then $I = \chi + \phi$, $\gamma = \chi - \phi$ hold and consequently $\chi = \frac{1}{2}(I + \gamma) = \frac{1}{2}(I - \mathcal{L}_X v)$ is valid. If L is a Lagrangian function then $d_\chi L = (\iota_\chi d - d\iota_\chi L =$

$\iota_\chi dL$ since $\iota_\chi L = 0$. Let now $V \in \mathcal{T}(\breve{T}M)$ be an arbitrary vector field then

$$d_\chi L(V) = \iota_\chi dL(V) = dL(\chi V) = \frac{1}{2} dL(V - \mathcal{L}_X v(V)) = \frac{1}{2}\{dL(V) - dL([X,V] - v[X,V])\}$$

holds. But now the assertion of the lemma directly follows by the preceding equalities.

It has been observed by J. Szilasi that a proof of the above Lemma can be obtained also by means of the Frölicher–Nijenhuis theory of vector–valued differential forms [F–N].

Theorem. *Let $X : \breve{T}M \to TTM$ be a second-order differential equation and $L : TM \to \mathbb{R}$ a Lagrangian function. Then X is associated with L as a Lagrangian vector field if and only if $d_\chi L = 0$ holds where χ is the horizontal projection field of the connection defined by $\gamma = -\mathcal{L}_X v$.*

Proof. Let $L : \breve{T}M \to \mathbb{R}$ be an arbitrary Lagrangian function then by the preceding Proposition the following holds

$$(d_v \iota_X + \iota_X d_v) L = \mathcal{L}_{vX} L + \iota_\gamma L,$$
$$\iota_X d_v L = AL,$$

since $vX = A$ and $\iota_X L = 0$, $\iota_\gamma L = 0$. Then by differentiation of the last equality the following ones are obtained

$$d\iota_X d_v L = dAL,$$
$$(\mathcal{L}_X - \iota_X d) d_v L = dAL,$$
$$\iota_X dd_v L = \mathcal{L}_X d_v L - dAL.$$

Therefore the Euler-Lagrange equation for L can be successively rewritten in the following equivalent forms

$$\iota_X dd_v L + d(AL - L) = 0$$
$$\mathcal{L}_X d_v L - dL = 0,$$
$$\mathcal{L}_X(\iota_v d - d\iota_v - dL) = 0,$$
$$\mathcal{L}_X \iota_v dL - dL = 0.$$

Consequently, the Euler-Lagrange equation is satisfied by X and L if and only if for arbitrary vector field $V \in \mathcal{T}(\breve{T}M)$ the following holds

$$(\mathcal{L}_X \iota_v dL - dL)(V) = XdL(vV) - dL(v[X,V]) - dL(V) = 0.$$

But then the preceeding Lemma applies and yields that L satisfies the Euler-Lagrange equation for a given X if and only if $d_\chi L = 0$ holds.

References

[F–N] A. Frölicher and A. Nijenhuis, Theory of vector valued differential forms I, *Indag. Math.* **18** (1956), 338–359; II, ibid **20** (1958), 414–429.

[Go] C. Godbillon, Géométrie Différentielle et Mécanique Analytique, *Paris*, 1969.

[K] J. Klein, Geometry of sprays. Lagrangian case. Principle of least curvature, *Proc. IUTAM–ISIMM Symposium on Modern Development in Analytical Mechanics*, *Turin*, 1982.

J. SZENTHE
EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF GEOMETRY
RÁKÓCZI U. 5
1088. BUDAPEST, HUNGARY

(Received November 16, 1992)