Publ. Math. Debrecen
60 / 1-2 (2002), 75–87

Generalized circles in Weyl spaces and their conformal mapping

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Abstract. In this paper, generalized circles in a Weyl space are defined and the system of differential equations satisfied by them is obtained. Furthermore, after having given the necessary and sufficient conditions for generalized circles to be preserved under a conformal mapping we obtained a covariant tensor of type $(0,4)$ which is invariant under such a conformal mapping.

1. Introduction

A differentiable manifold of dimension n having a conformal metric tensor g and a symmetric connection ∇ satisfying the compatibility condition

$$\nabla g = 2(T \otimes g) \tag{1.1}$$

where T is a 1-form (complementary covector field) is called a Weyl space which we denote it by $W_n(g,T)$. After the renormalisation

$$\bar{g} = \lambda^2 g \tag{1.2}$$

of the metric tensor g, T is transformed by the law

$$\bar{T} = T + d\ln\lambda. \tag{1.3}$$

An object A defined on $W_n(g,T)$ is called a satellite of g of weight $\{p\}$ if it admits a transformation of the form $\bar{A} = \lambda^p A$ under the renormalization

Mathematics Subject Classification: Primary 53A40; Secondary 53A30.
Key words and phrases: Weyl space, generalized circle, conformal mapping, generalized concircular mapping.

(1.2) of g ([1]–[3]). The prolonged derivative and the prolonged covariant derivative in the direction of the vector X of the satellite A of weight $\{p\}$ are, respectively defined by

$$\dot{\partial}_X A = \partial_X A - pT(X)A \tag{1.4}$$

and

$$\dot{\nabla}_X A = \nabla_X A - pT(X)A \tag{1.5}$$

where $\partial_X A$ is the derivative of A in the direction of X. By (1.2) and (1.5) it follows that for every X $\dot{\nabla}_X g = 0$. We note that prolonged differentiation and prolonged covariant differentiation preserve the weights of the satellites. Let $x \in U \subset W_n(g,T)$, $X \in T_x(U)$, $A \in \chi(U)$, and let $X = \sum_{k=1}^n X^k (\frac{\partial}{\partial x^k})_x$, $A = \sum_{i=1}^n A^i \frac{\partial}{\partial x^i}$, $T = \sum_{k=1}^n T_k dx^k$. Then (1.5) gives

$$X^k \dot{\nabla}_k A^i = X^k \nabla_k A^i - pT_k A^i, \quad \nabla_k = \nabla_{\frac{\partial}{\partial x^k}} \tag{1.6}$$

A scalar function f defined on $W_n(g,T)$ is called prolonged covariant constant along a curve C, with tangent vector $\underset{1}{\xi}$, if the condition

$$\dot{\nabla}_{\underset{1}{\xi}} f = \dot{\partial}_{\underset{1}{\xi}} f = 0 \tag{1.7}$$

holds true. Circles in Riemannian spaces are extensively studied by K. Yano [4] and K. Nomizu [5]. The definition of a circle given in a Riemannian space is not applicable in a Weyl space, since the Weyl connection does not preserve the weights of the satellites of the metric tensor g. Instead, we use the prolonged covariant differentiation due to the fact that it preserves the weight. As far as we know, generalized circles in Weyl spaces have not yet been studied. Let $W_n(g,T)$ be a subspace of the Weyl space $\bar{W}_m(\bar{g},\bar{T})$ and let ∇ and $\bar{\nabla}$ be the corresponding connections. Let $p \varepsilon W_n(g,T)$ and let U, $\bar{U}$ be the special coordinate neighborhoods of p. Then, the Gauss equation and the Weingarten equation for $W_n(g,T)$ are respectively

$$\bar{\nabla}_{\bar{X}} \bar{Y}|_U = \bar{\nabla}_X Y = \nabla_X Y + \alpha(X,Y) \tag{1.8}$$

$$\bar{\nabla}_{\bar{X}} \xi|_U = \bar{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi \tag{1.9}$$

where $X \varepsilon T_p(U)$, $Y \varepsilon \chi(U)$ and ξ is a vector field normal to $W_n(g,T)$ while $\bar{X}$, $\bar{Y}$ are extensions of X and Y to $\bar{U}$ [6]. We now find the expressions for Gauss and Weingarten equations in terms of prolonged covariant derivative. The prolonged covariant derivative of the vector field $Y \varepsilon \chi(U)$ of weight $\{-1\}$ in the direction of X is, according to (1.5)

$$\dot{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + \bar{T}(X) Y. \tag{1.15}$$

By (1.8), (1.10) becomes $\dot{\bar{\nabla}}_X Y = \nabla_X Y + \alpha(X,Y) + \bar{T}(X)Y$ from which it follows that $\tan \dot{\bar{\nabla}}_X Y = \nabla_X Y + \bar{T}(X)Y = \dot{\nabla}_X Y$, $\operatorname{nor} \dot{\bar{\nabla}}_X Y = \alpha(X,Y)$ and consequently we have

$$\dot{\bar{\nabla}}_X Y = \tan \dot{\bar{\nabla}}_X Y + \operatorname{nor} \dot{\bar{\nabla}}_X Y = \dot{\nabla}_X Y + \alpha(X,Y). \tag{1.11}$$

Similarly, the normal vector field ξ of weight $\{-1\}$ has the prolonged covariant derivative

$$\dot{\bar{\nabla}}_X \xi = \bar{\nabla}_X \xi + \bar{T}(X) \xi \tag{1.12}$$

in the direction of X. By the Weingarten equation (1.9), (1.12) takes the form $\dot{\bar{\nabla}}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi + \bar{T}(X)\xi$. Since $\tan \dot{\bar{\nabla}}_X \xi = -A_\xi X$, $\operatorname{nor} \dot{\bar{\nabla}}_X \xi = \nabla^{\perp}_X \xi + \bar{T}(X)\xi$, (1.12) reduces to $\dot{\bar{\nabla}}_X \xi = \tan \dot{\bar{\nabla}}_X \xi + \operatorname{nor} \dot{\bar{\nabla}}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi + \bar{T}(X)\xi$, or

$$\dot{\bar{\nabla}}_X \xi = -A_\xi X + \dot{\nabla}^{\perp}_X \xi. \tag{1.13}$$

We call $\dot{\nabla}^{\perp}_X$ the generalized normal connection.

2. Generalized circles in weyl spaces

Let C be a smooth curve belonging to the Weyl space $W_n(g,T)$ and let $\underset{1}{\xi}$ be the tangent vector to C at the point P normalized by the condition $g(\underset{1}{\xi}, \underset{1}{\xi}) = 1$.

Definition 1. A curve in $W_n(g,T)$ is called a generalized circle if there exist a vector field $\underset{2}{\xi}$, normalized by the condition $g(\underset{2}{\xi}, \underset{2}{\xi}) = 1$, along C and a positive prolonged covariant constant scalar function κ of weight $\{-1\}$ such that

$$\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi} = \kappa\underset{2}{\xi} \tag{2.1}$$

$$\dot{\nabla}_{\underset{1}{\xi}}\underset{2}{\xi} = -\kappa\underset{1}{\xi}. \tag{2.2}$$

We note that the equations (2.1) and (2.2) are invariant under a gauge transformation. Concerning generalized circles in $W_n(g,T)$ we have the following two theorems.

Theorem 1. *A generalized circle C satisfies the third order differential equation*

$$\dot{\nabla}^2_{\underset{1}{\xi}}\underset{1}{\xi} + g(\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}, \dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi})\underset{1}{\xi} = 0, \quad \underset{1}{\xi}^i = \frac{dx^i}{ds} \tag{2.3}$$

where x^i are the coordinates of a current point belonging to C and s is the parameter of C. Conversely, if the curve C satisfies the above differential equation, then it is either a generalized circle or a geodesic.

Proof. Suppose that C is a generalized circle. Then we have

$$\dot{\nabla}_{\underset{1}{\xi}}\kappa = 0. \tag{2.4}$$

Taking the prolonged covariant derivative of (2.1) in the direction of $\underset{1}{\xi}$ and using (2.2) and (2.4) we find that

$$\dot{\nabla}^2_{\underset{1}{\xi}}\underset{1}{\xi} = (\dot{\nabla}_{\underset{1}{\xi}}\kappa)\underset{2}{\xi} + \kappa(\dot{\nabla}_{\underset{1}{\xi}}\underset{2}{\xi}) = \kappa(-\kappa\underset{1}{\xi}) = -(\kappa^2)\underset{1}{\xi}. \tag{2.5}$$

On the other hand,

$$g(\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}, \dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}) = g(\kappa\underset{2}{\xi}, \kappa\underset{2}{\xi}) = \kappa^2 g(\underset{2}{\xi}, \underset{2}{\xi}) = \kappa^2 \tag{2.6}$$

so that the equation (2.5) becomes

$$\dot{\nabla}^2_{\underset{1}{\xi}}\underset{1}{\xi} + g(\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}, \dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi})\underset{1}{\xi} = 0. \tag{2.7}$$

Conversely, suppose that a smooth curve C satisfies the equation (2.3). If $\underset{1}{\xi}$ is the tangent vector to C, normalized by the condition $g(\underset{1}{\xi},\underset{1}{\xi}) = 1$, then

$$g(\underset{1}{\xi}, \dot\nabla_{\underset{1}{\xi}}\underset{1}{\xi}) = 0. \tag{2.8}$$

The derivative of (2.8) in the direction of $\underset{1}{\xi}$ is, by (2.3),

$$\begin{aligned}\underset{1}{\xi} g(\dot\nabla_{\underset{1}{\xi}}\underset{1}{\xi}, \dot\nabla_{\underset{1}{\xi}}\underset{1}{\xi}) &= 2g(\dot\nabla^2_{\underset{1}{\xi}}\underset{1}{\xi}, \dot\nabla_{\underset{1}{\xi}}\underset{1}{\xi}) = -2g(g(\dot\nabla_{\underset{1}{\xi}}\underset{1}{\xi}, \dot\nabla_{\underset{1}{\xi}}\underset{1}{\xi})\underset{1}{\xi}, \dot\nabla_{\underset{1}{\xi}}\underset{1}{\xi}) \\ &= -2g(\dot\nabla_{\underset{1}{\xi}}\underset{1}{\xi}, \dot\nabla_{\underset{1}{\xi}}\underset{1}{\xi}).g(\underset{1}{\xi}, \dot\nabla_{\underset{1}{\xi}}\underset{1}{\xi}) = 0.\end{aligned} \tag{2.9}$$

Suppose that g is positive definite and define the function k by

$$k = \sqrt{g(\dot\nabla_{\underset{1}{\xi}}\underset{1}{\xi}, \dot\nabla_{\underset{1}{\xi}}\underset{1}{\xi})}\ . \tag{2.10}$$

If $k = 0$, then $\dot\nabla_{\underset{1}{\xi}}\underset{1}{\xi} = 0$ so that C becomes a geodesic [7]. Suppose that $k \neq 0$ and define the vector field $\underset{2}{\xi}$ along C by

$$\underset{2}{\xi} = \frac{\dot\nabla_{\underset{1}{\xi}}\underset{1}{\xi}}{k} \tag{2.11}$$

which is normalized by the condition $g(\underset{2}{\xi},\underset{2}{\xi}) = 1$. Then we have

$$\dot\nabla_{\underset{1}{\xi}}\underset{1}{\xi} = k\underset{2}{\xi}. \tag{2.12}$$

Taking the prolonged covariant derivative of (2.11) in the direction of $\underset{1}{\xi}$ and using (2.3) and (2.11) we obtain

$$\begin{aligned}\dot\nabla_{\underset{1}{\xi}}\underset{2}{\xi} &= -\frac{\dot\nabla_{\underset{1}{\xi}}k}{k^2}\dot\nabla_{\underset{1}{\xi}}\underset{1}{\xi} + \frac{1}{k}\dot\nabla^2_{\underset{1}{\xi}}\underset{1}{\xi} = -\frac{\dot\nabla_{\underset{1}{\xi}}k}{k}\underset{2}{\xi} - \frac{1}{k}g(\dot\nabla_{\underset{1}{\xi}}\underset{1}{\xi}, \dot\nabla_{\underset{1}{\xi}}\underset{1}{\xi})\underset{1}{\xi} \\ &= -(\dot\nabla_{\underset{1}{\xi}}\ln k)\underset{2}{\xi} - k\underset{1}{\xi}.\end{aligned} \tag{2.13}$$

Since, by (2.9),

$$\dot{\nabla}_{\underset{1}{\xi}} \ln k = \frac{1}{2}\dot{\nabla}_{\underset{1}{\xi}} \ln g(\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}, \dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}) = \frac{1}{2}\,\frac{\dot{\nabla}_{\underset{1}{\xi}} g(\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}, \dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi})}{g(\dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi}, \dot{\nabla}_{\underset{1}{\xi}}\underset{1}{\xi})} = 0, \tag{2.14}$$

k is a prolonged covariant constant along C. From (2.13) it follows that $\dot{\nabla}_{\underset{1}{\xi}}\underset{2}{\xi} = -k\underset{1}{\xi}$ showing that C is a generalized circle. □

Theorem 2. *A smooth curve C considered as a 1-dimensional subspace of the Weyl space $W_n(g,T)$ will be a generalized circle if and only if it has a non-zero parallel mean curvature vector.*

PROOF. Let C be a generalized circle in $W_n(g,T)$ and α be the second fundamental form of C. Then, from equations (1.11) and (2.1) we find

$$\alpha(\underset{1}{\xi}, \underset{1}{\xi}) = \kappa\underset{2}{\xi} \tag{2.15}$$

from which it follows that $Tr\alpha = \eta = \alpha(\underset{1}{\xi}, \underset{1}{\xi}) = \kappa\underset{2}{\xi}$. Then, by using the Weingarten equation (1.13) we get $\dot{\dot{\nabla}}^{\perp}_{\underset{1}{\xi}}\eta = \dot{\dot{\nabla}}^{\perp}_{\underset{1}{\xi}}(\kappa\underset{2}{\xi}) = \dot{\dot{\nabla}}_{\underset{1}{\xi}}(\kappa\underset{2}{\xi}) + A_{\kappa\underset{2}{\xi}}\underset{1}{\xi} = \kappa\dot{\dot{\nabla}}_{\underset{1}{\xi}}\underset{2}{\xi} + \kappa A_{\underset{2}{\xi}}\underset{1}{\xi} = 0$ showing that η is parallel with respect to the generalized normal connection $\dot{\nabla}^{\perp}$. Conversely, suppose that a smooth curve C has a non-zero parallel mean curvature vector, say η. Then

$$\eta = \alpha(\underset{1}{\xi}, \underset{1}{\xi}). \tag{2.16}$$

Define the vector field $\underset{2}{\xi}$ by

$$\underset{2}{\xi} = \frac{\eta}{\kappa} \tag{2.17}$$

where κ is a positive prolonged covariant constant of weight $\{-1\}$. From (1.11) and (2.17) we have

$$\dot{\dot{\nabla}}_{\underset{1}{\xi}}\underset{1}{\xi} = \kappa\underset{2}{\xi}. \tag{2.18}$$

On the other hand, by using the Weingarten equation we obtain

$$\begin{aligned}\dot{\nabla}^{\perp}_{\underset{1}{\xi}} \eta &= \dot{\nabla}^{\perp}_{\underset{1}{\xi}}(\kappa \underset{2}{\xi}) = \dot{\nabla}_{\underset{1}{\xi}}(\kappa \underset{2}{\xi}) + A_{\kappa \underset{2}{\xi}} \underset{1}{\xi} = \kappa(\dot{\nabla}_{\underset{1}{\xi}} \underset{2}{\xi}) + \kappa A_{\underset{2}{\xi}} \underset{1}{\xi} \\ &= \kappa(\dot{\nabla}_{\underset{1}{\xi}} \underset{2}{\xi} + A_{\underset{2}{\xi}} \underset{1}{\xi}) = \kappa \dot{\nabla}^{\perp}_{\underset{1}{\xi}} \underset{2}{\xi} = 0\end{aligned} \tag{2.19}$$

so that the vector field $\underset{2}{\xi}$ is parallel with respect to the generalized normal connection. Since η and $\underset{2}{\xi}$ are parallel vector fields, the vector $\dot{\nabla}_{\underset{1}{\xi}} \underset{2}{\xi}$ is tangent to C. Taking the prolonged covariant derivative of $g(\underset{1}{\xi}, \underset{2}{\xi}) = 0$ in the direction of $\underset{1}{\xi}$ and using (2.18) we find that $g(\dot{\nabla}_{\underset{1}{\xi}} \underset{2}{\xi}, \underset{1}{\xi}) = -g(\underset{2}{\xi}, \dot{\nabla}_{\underset{1}{\xi}} \underset{1}{\xi}) = -\kappa$ from which it follows that $\dot{\nabla}_{\underset{1}{\xi}} \underset{2}{\xi} = -\kappa \underset{1}{\xi}$. Consequently, C is a generalized circle. □

3. Conformal mapping of Weyl spaces preserving generalized circles

Let τ be a conformal mapping of $W_n(g, T)$ onto $\bar{W}_n(\bar{g}, \bar{T})$. Then, at the corresponding points of these spaces we can make [2]

$$g = \bar{g}. \tag{3.1}$$

The covariant vector P defined by

$$P = T - \bar{T} \tag{3.2}$$

is called the vector of the conformal mapping. Clearly P has zero weight. Let ∇ and $\bar{\nabla}$ be the Weyl connections of $W_n(g, T)$ and $\bar{W}_n(\bar{g}, \bar{T})$ and let the connection coefficients be denoted by Γ^i_{jk} and $\bar{\Gamma}^i_{jk}$ respectively, then by (3.1) and (3.2) we have

$$\begin{aligned}\bar{\Gamma}^i_{jk} &= \Gamma^i_{jk} - \bar{g}^{im}(\bar{g}_{mj}\bar{T}_k + \bar{g}_{mk}\bar{T}_j - \bar{g}_{jk}\bar{T}_m) \\ &= \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} - [\delta^i_j(T_k - P_k) + \delta^i_k(T_j - P_j) - g^{im} g_{jk}(T_m - P_m)] \\ \bar{\Gamma}^i_{jk} &= \Gamma^i_{jk} + \delta^i_j P_k + \delta^i_k P_j - g^{im} g_{jk} P_m.\end{aligned} \tag{3.3}$$

Let C be a smooth curve in $W_n(g,T)$ and let $\bar{C}$ be its image under the conformal mapping τ. Denote the parameters of C and $\bar{C}$ by s and $\bar{s}$ respectively. Denote the coordinates of a current point p on C by x^i and those of the corresponding point $\bar{p}$ by $\bar{x}^i$. Then, for the tangent vectors $\underset{1}{\xi}^i$ and $\underset{1}{\bar{\xi}}^i$ at corresponding points we have

$$\underset{1}{\bar{\xi}}^i = \underset{1}{\xi}^i, \quad \left(\underset{1}{\xi}^i = \frac{dx^i}{ds}\right) \tag{3.4}$$

We assume $\underset{1}{\xi}$ to be normalized by the condition $g(\underset{1}{\xi},\underset{1}{\xi}) = 1$.

If generalized circles in $W_n(g,T)$ are transformed into generalized circles in $\bar{W}_n(\bar{g},\bar{T})$ under the conformal mapping τ, then τ is called a generalized concircular mapping. Concerning generalized concircular mappings we have the

Theorem 3. *The conformal mapping $\tau : W_n(g,T) \to \bar{W}_n(\bar{g},\bar{T})$ will be generalized concircular if and only if*

$$P_{kl} = \phi g_{kl}, \quad P_{kl} = \nabla_l P_k - P_k P_l + \frac{1}{2} g_{kl} g^{rs} P_r P_s$$

where ϕ is a scalar smooth function of weight $\{-2\}$ defined on $W_n(g,T)$.

Proof. Connections of $W_n(g,T)$ and $\bar{W}_n(\bar{g},\bar{T})$ are related to each other by the equation

$$\bar{\nabla}_i \underset{1}{\xi}^k = \nabla_i \underset{1}{\xi}^k + \gamma_{ij}^k \underset{1}{\xi}^j, \quad \gamma_{ij}^k = P_i \delta_j^k + P_j \delta_i^k - g_{ij} g^{kl} P_l. \tag{3.5}$$

Remembering that the weight of $\underset{1}{\xi}$ is $\{-1\}$ and using (3.2), (3.4) and (3.5) we find that

$$\begin{aligned} \dot{\bar{\nabla}}_i \underset{1}{\bar{\xi}}^k &= \dot{\bar{\nabla}}_i \underset{1}{\xi}^k = \bar{\nabla}_i \underset{1}{\xi}^k + \bar{T}_i \underset{1}{\xi}^k = \bar{\nabla}_i \underset{1}{\xi}^k + (T_i - P_i) \underset{1}{\xi}^k \\ &= \nabla_i \underset{1}{\xi}^k + \gamma_{ij}^k \underset{1}{\xi}^j + (T_i - P_i) \underset{1}{\xi}^k \\ &= \dot{\nabla}_i \underset{1}{\xi}^k + (P_j \delta_i^k - g_{ij} g^{kl} P_l) \underset{1}{\xi}^j. \end{aligned} \tag{3.6}$$

The prolonged covariant derivative of $\underset{1}{\bar{\xi}}$ in its direction is found to be

$$\text{(3.7)} \quad \begin{aligned} &\underset{1}{\bar{\xi}}^{i}\dot{\bar{\nabla}}_i\underset{1}{\bar{\xi}}^{k} = \underset{1}{\xi}^{i}\dot{\nabla}_i\underset{1}{\xi}^{k} + (P_j\underset{1}{\xi}^{j})(\delta_i^k\underset{1}{\xi}^{i}) - (g^{kl}P_l)g_{ij}\underset{1}{\xi}^{i}\underset{1}{\xi}^{j},\ \ g_{ij}\underset{1}{\xi}^{i}\underset{1}{\xi}^{j} = 1 \\ &\underset{1}{\bar{\xi}}^{i}\dot{\bar{\nabla}}_i\underset{1}{\bar{\xi}}^{k} = \underset{1}{\xi}^{i}\dot{\nabla}_i\underset{1}{\xi}^{k} + (P_j\underset{1}{\xi}^{j})\underset{1}{\xi}^{k} - g^{kl}P_l. \end{aligned}$$

By setting $\underset{1}{\bar{\xi}}^{i}\dot{\bar{\nabla}}_i\underset{1}{\bar{\xi}}^{k} = \frac{\dot{\delta}\underset{1}{\bar{\xi}}^{k}}{\delta\bar{s}}$, $\underset{1}{\xi}^{i}\dot{\nabla}_i\underset{1}{\xi}^{k} = \frac{\dot{\delta}\underset{1}{\xi}^{k}}{\delta s}$ the equation (3.7) becomes

$$\text{(3.8)} \qquad \frac{\dot{\delta}\underset{1}{\bar{\xi}}^{k}}{\delta\bar{s}} = \frac{\dot{\delta}\underset{1}{\xi}^{k}}{\delta s} + (P_j\underset{1}{\xi}^{j})\underset{1}{\xi}^{k} - g^{kl}P_l.$$

The equation (3.7) may be written in the form

$$\text{(3.9)} \qquad \underset{1}{\bar{\xi}}^{i}\bar{\nabla}_i\underset{1}{\bar{\xi}}^{k} = \underset{1}{\xi}^{i}\nabla_i\underset{1}{\xi}^{k} + 2(P_j\underset{1}{\xi}^{j})\underset{1}{\xi}^{k} - g^{kl}P_l.$$

Remembering that the weight of $\underset{1}{\bar{\xi}}^{i}\dot{\bar{\nabla}}_i\underset{1}{\bar{\xi}}^{k} = \frac{\dot{\delta}\underset{1}{\bar{\xi}}^{k}}{\delta\bar{s}}$ is $\{-2\}$ we find that

$$\text{(3.10)} \quad \begin{aligned} &\frac{\dot{\delta}^2\underset{1}{\bar{\xi}}^{k}}{\delta\bar{s}^2} = \frac{\dot{\delta}}{\delta\bar{s}}\Big(\frac{\dot{\delta}\underset{1}{\bar{\xi}}^{k}}{\delta\bar{s}}\Big) = \underset{1}{\xi}^{m}\dot{\bar{\nabla}}_m(\underset{1}{\bar{\xi}}^{i}\dot{\bar{\nabla}}_i\underset{1}{\bar{\xi}}^{k}) \\ &\frac{\dot{\delta}^2\underset{1}{\bar{\xi}}^{k}}{\delta\bar{s}^2} = \underset{1}{\bar{\xi}}^{m}[\bar{\nabla}_m(\underset{1}{\bar{\xi}}^{i}\dot{\bar{\nabla}}_i\underset{1}{\bar{\xi}}^{k}) + 2\bar{T}_m(\underset{1}{\bar{\xi}}^{i}\dot{\bar{\nabla}}_i\underset{1}{\bar{\xi}}^{k})] \\ &\frac{\dot{\delta}^2\underset{1}{\bar{\xi}}^{k}}{\delta\bar{s}^2} = \underset{1}{\bar{\xi}}^{m}[\bar{\nabla}_m u^k + 2(T_m - P_m)u^k], \quad u^k = \underset{1}{\bar{\xi}}^{i}\dot{\bar{\nabla}}_i\underset{1}{\bar{\xi}}^{k} \\ &\frac{\dot{\delta}^2\underset{1}{\bar{\xi}}^{k}}{\delta\bar{s}^2} = \underset{1}{\bar{\xi}}^{m}\bar{\nabla}_m u^k + 2T_m\underset{1}{\bar{\xi}}^{m}u^k - 2P_m\underset{1}{\bar{\xi}}^{m}u^k. \end{aligned}$$

Since, by (3.9) $\underset{1}{\bar{\xi}}^{m}\bar{\nabla}_m u^k = \underset{1}{\xi}^{m}\nabla_m u^k + 2(P_j\underset{1}{\xi}^{j})u^k - g^{kl}P_l$ the equation

(3.10) takes the form

$$\text{(3.11)}\qquad \frac{\dot{\delta}^2 \underset{1}{\bar{\xi}}^k}{\delta \bar{s}^2} = \underset{1}{\xi}^m \nabla_m (\underset{1}{\bar{\xi}}^i \dot{\bar{\nabla}}_i \underset{1}{\bar{\xi}}^k) + 2\underset{1}{\xi}^j P_j (\underset{1}{\bar{\xi}}^i \dot{\bar{\nabla}}_i \underset{1}{\xi}^k) - g^{kl} P_l + 2(T_m - P_m)\underset{1}{\bar{\xi}}^m (\underset{1}{\bar{\xi}}^i \dot{\bar{\nabla}}_i \underset{1}{\bar{\xi}}^k).$$

Taking the equation (3.7) into account, (3.11) can be written as

$$\text{(3.12)}\qquad \begin{aligned} \frac{\dot{\delta}^2 \underset{1}{\bar{\xi}}^k}{\delta \bar{s}^2} &= \underset{1}{\xi}^m \nabla_m [(\underset{1}{\xi}^i \dot{\nabla}_i \underset{1}{\xi}^k) + (P_h \underset{1}{\xi}^h)\underset{1}{\xi}^k - g^{kl} P_l] \\ &\quad + 2(\underset{1}{\xi}^j P_j)(\underset{1}{\xi}^i \dot{\nabla}_i \underset{1}{\xi}^k + P_h \underset{1}{\xi}^h \underset{1}{\xi}^k - g^{kl} P_l) \\ &\quad + 2(T_m - P_m)\underset{1}{\xi}^m (\underset{1}{\xi}^i \dot{\nabla}_i \underset{1}{\xi}^k + P_h \underset{1}{\xi}^h \underset{1}{\xi}^k - g^{kl} P_l) - g^{kl} P_l \\ \frac{\dot{\delta}^2 \underset{1}{\bar{\xi}}^k}{\delta \bar{s}^2} &= \underset{1}{\xi}^m \dot{\nabla}_m (\underset{1}{\xi}^i \dot{\nabla}_i \underset{1}{\xi}^k) + \underset{1}{\xi}^m \nabla_m (P_h \underset{1}{\xi}^h \underset{1}{\xi}^k - g^{kl} P_l) \\ &\quad + 2T_m \underset{1}{\xi}^m (P_h \underset{1}{\xi}^h \underset{1}{\xi}^k - g^{kl} P_l) - g^{kl} P_l \\ \frac{\dot{\delta}^2 \underset{1}{\bar{\xi}}^k}{\delta \bar{s}^2} &= \frac{\dot{\delta}^2 \underset{1}{\xi}^k}{\delta s^2} + \underset{1}{\xi}^m \dot{\nabla}_m (P_h \underset{1}{\xi}^h \underset{1}{\xi}^k - g^{kl} P_l) - g^{kl} P_l \end{aligned}$$

where we have used the property that $P_h \underset{1}{\xi}^h \underset{1}{\xi}^k - g^{kl} P_l$ is of weight $\{-2\}$. With the help of (3.4), (3.8) and (3.12) we get

$$\text{(3.13)}\qquad \begin{aligned} \frac{\dot{\delta}^2 \underset{1}{\bar{\xi}}^i}{\delta \bar{s}^2} + \bar{g}_{kj} \frac{\dot{\delta} \underset{1}{\bar{\xi}}^k}{\delta \bar{s}} \frac{\dot{\delta} \underset{1}{\bar{\xi}}^j}{\delta \bar{s}} \underset{1}{\xi}^i &= \frac{\dot{\delta}^2 \underset{1}{\xi}^i}{\delta s^2} + \underset{1}{\xi}^m \dot{\nabla}_m (P_h \underset{1}{\xi}^h \underset{1}{\xi}^i - g^{il} P_l) - g^{il} P_l \\ &\quad + g_{kj} \left(\frac{\dot{\delta} \underset{1}{\xi}^k}{\delta s} + P_h \underset{1}{\xi}^h \underset{1}{\xi}^k - g^{kl} P_l \right) \left(\frac{\dot{\delta} \underset{1}{\xi}^j}{\delta s} + P_m \underset{1}{\xi}^m \underset{1}{\xi}^j - g^{jm} P_m \right) \underset{1}{\xi}^i \\ &= \left(\frac{\dot{\delta}^2 \underset{1}{\xi}^i}{\delta s^2} + g_{kj} \frac{\dot{\delta} \underset{1}{\xi}^k}{\delta s} \frac{\dot{\delta} \underset{1}{\xi}^j}{\delta s} \underset{1}{\xi}^i \right) + \underset{1}{\xi}^m \dot{\nabla}_m (P_h \underset{1}{\xi}^h \underset{1}{\xi}^i - g^{ih} P_h) - g^{ih} P_h \end{aligned}$$

$$-\left[2P_h\left(\frac{\dot{\delta}\underset{1}{\xi}^k}{\delta s}\right)+(P_h\underset{1}{\xi}^h)^2-g^{jm}P_jP_m\right]\underset{1}{\xi}^i.$$

Therefore, generalized circles in $W_n(g,T)$ will be transformed into generalized circles in $\bar{W}_n(\bar{g},\bar{T})$ under τ, if and only if the condition

$$\underset{1}{\xi}^m\dot{\nabla}_m[P_h(\underset{1}{\xi}^i\underset{1}{\xi}^h-g^{ih})]-g^{ih}P_h-\left[2P_h\left(\frac{\dot{\delta}\underset{1}{\xi}^h}{\delta s}\right)+(P_h\underset{1}{\xi}^h)^2-g^{jm}P_jP_m\right]\underset{1}{\xi}^i=0$$

or, after simplification

$$(\underset{1}{\xi}^k\nabla_kP_h)(\underset{1}{\xi}^i\underset{1}{\xi}^h-g^{ih})+(\underset{1}{\xi}^hP_h)P_lg^{il}-(P_h\underset{1}{\xi}^h)^2\underset{1}{\xi}^i=0 \tag{3.14}$$

is satisfied. Introducing the notation $P_{hk}=\nabla_kP_h-P_hP_k+\frac{1}{2}g_{hk}g^{rs}P_rP_s$ we can transform (3.14) into the form

$$\phi\underset{1}{\xi}^i-P_{hk}g^{ih}\underset{1}{\xi}^k=0,\quad \phi=P_{hk}\underset{1}{\xi}^h\underset{1}{\xi}^k. \tag{3.15}$$

Transvecting (3.15) by g_{il} and remembering that $g_{il}g^{ih}=\delta^h_l$, by a suitable change of indices, (3.15) can be reduced to the form

$$(P_{hk}-\phi g_{hk})\underset{1}{\xi}^k=0. \tag{16}$$

In order that the system of homogeneous linear equations (3.16) be satisfied for any vector $\underset{1}{\xi}^k$ (i.e. for any generalized circle) it is necessary that $P_{hk}-\phi g_{hk}=0$. Conversely, it is clear that if $P_{hk}=\phi g_{hk}$ then the condition (3.14) is identically satisfied. This completes the proof of the theorem. □

4. Generalized concircular curvature tensor of a Weyl space

Let τ be the generalized concircular mapping of $W_n(g,T)$ onto $\bar{W}_n(\bar{g},\bar{T})$.

Theorem 4. *Let R_{pjkl} and R be respectively the components of the covariant curvature tensor and the scalar curvature of $W_n(g,T)$. Then the covariant tensor Z with components*

$$Z_{pjkl} = R_{pjkl} - \frac{R}{n(n-1)}(g_{jk}g_{pl} - g_{pk}g_{jl}) \tag{4.1}$$

is invariant under τ.

Proof. Let the components of the mixed tensor of $W_n(g,T)$ be R^p_{jkl} and let $\bar{R}^p_{jkl}$ be their images under τ. Then we have

$$\bar{R}^p_{jkl} = R^p_{jkl} + \delta^p_l P_{jk} - \delta^p_k P_{jl} + g_{jk}g^{pm}P_{ml} - g_{jl}g^{pm}P_{mk} + 2\delta^p_j \nabla_{[k}P_{l]} \tag{4.2}$$

where $P_{jk} = \nabla_k P_j - P_j P_k + \frac{1}{2} g_{jk} g^{rs} P_r P_s$. Since τ is generalized concircular we have $P_{jk} = \phi g_{jk}$ so that (4.2) transforms into

$$\bar{R}^p_{jkl} = R^p_{jkl} + 2\phi(g_{jk}\delta^p_l - g_{jl}\delta^p_k). \tag{4.3}$$

If contraction on the indices p and l in (4.3) is performed (4.3) becomes

$$\bar{R}_{jk} = R_{jk} + 2\phi(n-1)g_{jk} \tag{4.4}$$

where $\bar{R}_{jk} = \bar{R}^p_{jkp}$ and $R_{jk} = R^p_{jkp}$ are, respectively, the Ricci tensors of $\bar{W}_n(\bar{g},\bar{T})$ and $W_n(g,T)$. Since we can make $\bar{g} = g$ at corresponding points of $W_n(g,T)$ and $\bar{W}_n(\bar{g},\bar{T})$, transvecting (4.4) by $\bar{g}^{jk} = g^{jk}$ we find that $\bar{R} = R + 2\phi n(n-1)$ from which we obtain

$$2\phi = \frac{\bar{R} - R}{n(n-1)}, \quad (n \neq 1) \tag{4.5}$$

where R ve $\bar{R}$ are respectively the scalar curvatures of $W_n(g,T)$ and $\bar{W}_n(\bar{g},\bar{T})$. Substitution of ϕ in (4.3) gives $\bar{R}^p_{jkl} = R^p_{jkl} + \frac{\bar{R}-R}{n(n-1)}(\delta^p_l g_{jk} - \delta^p_k g_{jl})$ or

$$\bar{R}^p_{jkl} - \frac{\bar{R}}{n(n-1)}(\delta^p_l \bar{g}_{jk} - \delta^p_k \bar{g}_{jl}) = R^p_{jkl} - \frac{R}{n(n-1)}(\delta^p_l g_{jk} - \delta^p_k g_{jl}). \tag{4.6}$$

Multiplying (4.6) by $\bar{g}_{rp} = g_{rp}$ and summing for p we get $\bar{Z}_{rjkl} = Z_{rjkl}$ so that the proof of the theorem is completed. □

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(Received December 15, 2000; revised August 7, 2001)