Publ. Math. Debrecen
60 / 1-2 (2002), 89–99

A Stein–Tomas restriction theorem for general measures

By THEMIS MITSIS (Jyväskylä)

Abstract. We prove a general Stein–Tomas type restriction theorem for measures of given dimension and Fourier exponent.

1. Introduction

Let $d\sigma$ be normalized surface measure on the $(n-1)$-dimensional sphere S^{n-1}. For an integrable function $f : S^{n-1} \to \mathbb{C}$, consider the Fourier transform

$$\widehat{fd\sigma}(\xi) = \int_{S^{n-1}} f(x) e^{-2\pi i x\cdot\xi} d\sigma(x).$$

The classical Stein–Tomas restriction theorem is the following statement.

Theorem 1.1. *For every $p \geq 2(n+1)/(n-1)$ there exists a constant $C_{p,n} > 0$ such that*

$$\|\widehat{fd\sigma}\|_p \leq C_{p,n} \|f\|_{L^2(d\sigma)}$$

for all $f \in L^2(d\sigma)$. Moreover, the above range of exponents is best possible.

This result has been extended to various more general situations where the sphere is replaced by smooth manifolds satisfying certain curvature hypotheses. For proofs and more details the reader may consult Stein [2] and the references contained there.

Mathematics Subject Classification: 42B99.
Key words and phrases: Fourier transforms of measures, restriction theorems.
Supported by a Marie Curie Fellowship of the European Community under No. HPMFCT-2000-00442.

The purpose of this paper is to prove an analogous restriction theorem for general measures. By the term measure we will always mean a positive, finite, compactly supported Borel measure in $\mathbb{R}^n$. For such a measure μ, we denote by $\mathrm{spt}(\mu)$ its support. The Fourier transform of μ is defined by

$$\widehat{\mu}(\xi) = \int e^{-2\pi i x\cdot\xi} d\mu(x).$$

Similarly, if $f : \mathbb{R}^n \to \mathbb{C}$ is a μ-integrable function, we define

$$\widehat{fd\mu}(\xi) = \int f(x) e^{-2\pi i x\cdot\xi} d\mu(x).$$

Letting $B(x, r)$ be the closed ball of radius r centered at the point x, we can state our main result as follows.

Theorem 1.2. *Let μ be a measure in $\mathbb{R}^n$ such that*

$$\mu(B(x, r)) \leq r^\alpha, \quad \forall x \in \mathbb{R}^n,\ r > 0$$

and

$$|\widehat{\mu}(\xi)| \leq \frac{1}{|\xi|^{\beta/2}}$$

for some $0 < \alpha$, $\beta < n$. Then for every $p > 2(2n - 2\alpha + \beta)/\beta$, there exists a constant $C_{p,n,\alpha,\beta} > 0$ such that

$$\|\widehat{fd\mu}\|_p \leq C_{p,n,\alpha,\beta} \|f\|_{L^2(d\mu)}$$

for all $f \in L^2(d\mu)$.

The example of a bounded flat hypersurface, in which case restriction fails, makes it clear that the decay condition on $\widehat{\mu}$ is, in a certain sense, indispensable and that no general restriction theorem can be based only on dimensionality or even smoothness considerations. We will address this issue again in Section 3.

Throughout this paper, $x \lesssim y$ means $x \leq Cy$, where C is a positive constant depending on the context and whose value is irrelevant. Similarly, $x \simeq y$ means ($x \lesssim y$ & $y \lesssim x$). If A is a subset of $\mathbb{R}^n$ then χ_A is the indicator function. Finally, we will denote α-dimensional Hausdorff measure by $\mathcal{H}^\alpha$.

2. The main result

Theorem 1.2 will be a consequence of the following

Proposition 2.1. *Let* μ *be a measure in* $\mathbb{R}^n$ *such that*

$$\mu(B(x,r)) \le r^{\alpha}, \quad \forall x \in \mathbb{R}^n,\ r > 0 \tag{1}$$

and

$$|\widehat{\mu}(\xi)| \le \frac{1}{|\xi|^{\beta/2}} \tag{2}$$

for some $0 < \alpha$, $\beta < n$. *Then for every* $p > 2(2n - 2\alpha + \beta)/\beta$

$$\|\widehat{\mu} * f\|_p \lesssim \|f\|_{p'}$$

for all f *in the Schwartz space* $\mathcal{S}$. *Here* p' *is the conjugate exponent* $p' = p/(p-1)$.

Proof. Let $\check{\mu}(x) = \widehat{\mu}(-x)$ and $\tilde{f}(x) = f(-x)$. Then

$$\widehat{\mu} * f(x) = \check{\mu} * \tilde{f}(-x).$$

Therefore, it is enough to show that $\|\check{\mu} * f\|_p \lesssim \|f\|_{p'}$.

Let ψ be a $\mathcal{C}^{\infty}$ function which is equal to 1 when $|x| \ge 1$ and to 0 when $|x| \le 1/2$, and let $\phi(x) = \psi(2x) - \psi(x)$. Then

$$\operatorname{supp}(\phi) \subset \{x : 1/4 \le |x| \le 1\}$$

and

$$\sum_{j=0}^{\infty} \phi(2^{-j}x) = 1, \quad \text{if } |x| \ge 1.$$

We now decompose $\check{\mu}$ as follows.

$$\check{\mu} = K_{-\infty} + \sum_{j=0}^{\infty} K_j$$

where

$$K_j(x) = \phi(2^{-j}x)\check{\mu}(x),$$

$$K_{-\infty}(x) = \left(1 - \sum_{j=0}^{\infty} \phi(2^{-j}x)\right)\check{\mu}(x).$$

We are going to estimate $\|K_j * f\|_\infty$ and $\|K_j * f\|_2$.

Using (2) and the support property of ϕ we get

$$\|K_j * f\|_\infty \leq \|K_j\|_\infty \|f\|_1 \lesssim 2^{-j\frac{\beta}{2}} \|f\|_1. \tag{3}$$

To estimate $\|K_j * f\|_2$, let $\phi_j(x) = \phi(2^{-j}x)$ and choose $N > \alpha$. Then, using the property of distribution functions and the fact that $\widehat{\phi}$ is rapidly decreasing, we get

$$\begin{aligned}
|\widehat{K}_j(\xi)| = |\widehat{\phi_j} * \mu(\xi)| &= \left| \int \widehat{\phi_j}(\xi - y) d\mu(y) \right| \\
&\leq C_N 2^{jn} \int \frac{d\mu(y)}{(1 + |2^j(\xi - y)|)^N} \\
&= C_N 2^{jn} \int_0^\infty \mu\left(\left\{ y : \frac{1}{(1 + |2^j(\xi - y)|)^N} \geq r \right\} \right) dr \\
&= C_N 2^{jn} \int_0^\infty \mu\left(\left\{ y : 1 + 2^j|\xi - y| \leq \frac{1}{r^{1/N}} \right\} \right) dr \\
&= C_N 2^{jn} \int_0^1 \mu\left(\left\{ y : 1 + 2^j|\xi - y| \leq \frac{1}{r^{1/N}} \right\} \right) dr \\
&\leq C_N 2^{jn} \int_0^1 \mu\left(\left\{ y : |\xi - y| \leq \frac{1}{2^j r^{1/N}} \right\} \right) dr \\
&= C_N 2^{jn} \int_0^1 \mu\left(B\left(\xi, \frac{1}{2^j r^{1/N}} \right) \right) dr \\
\text{by (1)} &\leq C_N 2^{jn} 2^{-j\alpha} \int_0^1 \frac{dr}{r^{\alpha/N}} \\
&\lesssim 2^{j(n-\alpha)}.
\end{aligned}$$

It follows that $\|\widehat{K}_j\|_\infty \lesssim 2^{j(n-\alpha)}$. Therefore

$$\|K_j * f\|_2 = \|\widehat{K}_j \widehat{f}\|_2 \leq \|\widehat{K}_j\|_\infty \|\widehat{f}\|_2 \lesssim 2^{j(n-\alpha)} \|f\|_2. \tag{4}$$

Now, (3) and (4) allow us to think of convolution with the kernel K_j as an operator from L^1 to L^∞ and from L^2 to L^2. Therefore, we can interpolate between (3) and (4) using the Riesz–Thorin theorem. This gives

$$\|K_j * f\|_{p(\theta)} \lesssim A(\theta)\|f\|_{q(\theta)}, \quad 0 \le \theta \le 1,$$

where

$$p(\theta) = 2/\theta,\ q(\theta) = 2/(2-\theta), \quad A(\theta) = 2^{j(n-\alpha)\theta - j\frac{\beta}{2}(1-\theta)}.$$

Equivalently

$$\|K_j * f\|_p \lesssim 2^{jc(n,\alpha,\beta,p)}\|f\|_{p'}, \quad p \ge 2,$$

where

$$c(n,\alpha,\beta,p) = \frac{2(n-\alpha)}{p} - \frac{\beta}{2}\left(1 - \frac{2}{p}\right).$$

Further, note that since $K_{-\infty}$ is a $\mathcal{C}^\infty$ function with compact support we have

$$\|K_{-\infty} * f\|_p \lesssim \|f\|_{p'}, \quad p \ge 2$$

by Young's inequality.

To complete the proof, notice that the sum

$$K_{-\infty} * f + \sum_j K_j * f$$

converges pointwise to $\check{\mu} * f$ since f is a Schwartz function. Moreover, the exponent $c(n,\alpha,\beta,p)$ is negative provided that $p > 2(2n-2\alpha+\beta)/\beta$. Therefore

$$\begin{aligned}\|\check{\mu} * f\|_p &\le \|K_{-\infty} * f\|_p + \sum_{j=0}^{\infty}\|K_j * f\|_p \\ &\lesssim \sum_{j=0}^{\infty} 2^{jc(n,\alpha,\beta,p)}\|f\|_{p'} \lesssim \|f\|_{p'}. \qquad \square\end{aligned}$$

To prove Theorem 1.2, let $f \in L^2(d\mu)$ and $g \in \mathcal{S}$. Then

$$\begin{aligned}\int \widehat{fd\mu}(\xi)g(\xi)d\xi &= \int \widehat{g}(y)f(y)d\mu(y) \le \|\widehat{g}\|_{L^2(d\mu)}\|f\|_{L^2(d\mu)} \\ &= \|f\|_{L^2(d\mu)}\left(\int \widehat{g}(y)\overline{\widehat{g}}(y)d\mu(y)\right)^{1/2}\end{aligned}$$

$$= \|f\|_{L^2(d\mu)} \left(\int \widehat{\mu} * \overline{g}(x) g(x) dx \right)^{1/2}$$
$$\leq \|f\|_{L^2(d\mu)} \left(\|\widehat{\mu} * \overline{g}\|_p \|g\|_{p'} \right)^{1/2} \lesssim \|f\|_{L^2(d\mu)} \|g\|_{p'}$$

where the last inequality follows from Proposition 2.1. Since $\mathcal{S}$ is dense in $L^{p'}$, we conclude that

$$\|\widehat{fd\mu}\|_p \lesssim \|f\|_{L^2(d\mu)}.$$

3. Remarks

1. It is a standard fact that the condition $|\widehat{\mu}(\xi)| \leq |\xi|^{-\beta/2}$ implies that the Hausdorff dimension of the support of μ is at least β. This can be shown using the formulas

$$I_\alpha(\mu) = C(n, \alpha) \int \frac{|\widehat{\mu}(\xi)|^2}{|\xi|^{n-\alpha}} d\xi$$

and

$$\dim_H(A) = \sup\{\alpha : \exists \mu, \text{ such that } \operatorname{spt}(\mu) \subset A,\ I_\alpha(\mu) < \infty\},$$

for all Borel sets $A \subset \mathbb{R}^n$. Here

$$I_\alpha(\mu) = \iint \frac{d\mu(x) d\mu(y)}{|x-y|^\alpha}$$

is the α-energy of μ and $\dim_H$ denotes Hausdorff dimension (see Mattila [1]).

Another natural notion of dimension is the so-called *Fourier dimension*, denoted by $\dim_F$. It is defined as the unique number in $[0, n]$ such that for any $0 < s < \dim_F A$, there exists a non-zero measure μ with $\operatorname{spt}(\mu) \subset A$ and $\widehat{\mu}(\xi) \leq |\xi|^{-s/2}$, and that for $\dim_F A < s < n$, no such measure exists. By the remarks above, we have that for any Borel set $A \subset \mathbb{R}^n$, $\dim_F A \leq \dim_H A$. Sets A for which $\dim_F A = \dim_H A$ are called *Salem sets*.

We further note that $|\widehat{\mu}(\xi)| \leq |\xi|^{-\beta/2}$ implies that $\mu(B(x,r)) \lesssim r^{\beta/2}$. To see this, choose $\phi \in \mathcal{S}$ such that $\phi \geq 0$, $\phi \gtrsim 1$ on $B(0,1)$ and $\widehat{\phi} = 0$ outside $B(0, \delta)$ for some $\delta > 0$. Let

$$\phi_{x,r}(y) = \phi\left(\frac{x-y}{r}\right).$$

Then

$$\begin{aligned}\mu(B(x,r)) &\lesssim \int \phi_{x,r}(y)d\mu(y) = \int \widehat{\phi}_{x,r}(\xi)\widehat{\mu}(-\xi)d\xi \\ &\le r^n \int_{|r\xi|\le\delta} |\widehat{\phi}(r\xi)| \frac{1}{|\xi|^{\beta/2}} d\xi \\ &= r^{\beta/2} \int_{|\xi|\le\delta} |\widehat{\phi}(\xi)| \frac{1}{|\xi|^{\beta/2}} d\xi \lesssim r^{\beta/2}.\end{aligned}$$

Thus, we obtain the following version of Theorem 1.2 if we only know the rate of decay of $\widehat{\mu}$.

Corollary 3.1. *Let μ be a measure in $\mathbb{R}^n$ such that*

$$|\widehat{\mu}(\xi)| \le \frac{1}{|\xi|^{\beta/2}}$$

for some $0 < \beta < n$. Then for every $p > 4n/\beta$, there exists a constant $C_{p,n,\beta} > 0$ such that

$$\|\widehat{fd\mu}\|_p \le C_{p,n,\beta} \|f\|_{L^2(d\mu)}$$

for all $f \in L^2(d\mu)$.

2. The argument for the L^2 estimate in the proof of Proposition 2.1 was based only on dimensionality considerations. This suggests that there should be an L^2 bound for $\widehat{fd\mu}$ valid under very general conditions. This was first observed by Strichartz [3] in a different context. He showed that if μ satisfies $\mu(B(x,r)) \le r^\alpha$ then

$$\sup_{x_0\in\mathbb{R}^n,\, r>0} \frac{1}{r^{n-\alpha}} \int_{B(x_0,r)} |\widehat{fd\mu}(\xi)|^2 d\xi \lesssim \|f\|^2_{L^2(d\mu)}.$$

His approach involved the fractional Hardy–Littlewood maximal function. Here we will give a simple, direct proof of a more general result.

Theorem 3.1. *Let μ be a measure in $\mathbb{R}^n$ satisfying $\mu(B(x,r)) \le h(r)$, for some non-negative function h. Then there exists a constant $C > 0$ such that*

$$\int_{B(x_0,r)} |\widehat{fd\mu}(\xi)|^2 d\xi \lesssim r^n h(C/r) \|f\|^2_{L^2(d\mu)}$$

for all $x_0 \in \mathbb{R}^n$, $r > 0$, $f \in L^2(d\mu)$.

PROOF. We will prove the theorem in the case where $B(x_0, r)$ is centered at the origin. The general case then follows by multiplying f by a character.

Let ϕ be a radial Schwartz function such that $\phi \geq 1$ on $B(0,1)$, $\widehat{\phi} = 0$ outside $B(0,C)$, for some $C > 0$ and let $\phi_r(x) = \phi(x/r)$. Then

$$\begin{aligned}\int_{B(0,r)} |\widehat{fd\mu}(\xi)|^2 d\xi &\leq \int |\phi_r(\xi)\widehat{fd\mu}(-\xi)|^2 d\xi = \int |\widehat{\phi}_r * (fd\mu)(x)|^2 dx \\ &= \int \left| \int \widehat{\phi}_r(x-y) f(y) d\mu(y) \right|^2 dx \\ &\leq \iint |\widehat{\phi}_r(x-y)| d\mu(y) \int |\widehat{\phi}_r(x-y)| \, |f(y)|^2 d\mu(y) dx.\end{aligned}$$

For a fixed x we have

$$\begin{aligned}\int |\widehat{\phi}_r(x-y)| d\mu(y) &= r^n \int_{B(x,C/r)} |\widehat{\phi}(r(x-y))| d\mu(y) \\ &\leq r^n \|\widehat{\phi}\|_\infty \mu(B(x,C/r)) \lesssim r^n h(C/r).\end{aligned}$$

Therefore

$$\begin{aligned}\int_{B(0,r)} |\widehat{fd\mu}(\xi)|^2 d\xi &\lesssim r^n h(C/r) \int |f(y)|^2 \int r^n |\widehat{\phi}(r(x-y))| dx d\mu(y) \\ &= r^n h(C/r) \|\widehat{\phi}\|_1 \int |f(y)|^2 d\mu(y) \lesssim r^n h(C/r) \|f\|^2_{L^2(d\mu)}.\end{aligned}$$ □

3. For expository reasons it will be convenient to make the following definition. If $0 < \alpha, \beta < n$ then the *restriction exponent* $p(n, \alpha, \beta)$ is defined by

$$\begin{aligned}p(n,\alpha,\beta) = \inf\{q : (&\forall \mu \text{ with } \mu(B(x,r)) \leq r^\alpha \text{ and } |\widehat{\mu}(\xi)|^2 \leq 1/|\xi|^\beta) \\ &(\forall f \in L^2(d\mu))(\|\widehat{fd\mu}\|_q \lesssim \|f\|_{L^2(d\mu)})\}.\end{aligned}$$

For general values of α, β it is unknown whether Theorem 1.2 is optimal, that is, whether $p(n, \alpha, \beta) = 2(2n - 2\alpha + \beta)/\beta$. We do, however, have the following lower bound.

Proposition 3.1. *If* $0 < \alpha,\ \beta < n$ *then*

$$p(n,\alpha,\beta) \geq \frac{2n}{\alpha}.$$

PROOF. Let $A = \mathrm{spt}(\mu)$ and notice that $\mu(B(x,r)) \leq r^\alpha$ implies that μ is absolutely continuous with respect to $\mathcal{H}^\alpha \mid A$ with bounded density. Therefore $\mu = h\mathcal{H}^\alpha \mid A$ for some $h \in L^\infty(\mathcal{H}^\alpha \mid A)$. Let $\epsilon > 0$ be such that $0 < \mathcal{H}^\alpha(\{x \in A : h(x) \geq \epsilon\}) < \infty$ and put $C = \{x \in A : h(x) \geq \epsilon\}$. Then for $\mathcal{H}^\alpha$-almost all $x \in C$ (see MATTILA [1]) we have

$$\limsup_r \frac{\mathcal{H}^\alpha(C \cap B(x,r))}{r^\alpha} \geq 2^\alpha.$$

It follows that there exists $x_0 \in \mathrm{spt}(\mu)$ such that

$$\limsup_r \frac{\mu(B(x_0,r))}{r^\alpha} \geq \epsilon 2^\alpha.$$

Choose a sequence r_k such that $r_k \searrow 0$, $\mu(B(x_0,r_k)) \simeq r_k^\alpha$ and put

$$f_k = \chi_{B(x_0,r_k)}.$$

Now let ϕ be a Schwartz function which equals 1 on $B(0,1)$ and define

$$\phi_r(x) = \phi\left(\frac{x-x_0}{r}\right).$$

Suppose we have restriction for some exponent q. Then

$$\begin{aligned} r_k^\alpha \simeq \int f_k(x)d\mu(x) &= \int \phi_{r_k}(x) f_k(x) d\mu(x) = \int \widehat{\phi}_{r_k}(\xi)\widehat{f_k d\mu}(-\xi)d\xi \\ &= r_k^n \int e^{-2\pi i \xi\cdot x_0}\widehat{\phi}(r_k\xi)\widehat{f_k d\mu}(-\xi)d\xi \\ &\leq r_k^{\frac{n}{q}} \|\widehat{\phi}\|_{q'} \|\widehat{f_k d\mu}\|_q \lesssim r_k^{\frac{n}{q}} \|f_k\|_{L^2(d\mu)} \simeq r_k^{\frac{n}{q}+\frac{\alpha}{2}}. \end{aligned}$$

Since $r_k \searrow 0$ we conclude that

$$q \geq \frac{2n}{\alpha}. \qquad \square$$

Note that the optimality of the range of exponents in the original Stein–Tomas theorem is intimately related to the fact that $d\sigma$ has two special regularity properties: It is a rectifiable $(n-1)$-dimensional measure and its Fourier transform has the best possible decay rate. Here, by the term rectifiable measure, we mean a measure whose support can be covered by a countable union $\bigcup_{i=0}^{\infty} f_i(\mathbb{R}^{n-1})$ of graphs of Lipschitz maps $f_i : \mathbb{R}^{n-1} \to \mathbb{R}^n$, plus a set of zero $(n-1)$-dimensional Hausdorff measure (the reader is refered to Mattila [1] for a complete discussion of the notion of rectifiability). It might, therefore, be of some interest to try to determine the values of $\alpha < n$ for which there exists a measure μ in $\mathbb{R}^n$ such that

$$\mu(B(x,r)) \simeq r^{\alpha}, \quad x \in \operatorname{spt}(\mu),\ 0 \le r \le 1 \tag{5}$$

and

$$|\widehat{\mu}(\xi)| \le \frac{1}{|\xi|^{\alpha/2}}. \tag{6}$$

Clearly, the support of a measure satisfying the above conditions must be a Salem set. It is, however, unknown whether any such measure exists for non-integral values of α. On the other hand, we remark that if there exists a measure μ satisfying (5) and (6) with $\alpha < 2n/3$ then the support of μ fails to be translation invariant in the sense that

$$\mu \times \mu(\{(y,z) : x - y - z \in \operatorname{spt}(\mu)\}) = 0, \quad \forall x \in \operatorname{spt}(\mu).$$

To see this, suppose that for some $x_0 \in \operatorname{spt}(\mu)$ we have

$$\mu \times \mu\big(\{(y,z) : x_0 - y - z \in \operatorname{spt}(\mu)\}\big) > 0$$

and let $\phi_{x_0,r}$ be as in the discussion preceding Corollary 3.1. Then

$$\begin{aligned} r^{\alpha} &\lesssim \iint \mu(B(x_0 - y - z, r))d\mu(y)d\mu(z) = \mu * \mu * \mu(B(x_0, r)) \\ &\lesssim \int \phi_{x_0,r}(y)d(\mu * \mu * \mu)(y) \le r^n \int_{|r\xi| \le \delta} |\widehat{\phi}(r\xi)|\,|\widehat{\mu}(-\xi)|^3 d\xi \\ &\le r^{3\alpha/2} \int_{|\xi| \le \delta} |\widehat{\phi}(\xi)| \frac{1}{|\xi|^{3\alpha/2}} d\xi \lesssim r^{3\alpha/2} \end{aligned}$$

for all $r \leq 1$, which is a contradiction.

No information on the nature of the geometric restrictions that (5) and (6) impose on μ, other than the above-essentially trivial-remark, is currently available.

References

[1] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces, *Cambridge University Press*, 1995.

[2] E. M. Stein, Harmonic Analysis, *Princeton University Press*, 1993.

[3] R. S. Strichartz, Fourier asymptotics of fractal measures, *J. Funct. Anal.* **89** (1990), 154–187.

THEMIS MITSIS
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF JYVÄSKYLÄ
P.O. BOX 35
FIN–40351 JYVÄSKYLÄ
FINLAND

E-mail: mitsis@math.jyu.fi

(Received January 19, 2001; revised August 31, 2001)