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The view-obstruction problem for polygons

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Abstract. In this paper we solve the view-obstruction problem for regular polygons inscribed in the unit circle with one vertex at $(1,0)$.

1. Introduction

Let C be a convex body in $\mathbb{R}^n$. For $\alpha > 0$, define

$$\Delta(\alpha, C) = \left\{\alpha C + \left(m_1 + \tfrac{1}{2}, \dots, m_n + \tfrac{1}{2}\right) : m_1, \dots, m_n \in \mathbb{Z}\right\},$$

where $\mathbb{Z}$ is the set of all non-negative integers.

For positive real numbers $a_1, \dots, a_n$, let $L(a_1, \dots, a_n)$ be the line

$$x_i = a_i t \quad (i = 1, 2, \dots, n).$$

The view-obstruction problem for C is to determine the infimum $K(C)$ of positive real numbers α for which $\Delta(\alpha, C) \cap L(a_1, \dots, a_n) \neq \emptyset$ for all n-tuples $(a_1, \dots, a_n)$ of positive real numbers. Here, and what follows, $\emptyset$ denotes the empty set.

The view-obstruction problem is given its name by T. W. Cusick [1] in 1973 with restricted C. The problems for spheres and cubes have been widely studied (see [2]–[18], [20]). In this paper, we consider the view-obstruction problem for regular polygons.

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Let C_n be the regular n-gon inscribed in unit circle $x_1^2 + x_2^2 = 1$ with one vertex at $(1, 0)$. Then n vertices of C_n are

$$\left(\cos \frac{2s\pi}{n}, \sin \frac{2s\pi}{n} \right), \qquad s = 0, 1, \dots, n-1.$$

Define $\alpha(a_1, a_2)$ by

$$\alpha(a_1, a_2)^{-1} = 2 \max_{0 \le s \le n-1} \left| a_2 \cos \frac{2s\pi}{n} - a_1 \sin \frac{2s\pi}{n} \right| .$$

We prove the following result.

Theorem. *For $n \ge 3$ we have*

$$K(C_n) = \max\{\alpha(1,2),\ \alpha(2,1)\}.$$

2. A proof

If a_1 and a_2 are positive real numbers and a_1/a_2 is irrational, then, for any $\varepsilon > 0$, by Kronecker's theorem, there is a half positive integer point $(m_1 + \frac{1}{2}, m_2 + \frac{1}{2})$ with the distance from $L(a_1, a_2)$ being less than ε. Thus, we may assume that a_1/a_2 is rational. Further, we may assume that a_1 and a_2 are positive integers with $(a_1, a_2) = 1$. If both a_1 and a_2 are positive odd integers, then $L(a_1, a_2)$ passes through $(m_1 + \frac{1}{2}, m_2 + \frac{1}{2})$ with $m_1 = (a_1 - 1)/2$ and $m_2 = (a_2 - 1)/2$. Now we assume that $2 \mid a_1 a_2$ (thus $a_1 \ne a_2$). Let $0 \le s_1 \le n-1$ and $\delta = -1$ or 1 with

$$\alpha(a_1, a_2)^{-1} = 2\delta \left(a_2 \cos \frac{2s_1\pi}{n} - a_1 \sin \frac{2s_1\pi}{n} \right).$$

Since $(a_1, a_2) = 1$, we may take positive integers m_1, m_2 such that

$$a_2 m_1 - a_1 m_2 + \frac{1}{2}(a_2 - a_1) = -\frac{1}{2}\delta.$$

Thus

$$a_2 m_1 - a_1 m_2 + \frac{1}{2}(a_2 - a_1) + \alpha(a_1, a_2) \left(a_2 \cos \frac{2s_1\pi}{n} - a_1 \sin \frac{2s_1\pi}{n} \right) = 0.$$

That is, $L(a_1, a_2)$ goes through
$(m_1 + \frac{1}{2}, m_2 + \frac{1}{2}) + \alpha(a_1, a_2)(\cos \frac{2s_1\pi}{n}, \sin \frac{2s_1\pi}{n})$. Hence

$$L(a_1, a_2) \cap \Delta(\alpha(a_1, a_2), C_n) \neq \emptyset. \tag{1}$$

To complete the proof, we need a lemma.

Lemma. *If a_1, a_2 are distinct positive integers with $(a_1, a_2) = 1$ and $2 \mid a_1a_2$, then*

$$\alpha(a_1, a_2) \le \alpha(2, 1), \quad \textit{if } a_1 > a_2;$$

$$\alpha(a_1, a_2) \le \alpha(1, 2), \quad \textit{if } a_1 < a_2.$$

Proof. We give only the proof for the case $a_1 > a_2$. A proof for the case $a_1 < a_2$ can be given similarly.

For $n = 3$, we have

$$\alpha(a_1, a_2)^{-1} = 2 \max_{0 \le s \le 2} \left| a_2 \cos \frac{2s\pi}{3} - a_1 \sin \frac{2s\pi}{3} \right| = a_2 + \sqrt{3}a_1.$$

By $a_1 > a_2$ we have $\sqrt{3}(a_1 - 2) \ge 0 \ge 1 - a_2$. Combining with the above equality, we have $\alpha(a_1, a_2) \le \alpha(2, 1)$. Now suppose that $n \ge 4$.

Take $0 < \theta(a_1, a_2) < \frac{\pi}{2}$ with

$$a_2 \cos \frac{2s\pi}{n} - a_1 \sin \frac{2s\pi}{n} = \sqrt{a_1^2 + a_2^2} \sin \left(\theta(a_1, a_2) - \frac{2s\pi}{n} \right).$$

Noting that $n \ge 4$ and $\theta(a_1, a_2)$ does not depend on s, we may take an integer s' such that $0 \le s' \le n - 1$ and

$$-\frac{\pi}{2} - \frac{\pi}{4} \le \theta(a_1, a_2) - \frac{2s'\pi}{n} < -\frac{\pi}{2} + \frac{\pi}{4}.$$

Thus, if $\{a_1, a_2\} \neq \{2, 1\}$, then

$$\max_{0 \le s \le n-1} \left| a_2 \cos \frac{2s\pi}{n} - a_1 \sin \frac{2s\pi}{n} \right| \ge \sqrt{a_1^2 + a_2^2} \cos \frac{\pi}{4} \ge \sqrt{13} \cdot \frac{\sqrt{2}}{2}$$
$$> \sqrt{5} \ge \max_{0 \le s \le n-1} \left| \cos \frac{2s\pi}{n} - 2 \sin \frac{2s\pi}{n} \right|.$$

Hence $\alpha(a_1, a_2) \le \alpha(2,1)$. This completes the proof of the lemma. □

Now, we return to prove the theorem. By the lemma and (1) we have

$$L(a_1, a_2) \cap \Delta(\alpha(2,1), C_n) \neq \emptyset, \quad \text{if } a_1 > a_2;$$
$$L(a_1, a_2) \cap \Delta(\alpha(1,2), C_n) \neq \emptyset, \quad \text{if } a_1 < a_2.$$

If $0 < \alpha < \alpha(2,1)$ then, for any β with $0 < \beta \le \alpha$ and any integers m_1, m_2, we have

$$\beta\left|\cos\frac{2s\pi}{n} - 2\sin\frac{2s\pi}{n}\right| < \alpha(2,1)\left|\cos\frac{2s\pi}{n} - 2\sin\frac{2s\pi}{n}\right| \le \frac{1}{2} \le \left|m_1 - 2m_2 - \frac{1}{2}\right|.$$

This means that $L(2,1)$ cannot pass through $(m_1 + \frac{1}{2}, m_2 + \frac{1}{2}) + \beta(\cos\frac{2s\pi}{n}, \sin\frac{2s\pi}{n})$ for any $0 < \beta \le \alpha$, $0 \le s \le n-1$ and any integers m_1, m_2. Hence $L(2,1) \cap \Delta(\alpha, C_n) = \emptyset$. Similarly, $L(1,2) \cap \Delta(\alpha, C_n) = \emptyset$ for $0 < \alpha < \alpha(1,2)$. The above arguments imply that

$$K(C_n) = \max\{\alpha(1,2),\ \alpha(2,1)\}.$$

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