Publ. Math. Debrecen
60 / 1-2 (2002), 107–114

A rigidity theorem for hypersurfaces in a sphere

By LIU XIMIN (Dalian)

Abstract. Let M^n be a closed hypersurface in the unit sphere S^{n+1}. Denote by $|h|^2$ and H the square of the length of its second fundamental form and the mean curvature, respectively, suppose that $|\nabla h|^2 \geq n^2|\nabla H|^2$. If $|h|^2 < 2\sqrt{n-1}$, M^n is a small hypersphere in S^{n+1}. We also characterize all M^n with $|h|^2 = 2\sqrt{n-1}$. When M^n has constant mean curvature, it is just the result of Hou [3].

1. Introduction

Let S^{n+1} be an $(n+1)$-dimensional unit sphere with constant sectional curvature 1, let M^n be an n-dimensional closed hypersurface in S^{n+1}, and $e_1, \dots, e_n$ a local orthonormal frame field on M^n, $\omega_1, \dots, \omega_n$ its dual coframe field. Then the second fundamental form of M^n is

$$h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j.$$

Denote by H the mean curvature and $|h|^2$ the square of the length of the second fundamental form. As it is well known, there are many rigidity results for minimal hypersurfaces or hypersurfaces with constant mean curvature H in S^{n+1} by use of J. Simons' method, for example, see [1], [4], [6], [8] etc. In [3], Hou proved the following theorem:

Mathematics Subject Classification: 53C42, 53A10.
Key words and phrases: hypersurface, sphere, second fundamental form, mean curvature.
This work is supported in part by the National Natural Science Foundation of China.

Theorem A. *Let M^n be a closed hypersurface of constant mean curvature in S^{n+1}. Then*

(1) *If $|h|^2 < 2\sqrt{n-1}$, M^n is a small hypersphere $S^n(r)$ of radius $r = \sqrt{\frac{n}{n+|h|^2}}$.*

(2) *If $|h|^2 = 2\sqrt{n-1}$, M^n is either a small hypersphere $S^n(r_0)$ or $H(r)$-torus $S^1(r) \times S^{n-1}(s)$, where $r_0^2 = \frac{n}{n+2\sqrt{n-1}}$, $r^2 = \frac{1}{\sqrt{n-1}+1}$ and $s^2 = \frac{\sqrt{n-1}}{\sqrt{n-1}+1}$.*

In the present paper, we would like to use Cheng–Yau's self-adjoint operator $\Box$ to prove the following general rigidity theorem.

Theorem B. *Let M^n be an n-dimensional closed hypersurface in S^{n+1} with $|\nabla h|^2 \geq n^2|\nabla H|^2$. Then*

(1) *If $|h|^2 < 2\sqrt{n-1}$, M^n is a small hypersphere $S^n(r)$ of radius $r = \sqrt{\frac{n}{n+|h|^2}}$.*

(2) *If $|h|^2 = 2\sqrt{n-1}$, M^n is either a small hypersphere $S^n(r_0)$ or $H(r)$-torus $S^1(r) \times S^{n-1}(s)$, where $r_0^2 = \frac{n}{n+2\sqrt{n-1}}$, $r^2 = \frac{1}{\sqrt{n-1}+1}$ and $s^2 = \frac{\sqrt{n-1}}{\sqrt{n-1}+1}$.*

Obviously, when M^n has constant mean curvature, Theorem B becomes Theorem A, so Theorem B is an extension of Theorem A.

2. Preliminaries

Let M^n be an n-dimensional closed hypersurface in S^{n+1}. We choose a local orthonormal frame $e_1, \ldots, e_{n+1}$ in S^{n+1} such that at each point of M^n, $e_1, \ldots, e_n$ span the tangent space of M^n and form an orthonormal frame there. Let $\omega_1, \ldots, \omega_{n+1}$ be its dual coframe. In this paper, we use the following convention on the range of indices:

$$1 \leq A, B, C, \cdots \leq n+1; \qquad 1 \leq i, j, k, \cdots \leq n.$$

Then the structure equations of S^{n+1} are given by

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{1}$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD}\omega_C \wedge \omega_D, \tag{2}$$

$$K_{ABCD} = (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{3}$$

Restrict these forms to M^n, we have

$$\omega_{n+1} = 0. \tag{4}$$

From Cartan's lemma we can write

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{5}$$

From these formulas, we obtain the structure equations of M^n:

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{6}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{7}$$

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{8}$$

where R_{ijkl} are the components of the curvature tensor of M^n and

$$h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j \tag{9}$$

is the second fundamental form of M^n. We also have

$$R_{ij} = (n-1)\delta_{ij} + nHh_{ij} - \sum_k h_{ik}h_{kj}, \tag{10}$$

$$n(n-1)(R-1) = n^2H^2 - |h|^2, \tag{11}$$

where R is the normalized scalar curvature, and H the mean curvature.

Define the first and the second covariant derivatives of h_{ij}, say h_{ijk} and h_{ijkl} by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}, \tag{12}$$

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_m h_{mjk}\omega_{mi} + \sum_m h_{imk}\omega_{mj} + \sum_m h_{ijm}\omega_{mk}. \tag{13}$$

Then we have the Codazzi equation

$$h_{ijk} = h_{ikj}, \tag{14}$$

and the Ricci's identity

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj} R_{mikl} + \sum_m h_{im} R_{mjkl}. \tag{15}$$

For a C^2-function f defined on M^n, we define its gradient and Hessian f_{ij} by the following formulas

$$df = \sum_i f_i \omega_i, \qquad \sum_j f_{ij}\omega_j = df_i + \sum_j f_j \omega_{ji}. \tag{16}$$

The Laplacian of f is defined by $\Delta f = \sum_i f_{ii}$.

Let $\phi = \sum_{ij} \phi_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor defined on M^n, where

$$\phi_{ij} = nH\delta_{ij} - h_{ij}. \tag{17}$$

Following Cheng–Yau [2], we introduce an opertator $\Box$ associated to ϕ acting on any C^2-function f by

$$\Box f = \sum_{i,j} \phi_{ij} f_{ij} = \sum_{i,j} (nH\delta_{ij} - h_{ij}) f_{ij}. \tag{18}$$

Since ϕ_{ij} is divergence-free, it follows [2] that the operator $\Box$ is self-adjoint relative to the L^2 inner product of M^n, i.e.,

$$\int_M f\Box g = \int_M g\Box f. \tag{19}$$

We can choose a local frame field $e_1, \dots, e_n$ at any point $p \in M$, such that $h_{ij} = \lambda_i \delta_{ij}$, by use of (18) and (11), we have

$$\begin{aligned} \Box(nH) &= nH\Delta(nH) - \sum_i \lambda_i (nH)_{ii} \\ &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i \lambda_i (nH)_{ii} \\ &= \frac{1}{2}n(n-1)\Delta R + \frac{1}{2}\Delta|h|^2 - n^2|\nabla H|^2 - \sum_i \lambda_i (nH)_{ii}. \end{aligned} \tag{20}$$

On the other hand, through a standard calculation by use of (14) and (15), we get

$$\frac{1}{2}\Delta|h|^2 = \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{21}$$

Putting (21) into (20), we have

$$\Box(nH) = \frac{1}{2}n(n-1)\Delta R + |\nabla h|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{22}$$

Because M^n is closed, we obtain the following formula by integrating (22) and by noting $\int_{M^n} \Delta R = 0$ and $\int_{M^n} \Box(nH) = 0$

$$\int_{M^n} \Big[|\nabla h|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2\Big] = 0. \tag{23}$$

From (8), we have $R_{ijij} = 1 - \lambda_i\lambda_j$, $i \neq j$, and by putting this into (23), we obtain

$$\int_{M^n} \Big[|\nabla h|^2 - n^2|\nabla H|^2 + n|h|^2 - n^2H^2 - |h|^4 + nH\sum_i \lambda_i^3\Big] = 0. \tag{24}$$

Let $\mu_i = \lambda_i - H$ and $|Z|^2 = \sum_i \mu_i^2$, we have

$$\sum_i \mu_i = 0, \quad |Z|^2 = |h|^2 - nH^2, \tag{25}$$

$$\sum_i \lambda_i^3 = \sum_i \mu_i^3 + 3H|Z|^2 + nH^3. \tag{26}$$

From (24)–(26), we get

$$\int_{M^n} \Big[|\nabla h|^2 - n^2|\nabla H|^2 + |Z|^2(n + nH^2 - |Z|^2) + nH\sum_i \mu_i^3\Big] = 0. \tag{27}$$

We need the following algebraic lemma due to M. Okumuru [7] (see also [1]).

Lemma 2.1. *Let μ_i, $i = 1, \dots, n$, be real numbers such that* $\sum_i \mu_i = 0$ *and* $\sum_i \mu_i^2 = \beta^2$, *where* $\beta = \text{constant} \geq 0$. *then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{28}$$

and the equality holds in (28) *if and only if at least* $(n-1)$ *of the* μ_i *are equal.*

By use of Lemma 2.1, we have

$$\int_{M^n}\Big[|\nabla h|^2 - n^2|\nabla H|^2 + |Z|^2(n + nH^2 - |Z|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||Z|\Big] \leq 0. \tag{29}$$

3. Proof of Thoerem B

Now consider the quadratic form $Q(u,t) = u^2 - \frac{n-2}{\sqrt{n-1}}ut - t^2$. By the orthogonal transformation

$$\begin{cases} \bar{u} = \dfrac{1}{\sqrt{2n}}\{(1+\sqrt{n-1}\,)u + (1-\sqrt{n-1}\,)t\} \\ \bar{t} = \dfrac{1}{\sqrt{2n}}\{(\sqrt{n-1}-1)u + (\sqrt{n-1}+1)t\}. \end{cases}$$

$Q(u,t)$ turns into $Q(u,t) = \frac{n}{2\sqrt{n-1}}(\bar{u}^2 - \bar{t}^{\,2})$, where $\bar{u}^2 + \bar{t}^{\,2} = u^2 + t^2 = |h|^2$.

Take $u = \sqrt{n}\,|H|$, $t = |Z|$, then

$$n + nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||Z| - |Z|^2 = n + Q(u,t) = nc + \frac{n(\bar{u}^2 - \bar{t}^{\,2})}{2\sqrt{n-1}}$$

$$= n + \frac{n(-\bar{u}^2 - \bar{t}^{\,2})}{2\sqrt{n-1}} + \frac{n\bar{u}^2}{\sqrt{n-1}} \geq n - \frac{n}{2\sqrt{n-1}}|h|^2. \tag{30}$$

From (29) and (30) we have

$$0 \geq \int_{M^n}\Big\{(|\nabla h|^2 - n^2|\nabla H|^2) + |Z|^2\Big[n - \frac{n}{2\sqrt{n-1}}|h|^2\Big]\Big\}. \tag{31}$$

By the assumption of theorem $|\nabla h|^2 \geq n^2|\nabla H|^2$, we know if $|h|^2 < 2\sqrt{n-1}$, we have $|Z|^2 \equiv 0$, which means that M^n is totally umbilical and hence is a small hypersphere $S^n(r)$, where $r = \sqrt{\frac{n}{n+|h|^2}}$.

If $|h|^2 = 2\sqrt{n-1}$, then equality holds in Lemma 2.1, and it follows that at least $(n-1)$ of λ_i's are equal to one another. When $\lambda_1 = \lambda_2 = \cdots = \lambda_n$, M^n is totally umbilical and hence a small hypersphere $S^n(r_0)$ where $r_0 = \frac{n}{n+2\sqrt{n-1}}$. When M^n is not totally umbilical, there are exactly $(n-1)$ of λ_i's that are equal to one another. Then from [3] we know that M^n is $H(r)$-torus $S^1(r) \times S^{n-1}(s)$, where $r^2 = \frac{1}{\sqrt{n-1}+1}$ and $s^2 = \frac{\sqrt{n-1}}{\sqrt{n-1}+1}$. This completes the proof of Theorem B.

From Theorem B, we have the following corollary immediately.

Corollary 1 [3]**.** *Let M^n be an n-dimensional closed hypersurface with constant mean curvature in S^{n+1}. Then*

(1) *If $|h|^2 < 2\sqrt{n-1}$, M^n is a small hypersphere $S^n(r)$ of radius $r = \sqrt{\frac{n}{n+|h|^2}}$.*

(2) *If $|h|^2 = 2\sqrt{n-1}$, M^n is either a small hypersphere $S^n(r_0)$ or $H(r)$-torus $S^1(r) \times S^{n-1}(s)$, where $r_0^2 = \frac{n}{n+2\sqrt{n-1}}$, $r^2 = \frac{1}{\sqrt{n-1}+1}$ and $s^2 = \frac{\sqrt{n-1}}{\sqrt{n-1}+1}$.*

Corollary 2. *Let M^n be an n-dimensional closed hypersurface with constant normalized scalar curvature R in S^{n+1}. Suppose $R \geq 1$, then*

(1) *If $|h|^2 < 2\sqrt{n-1}$, M^n is a small hypersphere $S^n(r)$ of radius $r = \sqrt{\frac{n}{n+|h|^2}}$.*

(2) *If $|h|^2 = 2\sqrt{n-1}$, M^n is either a small hypersphere $S^n(r_0)$ or $H(r)$-torus $S^1(r) \times S^{n-1}(s)$, where $r_0^2 = \frac{n}{n+2\sqrt{n-1}}$, $r^2 = \frac{1}{\sqrt{n-1}+1}$ and $s^2 = \frac{\sqrt{n-1}}{\sqrt{n-1}+1}$.*

Proof. From (11),

$$n^2H^2 - \sum_{i,j} h_{ij}^2 = n(n-1)(R-1).$$

Taking the covariant derivative of the above expression, and using the fact $R = \text{constant}$, we get

$$n^2HH_k = \sum_{i,j} h_{ij}h_{ijk}.$$

By Cauchy–Schwarz inequality, we have

$$\sum_k n^4H^2(H_k)^2 = \sum_k \Big(\sum_{i,j} h_{ij}h_{ijk}\Big)^2 \leq \Big(\sum_{i,j} h_{ij}^2\Big) \sum_{i,j,k} h_{ijk}^2, \tag{32}$$

that is

$$n^4H^2|\nabla H|^2 \le |h|^2|\nabla h|^2. \tag{33}$$

On the other hand, from $R \ge 1$, we have $n^2H^2 - |h|^2 \ge 0$. Thus

$$|\nabla h|^2 \ge n^2|\nabla H|^2,$$

so Corollary 2 follows from Theorem B. □

Acknowledgement. This note was written during the author's stay at the Max-Planck-Institut für Mathematik in Bonn. The author would like to express his thanks to the institute for its hospitality and support.

References

[1] H. Alencar and M. P. do Carmo, Hypersurfaces with constant mean curvature in spheres, *Proc. Amer. Math. Soc.* **120** (1994), 1223–1229.

[2] S. Y. Cheng and S. T. Yau, Hypersurfaces with constant scalar curvature, *Math. Ann.* **225** (1977), 195–204.

[3] Z. H. Hou, Hypersurfaces in sphere with constant mean curvature, *Proc. Amer. Math. Soc.* **125** (1997), 1193–1196.

[4] H. B. Lawson, Jr., Local rigidity theorems for minimal hypersurfaces, *Ann. of Math.* (2), **89** (1969), 187–197.

[5] H. Li, Global rigidity theorems of hypersurfaces, *Ark. Mat.* **35** (1997), 327–351.

[6] K. Nomizu and B. Smyth, A formula for Simon's type and hypersurfaces, *J. Diff. Geom.* **3** (1969), 367–377.

[7] M. Okumuru, Hypersurfaces and a pinching problem on the second fundamental tensor, *Amer. J. Math.* **96** (1974), 207–213.

[8] J. Simons, Minimal varieties in Riemannian manifolds, *Ann. of Math.* (2) **88** (1968), 62–105.

LIU XIMIN
DEPARTMENT OF APPLIED MATHEMATICS
DALIAN UNIVERSITY OF TECHNOLOGY
DALIAN 116024
CHINA

E-mail: xmliu@dlut.edu.cn

(Received January 31, 2001)