Compactness criteria for the weak convergence
of vector measures in locally convex spaces

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Abstract. We give some compactness and sequential compactness criteria for a set of Radon vector measures in certain locally convex spaces with respect to the weak convergence of vector measures. Our results contain Prokhorov–LeCam's criteria for real measures and apply to the cases which are not covered by März–Shortt's criteria for vector measures in a Banach space. Especially, our criteria apply to the cases that vector measures take values in the space $\mathscr{S}$ of all rapidly decreasing, infinitely differentiable functions, the space $\mathscr{D}$ of all test functions, and the strong duals of those spaces.

1. Introduction

In 1956, YU. V. PROKHOROV [15] gave a compactness criterion for a set of finite non-negative measures on a complete separable metric space with respect to the weak convergence of measures. This criterion was extended by LeCam [12] to real measures on a completely regular space whose compact subsets are all metrizable (see also VARADARAJAN [20], SMOLYANOV and FOMIN [16], and VAKHANIA *et al.* [19]). These results still play an important roll in the study of stochastic convergence in probability theory.

Recently, DEKIERT [4] introduced the notion of weak convergence of vector measures, and MÄRZ and SHORTT [14] gave sequential compactness criteria for Banach space valued vector measures on a metric space.

The purpose of this paper is to give some compactness and sequential compactness criteria for a set of Radon vector measures in certain locally convex spaces, which contain Prokhorov–LeCam's criteria for real measures and apply to the cases which are not covered by März–Shortt's criteria for vector measures in a Banach space. Especially, our criteria apply to the cases that vector measures take values in the space $\mathscr{S}$ of all rapidly decreasing, infinitely differentiable functions, the space $\mathscr{D}$ of all test functions, and the strong duals of those spaces.

In Section 2 we prepare notation and definitions, and recall some necessary results concerning vector measures and an integral of scalar functions with respect to vector measures.

In Section 3 we give a general compactness criterion for a set of vector measures, which take values in a sequentially complete locally convex space and are defined on an arbitrary completely regular space. In the case that vector measures take values in a semi-reflexive space or a semi-Montel space, we show that the relative compactness of a set of Radon vector measures follows from that of the corresponding set of real measures. In this case, we also show that the same is true of the sequential completeness for the space of all Radon vector measures.

In Section 4, using the criteria in Section 3, we show that a set of Radon vector measures is relatively sequentially compact if the corresponding set of real measures is uniformly bounded and uniformly tight under an additional assumption of separability.

In this paper, all the topological spaces and topological linear spaces are Hausdorff, and the scalar fields of topological linear spaces are taken to be the field $\mathbb{R}$ of all real numbers.

2. Notation and preliminaries

Let S be a completely regular space. Denote by $\mathcal{B}(S)$ the σ-field of Borel subsets of S and by $C(S)$ the Banach space of all bounded, continuous real functions on S with the norm $\|f\| \equiv \sup_{s \in S} |f(s)|$. Denote by χ_A the indicator function of a set A. Let X be a locally convex space and X^* the topological dual of X. Denote by $\langle x, x^* \rangle$ the natural duality between X and X^*.

A finitely additive set function $\mu : \mathcal{B}(S) \to X$ is called a *vector measure* if it is countably additive for the original topology of X, i.e., for any

sequence $\{E_n\}$ of pairwise disjoint subsets of $\mathcal{B}(S)$, we have $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$, where the series is unconditionally convergent with respect to the original topology of X.

If μ is a vector measure and $x^* \in X^*$, then $x^*\mu$ defined by $(x^*\mu)(E) = \langle \mu(E), x^* \rangle$, $E \in \mathcal{B}(S)$, is a real measure. Conversely, a theorem of Orlicz and Pettis ensures that a finitely additive set function $\mu : \mathcal{B}(S) \to X$ is countably additive if so is $x^*\mu$ for every $x^* \in X^*$.

We say that a vector measure $\mu : \mathcal{B}(S) \to X$ is *Radon* if for each $x^* \in X^*$, $x^*\mu$ is Radon, i.e., for any $\varepsilon > 0$ and $E \in \mathcal{B}(S)$, there exists a compact subset K of E such that $|x^*\mu|(E - K) < \varepsilon$, where $|x^*\mu|$ denotes the total variation of $x^*\mu$. Denote by $\mathcal{M}_t(S; X)$ the set of all Radon vector measures $\mu : \mathcal{B}(S) \to X$. When $X = \mathbb{R}$, we write $\mathcal{M}_t(S)$ instead of $\mathcal{M}_t(S; \mathbb{R})$. Then, $\mathcal{M}_t(S)$ is a Banach space with the total variation norm $\|m\| \equiv |m|(S)$, and is isometrically embedded into $C(S)^*$ by the natural embedding θ defined by

$$(2.1) \qquad \theta(m)(f) = \int_S f \, dm, \quad m \in \mathcal{M}_t(S), \ f \in C(S)$$

(this can be proved by a similar argument in a proof of Theorem IV.6.2 of DUNFORD and SCHWARTZ [6]).

Let $\{m_\alpha\}$ be a net in $\mathcal{M}_t(S)$ and $m \in \mathcal{M}_t(S)$. We say that $\{m_\alpha\}$ *converges weakly to m*, and write $m_\alpha \xrightarrow{w} m$, if for every $f \in C(S)$, we have

$$\int_S f \, dm_\alpha \to \int_S f \, dm.$$

In what follows, we always equip $\mathcal{M}_t(S)$ with the topology determined by this weak convergence and call it the *weak topology of measures*.

A subset $\mathcal{V}$ of $\mathcal{M}_t(S; X)$ is said to be *uniformly bounded* if $\sup_{\mu \in \mathcal{V}} |x^*\mu|(S) < \infty$ for every $x^* \in X^*$. Then, the principle of uniform boundedness (see Corollary II.3.21 of [6]) ensures that $\mathcal{V}$ is uniformly bounded if and only if $\sup_{\mu \in \mathcal{V}} |\int_S f \, dx^*\mu| < \infty$ for every $x^* \in X^*$ and $f \in C(S)$. For further information on vector measures see DIESTEL and UHL [5], LEWIS [13], and KLUVÁNEK and KNOWLES [11].

In this paper, we need an integral of measurable real functions with respect to vector measures. Let $\mu : \mathcal{B}(S) \to X$ be a vector measure. A Borel measurable real function f on S is said to be *μ-integrable* if (a) f is

$x^*\mu$-integrable for each $x^* \in X^*$, and (b) for each $E \in \mathcal{B}(S)$, there exists an element of X, denoted by $\int_E f d\mu$, such that

$$\left\langle \int_E f d\mu, x^* \right\rangle = \int_E f dx^* \mu$$

for every $x^* \in X^*$. This type of integral is defined by LEWIS [13], and we refer the reader to [13] and [11] for the properties of the integral. We note here that (1) the integral is linear in f, (2) if f is μ-integrable, then the indefinite integral $\lambda(E) = \int_E f d\mu$ is a vector measure on $\mathcal{B}(S)$, and (3) if X is sequentially complete, then every bounded, Borel measurable real function on S is μ-integrable.

3. Compactness criteria

We introduce the notion of weak convergence of vector measures. Assume that X is a sequentially complete locally convex space. Let $\{\mu_\alpha\}$ be a net in $\mathcal{M}_t(S; X)$ and $\mu \in \mathcal{M}_t(S; X)$. We say that $\{\mu_\alpha\}$ *converges weakly to* μ, and write $\mu_\alpha \xrightarrow{w} \mu$, if for every $f \in C(S)$, we have

$$(3.1) \qquad \int_S f d\mu_\alpha \to \int_S f d\mu \quad \text{for the original topology of } X.$$

When the convergence (3.1) is valid for the weak topology $\sigma(X, X^*)$ of X, we say that that $\{\mu_\alpha\}$ *converges σ-weakly to* μ, and write $\mu_\alpha \xrightarrow{\sigma w} \mu$.

The topologies determined by the weak convergence and the σ-weak convergence are called the *weak topology of vector measures* (for short, WTVM) and the *σ-weak topology of vector measures* (for short, σ-WTVM), respectively. These topologies are natural analogies of that defined by DEKIERT [4] for vector measures in Banach spaces, and coincide with the usual weak topology of measures in the case that $X = \mathbb{R}$ (see [15], [12], [20], and [19]).

The following theorem gives a general compactness criterion for a set of Radon vector measures with respect to the σ-WTVM.

Theorem 1. *Let S be a completely regular space and X a sequentially complete locally convex space. Assume that $\mathcal{V} \subset \mathcal{M}_t(S; X)$ satisfies the following two conditions* (a) *and* (b):

(a) *For each $x^* \in X^*$, the set $x^*(\mathcal{V}) \equiv \{x^*\mu : \mu \in \mathcal{V}\}$ is relatively compact in $\mathcal{M}_t(S)$.*

(b) *The set $\left\{\int_S f d\mu : f \in C(S), \|f\| \leq 1, \mu \in \mathcal{V}\right\}$ is relatively weakly compact in X.*

Then, $\mathcal{V}$ is relatively compact in $\mathcal{M}_t(S; X)$ with respect to the σ-WTVM.

To prove Theorem 1, we need a type of Riesz representation theorem for weakly compact operators. Let $\mu \in \mathcal{M}_t(S; X)$. If X is sequentially complete, then we can define a continuous linear operator $T_\mu : C(S) \to X$ by

$$(3.2) \qquad T_\mu(f) = \int_S f d\mu, \quad f \in C(S),$$

which is called the operator *determined by* μ. Recall that a linear operator $T : C(S) \to X$ is said to be *weakly compact* if it maps every bounded subset of $C(S)$ into a relatively weakly compact subset of X. If S is compact, every weakly compact operator from $C(S)$ into X is represented by a Radon vector measure with values in X (see Theorem 3.1 of [13] and also Proposition 1 of [10]). The following proposition is an extension of this representation theorem to the case that the underlying space S is not necessarily compact.

Proposition 1 (cf. [1], [13], [10], [16], [9]). *Let S be a completely regular space and X a locally convex space. Assume that a weakly compact operator $T : C(S) \to X$ satisfies the following condition $(^*)$: For each $\varepsilon > 0$ and $x^* \in X^*$, there exists a compact subset K of S such that $|\langle T(f), x^* \rangle| \leq \varepsilon \|f\|$ for all $f \in C(S)$ with $f(K) = 0$.*
Then, there exists a vector measure $\mu : \mathcal{B}(S) \to X$ such that

(a) *μ is Radon,*

(b) *the closed absolutely convex hull of the range $R(\mu) \equiv \{\mu(B) : B \in \mathcal{B}(S)\}$ of μ is weakly compact,*

(c) *every bounded, Borel measurable, real valued function on S is μ-integrable,*

(d) *$T(f) = \int_S f d\mu$ for all $f \in C(S)$, and*

(e) *$T^*x^* = \theta(x^*\mu)$ for all $x^* \in X^*$, where $\theta : \mathcal{M}(S) \to C(S)^*$ is the natural embedding defined by (2.1).*

Conversely, if a vector measure $\mu : \mathcal{B}(S) \to X$ *satisfies* (a), (b) *and* (c), *then* (d) *defines a weakly compact operator which satisfies* (*) *and* (e). *Further,* μ *is uniquely determined by* (a) *and* (d). *If* X *is quasi-complete, then* (b) *and* (c) *are automatically satisfied for all vector measures.*

PROOF. Because of the lack of a reference in a convenient form, we prove Proposition 1 using Theorem III.2.1 of [3] and an idea in the proof of Theorem 3.1 of [13].

Let $T : C(S) \to X$ be a weakly compact operator. By Proposition 17.2.4 of JARCHOW [8], the second adjoint T^{**} maps $C(S)^{**}$ into X, and is an extension of T.

For each bounded, Borel measurable real function g on S, we put

$$(3.3) \qquad \hat{g}(\theta(m)) = \int_S g \, dm, \qquad m \in \mathcal{M}_t(S).$$

Since the natural embedding $\theta : \mathcal{M}_t(S) \to C(S)^*$ defined by (2.1) is an isometric isomorphism, it is easy to see that $\hat{g}$ is a bounded linear functional on the linear subspace $\theta(\mathcal{M}_t(S))$ of $C(S)^*$. Therefore, by Hahn–Banach theorem, there exists an extension $\tilde{g} \in C(S)^{**}$ of $\hat{g}$ such that $\|\tilde{g}\| = \|\hat{g}\|$.

Fix $x^* \in X^*$ for a moment. By (*), for each $\varepsilon > 0$, there exists a compact subset K of S such that

$$|\langle f, T^*(x^*) \rangle| = |\langle T(f), x^* \rangle| \le \varepsilon \|f\|$$

for all $f \in C(S)$ with $f(K) = 0$. Since $T^*(x^*) \in C(S)^*$, by Theorem III.2.1 of [3] we can find a $m_x^* \in \mathcal{M}_t(S)$ such that

$$\langle T(f), x^* \rangle = \langle f, T^*(x^*) \rangle = \int_S f \, dm_x^*$$

for all $f \in C(S)$, so that we have

$$(3.4) \qquad\qquad T^*(x^*) = \theta(m_x^*)$$

for all $x^* \in X^*$.

Define the set function $\mu : \mathcal{B}(S) \to X$ by

$$(3.5) \qquad\qquad \mu(E) = T^{**}(\widetilde{\chi_E})$$

for all $E \in \mathcal{B}(S)$. Then, it is well-defined, i.e., if $\widetilde{\widetilde{\chi_E}}$ is another extension of χ_E, then we have $T^{**}(\widetilde{\chi_E}) = T^{**}(\widetilde{\widetilde{\chi_E}})$. For, by (3.4) we have $T^* X^* \subset \theta(\mathcal{M}_t(S))$. Hence, for each $x^* \in X^*$, we have

$$\langle T^{**}(\widetilde{\chi_E}), x^* \rangle = \langle T^*(x^*), \widetilde{\chi_E} \rangle = m_x^*(E) = \left\langle T^*(x^*), \widetilde{\widetilde{\chi_E}} \right\rangle = \left\langle T^{**}(\widetilde{\widetilde{\chi_E}}), x^* \right\rangle,$$

which implies that $T^{**}(\widetilde{\chi_E}) = T^{**}(\widetilde{\widetilde{\chi_E}})$.

By (3.3)–(3.5), for each $x^* \in X^*$ and $E \in \mathcal{B}(S)$, we have

$$m_x^*(E) = \widetilde{\chi_E}(\theta(m_x^*)) = \langle T^*(x^*), \widetilde{\chi_E} \rangle = \langle T^{**}(\widetilde{\chi_E}), x^* \rangle$$
$$= \langle \mu(E), x^* \rangle = (x^* \mu)(E),$$

and this implies

(3.6) $$x^* \mu = m_x^*.$$

Since $x^* \in X^*$ is arbitrary, (e) follows from (3.4) and (3.6). The countable additivity of μ follows from (3.6) and Theorem 1.1 of [13]. Since m_x^* is Radon for each $x^* \in X^*$, (a) follows from (3.6). The proof that μ satisfies (b), (c), and (d) is exactly the same as the corresponding proof of Theorem 3.1 of [13].

Conversely, suppose that μ satisfies (a), (b), and (c). Then, the proof of the weak compactness of the operator defined by (d) is the same as the proof of Theorem 3.1 of [13]. Condition (*) follows from (a).

Assume that X is quasi-complete. Then, X is sequentially complete, so that (c) follows. Since the range $R(\mu)$ is weakly compact, the closed absolutely convex hull of $R(\mu)$ is also weakly compact by Krein's theorem (see Theorem IV.11.4 of SCHAEFER [17] and Remark of TWEDDLE [18]).

Finally, the proof of the uniqueness of μ follows from Corollary 2 to Proposition I.3.8 of [19]. $\square$

PROOF of Theorem 1. For each $\mu \in \mathcal{M}_t(S; X)$, we define a continuous linear operator $T_\mu : C(S) \to X$ by

$$T_\mu(f) = \int_S f \, d\mu, \quad f \in C(S).$$

Then it follows from (b) of Theorem 1 that T_μ is weakly compact for every $\mu \in \mathcal{V}$. Let X_σ be the space X with the weak topology $\sigma(X, X^*)$. Denote

by $\mathcal{L}(C(S), X_\sigma)$ the space of all continuous linear operators from $C(S)$ into X_σ, and by $\mathcal{L}_s(C(S), X_\sigma)$ the same space with the topology of simple convergence. We also denote by $X^{C(S)}$ the set of all mappings from $C(S)$ into X. Put $\mathcal{H} = \{T_\mu : \mu \in \mathcal{V}\}$ and denote by $\mathcal{H}_1$ the closure of $\mathcal{H}$ in $X^{C(S)}$ for the topology of simple convergence. Then, it follows from (b) and Tychonoff's theorem that $\mathcal{H}_1$ is compact in $X^{C(S)}$ for the topology of simple convergence. To prove that $\mathcal{H}$ is a relatively compact subset of $\mathcal{L}_s(C(S), X_\sigma)$, we have only to show that $\mathcal{H}_1 \subset \mathcal{L}(C(S), X_\sigma)$. Since (b) implies that the set $\{\langle T_\mu(f), x^* \rangle : \mu \in \mathcal{V}\}$ is bounded for each $x^* \in X^*$ and $f \in C(S)$, it follows from Banach–Steinhaus theorem (see, e.g., Theorem III.4.2 of [17]) that $\mathcal{H}$ is an equicontinuous subset of $\mathcal{L}(C(S), X_\sigma)$. Then $\mathcal{H}_1 \subset \mathcal{L}(C(S), X_\sigma)$ by Theorem III.4.3 of [17]. Thus, we have finished the proof of the relative compactness of $\mathcal{H}$, so that for any net $\{\mu_\alpha\}$ of $\mathcal{V}$, we can find a subnet $\{\mu_{\alpha'}\}$ of $\{\mu_\alpha\}$ and an operator $T \in \mathcal{L}(C(S), X_\sigma)$ such that

$$(3.7) \qquad \langle T(f), x^* \rangle = \lim_{\alpha'} \langle T_{\mu_{\alpha'}}(f), x^* \rangle = \lim_{\alpha'} \left\langle \int_S f d\mu_{\alpha'}, x^* \right\rangle$$

for all $x^* \in X^*$ and $f \in C(S)$.

Now we shall prove that T is weakly compact and satisfies assumption (*) of Proposition 1. Put $D = \{f \in C(S) : \|f\| \leq 1\}$. Then it follows from (b) of Theorem 1 that the set $\bigcup_{\alpha'} T_{\mu_{\alpha'}}(D)$ is relatively weakly compact in X. On the other hand, by (3.7) it is easy to see that $T(D)$ is contained in the closure of $\bigcup_{\alpha'} T_{\mu_{\alpha'}}(D)$ for the weak topology $\sigma(X, X^*)$. Thus, $T(D)$ is relatively weakly compact in X, and this implies that T is weakly compact.

Next we show that T satisfies assumption (*) of Proposition 1. Fix $\varepsilon > 0$ and $x^* \in X^*$. By (3.7), we have

$$(3.8) \qquad |\langle T(f), x^* \rangle| = \lim_{\alpha'} \left| \left\langle \int_S f d\mu_{\alpha'}, x^* \right\rangle \right| = \lim_{\alpha'} \left| \int_S f dx^* \mu_{\alpha'} \right|$$

for all $f \in C(S)$. On the other hand, by (a) of Theorem 1, there exists a further subnet $\{m_{\alpha''}\}$ of $\{x^* \mu_{\alpha'}\}$ and a $m \in \mathcal{M}_t(S)$ such that

$$(3.9) \qquad\qquad\qquad m_{\alpha''} \xrightarrow{w} m.$$

Since m is Radon, there exists a compact subset K of S such that

$$(3.10) \qquad\qquad\qquad |m|(S - K) < \varepsilon.$$

Fix $f \in C(S)$ with $f(K) = 0$. Then, it follows from (3.8), (3.9) and (3.10) that

$$|\langle T(f), x^* \rangle| = \lim_{\alpha''} \left| \int_S f \, dm_{\alpha''} \right| = \left| \int_S f \, dm \right|$$

$$= \left| \int_{S-K} f \, dm \right| \leq \|f\| \cdot |m| \, (S - K) \leq \varepsilon \|f\|,$$

and this implies that T satisfies assumption $(*)$ of Proposition 1. Thus, by Proposition 1, we can find a vector measure $\mu \in \mathcal{M}_t(S; X)$ such that

$$T(f) = \int_S f \, d\mu$$

for all $f \in C(S)$. Hence by (3.7), for each $x^* \in X^*$ we have

$$\lim_{\alpha'} \left\langle \int_S f \, d\mu_{\alpha'}, x^* \right\rangle = \left\langle \int_S f \, d\mu, x^* \right\rangle,$$

and this implies the relative weak compactness of $\mathcal{V}$.   □

We say that $M \subset \mathcal{M}_t(S)$ is *uniformly tight* if for each $\varepsilon > 0$, there exists a compact subset K of S such that $|m|(S - K) < \varepsilon$ for all $m \in M$. Then it is well-known that every uniformly bounded and uniformly tight subset M of $\mathcal{M}_t(S)$ is relatively compact in $\mathcal{M}_t(S)$ (see [16], [12], and [20]).

A locally convex space X for which $X = X^{**}$ (more precisely for which the canonical embedding of X into X^{**} is surjective) is said to be *semi-reflexive*. It is known that bounded sets in a semi-reflexive space are relatively weakly compact, and every semi-reflexive space is quasi-complete, and hence sequentially complete (see IV.5.5 and Corollary 1 to IV.5.5 of [17]). We say that a locally convex space X is a *semi-Montel space* if every bounded subset of X is relatively compact. In such a space, $\sigma(X, X^*)$ and the original topology of X coincide on bounded sets, so that X is in particular quasi-complete with respect to the weak topology $\sigma(X, X^*)$. Hence it is semi-reflexive by IV.5.5 of [17].

The following contains Prokhorov-LeCam's compactness criteria for real measures and applies to the cases that vector measures take values in $\mathscr{S}$, $\mathscr{D}$, and the strong duals of those spaces, which are important examples of semi-Montel spaces.

Corollary 1. *Let S be a completely regular space and X a semi-reflexive locally convex space. Let $\mathcal{V} \subset \mathcal{M}_t(S; X)$ and assume that for each $x^* \in X^*$, $x^*(\mathcal{V})$ is relatively compact in $\mathcal{M}_t(S)$ (this condition is satisfied if, for instance, $x^*(\mathcal{V})$ is uniformly bounded and uniformly tight). Then, $\mathcal{V}$ is relatively compact in $\mathcal{M}_t(S; X)$ with respect to the σ-WTVM. When X is a semi-Montel space, $\mathcal{V}$ is relatively compact with respect to the WTVM.*

PROOF. Put $W = \left\{ \int_S f d\mu : f \in C(S), \ \|f\| \le 1, \mu \in \mathcal{V} \right\}$. Since $x^*(\mathcal{V})$ is relatively compact for each $x^* \in X^*$, for every $f \in C(S)$ we have

$$\sup_{\mu \in \mathcal{V}} \left| \int_S f dx^* \mu \right| < \infty,$$

so that it is easy to see that the set W is a bounded subset of X. Thus, it follows form Theorem 1 that $\mathcal{V}$ is relatively compact with respect to the σ-WTVM.

Assume that X is a semi-Montel space. Since every semi-Montel space is semi-reflexive, $\mathcal{V}$ is relatively compact with respect to the σ-WTVM. Consequently, for any net $\{\mu_\alpha\}$ in $\mathcal{V}$ there exist a subnet $\{\mu_{\alpha'}\}$ of $\{\mu_\alpha\}$ and a $\mu \in \mathcal{M}_t(S; X)$ such that for every $f \in C(S)$, we have

$$\int_S f d\mu_{\alpha'} \to \int_S f d\mu \quad \text{for the weak topology } \sigma(X, X^*) \text{ of } X.$$

In the case that $\|f\| \le 1$, the set $\{\int_S f d\mu_{\alpha'}\}$ is contained in the bounded subset W of X. Since X is a semi-Montel space, we have

$$\int_S f d\mu_{\alpha'} \to \int_S f d\mu \quad \text{for the original topology of } X$$

(see, e.g., [8, p. 230]), and we conclude that $\mathcal{V}$ is relatively compact with respect to the WTVM. For general f, we have only to consider $f/\|f\|$ instead of f. $\qquad\square$

We say that $\mathcal{M}_t(S)$ is *sequentially complete* if it is sequentially complete with respect to the usual weak topology of measures. As an application of Corollary 1, we have a criterion for sequential completeness of vector measures.

Corollary 2. *Let S be a completely regular space and X a semi-reflexive locally convex space. If $\mathcal{M}_t(S)$ is sequentially complete (this is satisfied, if S is a R-space in the sense of Dalecky–Smolyanov–Fomin [3]), then $\mathcal{M}_t(S; X)$ is sequentially complete with respect to the σ-WTVM. When X is a semi-Montel space, the same conclusion holds with respect to the WTVM.*

PROOF. We give a proof only in the case that X is a semi-Montel space. A proof of the case that X is semi-reflexive is similar.

Assume that X is a semi-Montel space. Let $\{\mu_n\}$ be a Cauchy sequence in $\mathcal{M}_t(S; X)$ with respect to the WTVM, and put $\mathcal{V} = \{\mu_n\}$. Then, for each $x^* \in X^*$, the sequence $\{x^*\mu_n\}$ is Cauchy in $\mathcal{M}_t(S)$. Since $\mathcal{M}_t(S)$ is sequentially complete by assumption, $x^*(\mathcal{V}) = \{x^*\mu_n\}$ converges in $\mathcal{M}_t(S)$, so that it is relatively compact in $\mathcal{M}_t(S)$. It follows from Corollary 1 that $\mathcal{V}$ is relatively compact in $\mathcal{M}_t(S; X)$. Therefore, the closure of $\mathcal{V}$ in $\mathcal{M}_t(S; X)$ is compact and hence sequentially complete. Thus $\{\mu_n\}$ converges in $\mathcal{M}_t(S; X)$ and the proof is complete.   □

4. Sequential compactness criteria

In this section, we turn our attention to sequential compactness criteria for vector measures with values in a semi-reflexive space or a semi-Montel space. The following theorem contains Prokhorov–LeCam's sequential compactness criteria for real measures (see [15], [12], and [16]) and applies to the following cases which are not covered by März–Shortt's criteria [14]: (1) The topological space S on which the vector measures are defined is not necessarily metrizable; and (2) the locally convex space X in which the vector measures take values is not necessarily normable.

Theorem 2. *Let S be a completely regular space whose compact subsets are all metrizable. Let X be a semi-reflexive space whose topological dual X^* has a countable set which separates points of X. Let $\mathcal{V} \subset \mathcal{M}_t(S; X)$ and assume that for each $x^* \in X^*$, $x^*(\mathcal{V})$ is uniformly bounded and uniformly tight. Then, $\mathcal{V}$ is relatively compact and metrizable, so that it is relatively sequentially compact in $\mathcal{M}_t(S; X)$ with respect to the σ-WTVM. When X is a semi-Montel space, the same conclusion holds with respect to the WTVM.*

Remark 1. It is routine to check that the condition that X^* has a countable set which separates points of X is equivalent to X^* being separable for the weak topology $\sigma(X^*, X)$.

Remark 2. Theorem 2 extends Corollary 1.6 of [14] to the cases that vector measures take values in a semi-reflexive space or a semi-Montel space, though we assume an additional assumption of separability. However, it should be remarked that the proof of Corollary 1.6 of [14] requires the following type of uniform tightness condition: Let X be a Banach space. $\mathcal{V} \subset \mathcal{M}_t(S; X)$ is *uniformly tight* if for each $\varepsilon > 0$, there exists a compact subset K of S such that $\sup_{\mu \in \mathcal{V}} \|\mu\|(S - K) < \varepsilon$, where $\|\mu\|$ is the semivariation of μ defined by $\|\mu\|(E) = \sup_{\|x^*\| \leq 1} |x^*\mu|(E)$, $E \in \mathcal{B}(S)$.

To prove the theorem, we need the following lemma:

Lemma 1. *Let S be a space as in Theorem 2 above and X a locally convex space. Let $\mathcal{V} \subset \mathcal{M}_t(S; X)$ satisfy conditions in Theorem 2. Then, for each $x^* \in X^*$, there exists a countable subset I of $C(S)$ which satisfies the following condition: For any $\varepsilon > 0$ and $f \in C(S)$, we can find a function $g \in I$ such that*

$$(4.1) \qquad \left| \int_S (f - g) dx^* \mu \right| \leq \varepsilon \left(|x^* \mu|(S) + 2\|f\| + \varepsilon \right)$$

for all $\mu \in \mathcal{V}$.

PROOF. Fix $x^* \in X^*$. By assumption, there exists a sequence $\{K_n\}$ of compact subsets of S such that

$$(4.2) \qquad \sup_{\mu \in \mathcal{V}} |x^* \mu|(S - K_n) < \frac{1}{n}.$$

Since each K_n is metrizable, $C(K_n)$ is separable. Fix $n \geq 1$ for a moment, and let $\{g_{i,n}\}_{i=1}^{\infty}$ be a countable dense subset of $C(K_n)$. Then, each $g_{i,n}$ has an extension $\tilde{g}_{i,n} \in C(S)$ such that

$$(4.3) \qquad \|\tilde{g}_{i,n}\| = \|g_{i,n}\|_{K_n} \equiv \sup_{s \in K_n} |g_{i,n}(s)|.$$

Put $I = \{\tilde{g}_{i,n}\}_{i,n=1}^{\infty}$. Fix $f \in C(S)$ and $\varepsilon > 0$, and choose n_0 such that $1/n_0 < \varepsilon$. We set $f_{n_0} = f\!\upharpoonright_{K_{n_0}}$ (the restriction of f onto K_{n_0}) $\in C(K_{n_0})$, then there exists a $g_{i_0, n_0} \in C(K_{n_0})$ such that

$$(4.4) \qquad \|f_{n_0} - g_{i_0, n_0}\|_{K_{n_0}} < \frac{1}{n_0},$$

since $\{g_{i,n_0}\}_{i=1}^{\infty}$ is dense in $C(K_{n_0})$.

On the other hand, by (4.3) and (4.4) we have

$$\|f - \tilde{g}_{i_0,n_0}\| \leq \|f\| + \|\tilde{g}_{i_0,n_0}\| = \|f\| + \|g_{i_0,n_0}\|_{K_{n_0}}$$

$$\leq \|f\| + \left(\frac{1}{n_0} + \|f_{n_0}\|_{K_{n_0}}\right) \leq 2\|f\| + \frac{1}{n_0}.$$

By (4.2), (4.4), and the inequality above, for each $\mu \in \mathcal{V}$ we have

$$\left|\int_S (f - \tilde{g}_{i_0,n_0})dx^*\mu\right| \leq \left|\int_{K_{n_0}} (f_{n_0} - g_{i_0,n_0})dx^*\mu\right| + \left|\int_{S-K_{n_0}} (f - \tilde{g}_{i_0,n_0})dx^*\mu\right|$$

$$\leq |x^*\mu|(K_{n_0}) \cdot \|f_{n_0} - g_{i_0,n_0}\|_{K_{n_0}} + |x^*\mu|(S - K_{n_0}) \cdot \|f - \tilde{g}_{i_0,n_0}\|$$

$$\leq \frac{1}{n_0}|x^*\mu|(S) + \frac{1}{n_0}\|f - \tilde{g}_{i_0,n_0}\| \leq \frac{1}{n_0}|x^*\mu|(S) + \frac{1}{n_0}\left(2\|f\| + \frac{1}{n_0}\right)$$

$$\leq \varepsilon\left(|x^*\mu|(S) + 2\|f\| + \varepsilon\right).$$

Hence, the proof of Lemma 1 is complete if we put $g = \tilde{g}_{i_0,n_0} \in I$. □

PROOF of Theorem 2. Denote by τ the σ-WTVM on $\mathcal{M}_t(S;X)$. Let $\mathcal{V}_1$ be the τ-closure of $\mathcal{V}$. Then, by Corollary 1, $\mathcal{V}_1$ is τ-compact in $\mathcal{M}_t(S;X)$, so that we have only to show that the relative topology of τ onto $\mathcal{V}_1$, denoted by τ_1, is metrizable. For this end, we show that there exists a metric topology on $\mathcal{V}_1$ which is coarser than τ_1 (see, e.g., Lemma I.5.8 of [6]).

Note that for each $x^* \in X^*$, $x^*(\mathcal{V}_1)$ is the closure of $x^*(\mathcal{V})$ in $\mathcal{M}_t(S)$. Hence it follows from Proposition 11 of [2], Chapter IX, §5, no. 5, $\mathcal{V}_1$ itself satisfies conditions in Theorem 2. Then, we have the following

Lemma 2. *For each $x^* \in X^*$, there exists a semi-metric d_x^* on $\mathcal{V}_1$ which satisfies the following two conditions* (a) *and* (b):

(a) *The relative topology τ_1 on $\mathcal{V}_1$ is finer than the topology generated by d_x^*.*

(b) *Let $\mu_1, \mu_2 \in \mathcal{V}_1$. Then, $d_x^*(\mu_1, \mu_2) = 0$ implies that $x^*\mu_1 = x^*\mu_2$.*

PROOF. Fix $x^* \in X^*$. Let $I = \{g_m\}_{m=1}^{\infty}$ be a countable subset of $C(S)$ in Lemma 1. Let $\{x_l^*\}_{l=1}^{\infty}$ be a countable subset of X^* which separates points of X.

Define a semi-metric d_x^* on $\mathcal{V}_1$ by

$$d_x^*(\mu_1, \mu_2) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{2^l} \cdot \frac{1}{2^m} \cdot \frac{|\langle \int_S g_m d\mu_1 - \int_S g_m d\mu_2, x_l^* \rangle|}{1 + |\langle \int_S g_m d\mu_1 - \int_S g_m d\mu_2, x_l^* \rangle|}$$

for all $\mu_1, \mu_2 \in \mathcal{V}_1$.

It is easy to prove (a), so that we prove (b). Assume that $d_x^*(\mu_1, \mu_2)=0$, $\mu_1, \mu_2 \in \mathcal{V}_1$. Then we have

$$\left\langle \int_S g_m d\mu_1 - \int_S g_m d\mu_2, x_l^* \right\rangle = 0$$

for all $l \geq 1$ and $m \geq 1$. Since $\{x_l^*\}_{l=1}^{\infty}$ separates points of X, we have

$$(4.5) \qquad \int_S g_m d\mu_1 = \int_S g_m d\mu_2$$

for all $m \geq 1$.

Fix $f \in C(S)$ and $\varepsilon > 0$. By Lemma 1, there exists a $g_{m_0} \in I$ such that

$$(4.6) \qquad \left| \int_S (f - g_{m_0}) dx^* \mu \right| \leq \varepsilon \left(|x^*\mu|(S) + 2\|f\| + \varepsilon \right)$$

for all $\mu \in \mathcal{V}_1$. Thus, by (4.5) and (4.6) we have

$$\left| \int_S f dx^* \mu_1 - \int_S f dx^* \mu_2 \right| \leq \left| \int_S (f - g_{m_0}) dx^* \mu_1 \right|$$

$$+ \left| \int_S g_{m_0} dx^* \mu_1 - \int_S g_{m_0} dx^* \mu_2 \right| + \left| \int_S (g_{m_0} - f) dx^* \mu_2 \right|$$

$$\leq \varepsilon \left(|x^*\mu_1|(S) + 2\|f\| + \varepsilon \right) + \varepsilon \left(|x^*\mu_2|(S) + 2\|f\| + \varepsilon \right).$$

Since ε is arbitrary, we have

$$\int_S f dx^* \mu_1 = \int_S f dx^* \mu_2$$

for all $f \in C(S)$. Since $x^*\mu_1$ and $x^*\mu_2$ are Radon, it follows from Corollary 2 to Proposition I.3.8 of [19] that $x^*\mu_1 = x^*\mu_2$. $\square$

We complete the proof of Theorem 2. Let $\{x_n^*\}_{n=1}^\infty$ be a countable subset of X^* which separates points of X. Put $d_n = d_{x_n^*}$ for simplicity and define a semi-metric d on $\mathcal{V}_1$ by

$$d(\mu_1, \mu_2) = \sum_{n=1}^\infty \frac{1}{2^n} \cdot \frac{d_n(\mu_1, \mu_2)}{1 + d_n(\mu_1, \mu_2)}$$

for all $\mu_1, \mu_2 \in \mathcal{V}_1$. Then, it is easy to verify that τ_1 is finer than the topology generated by the semi-metric d. To prove that d is actually a metric, we assume that $d(\mu_1, \mu_2) = 0$, $\mu_1, \mu_2 \in \mathcal{V}_1$. Then, $d_n(\mu_1, \mu_2) = 0$ for all $n \geq 1$, so that by (b) of Lemma 2, we have $x_n^*\mu_1 = x_n^*\mu_2$ for all $n \geq 1$. Since $\{x_n^*\}$ separates points of X, we conclude that $\mu_1 = \mu_2$, and the proof of Theorem 2 is complete in the case that X is semi-reflexive.

Assume that X is a semi-Montel space and denote by ξ the WTVM on $\mathcal{M}_t(S; X)$. Let $\mathcal{V}_2$ be the ξ-closure of $\mathcal{V}$. Then, the same argument as is used above shows that $\mathcal{V}_2$ is ξ-compact and the relative topology of ξ onto $\mathcal{V}_2$ is metrizable. Consequently, $\mathcal{V}$ is relatively compact with respect to the WTVM. $\square$

The following result gives a sequential compactness criterion for vector measures with values in $\mathscr{S}$, $\mathscr{D}$, and the strong duals $\mathscr{S}_\beta^*$ and $\mathscr{D}_\beta^*$.

Corollary 3. *Let S be a completely regular space whose compact subsets are all metrizable. Let Φ be a Fréchet–Montel space or the strict inductive limit of an increasing sequence of Fréchet–Montel spaces or the strong duals of those spaces. Let $\mathcal{V} \subset \mathcal{M}_t(S; \Phi)$ and assume that for each $x^* \in \Phi^*$, $x^*(\mathcal{V})$ is uniformly bounded and uniformly tight. Then, $\mathcal{V}$ is relatively compact and metrizable, so that it is relatively sequentially compact in $\mathcal{M}_t(S; \Phi)$ with respect to the WTVM.*

PROOF. It is well-known that Φ is a Montel space. By Corollary V.1.18 of [7], Φ and Φ_β^* are Suslin spaces, so that they are separable by Theorem III.1.2 of [7]. Thus, Φ_β^* has a countable subset which separates points of Φ. Consequently, the relative sequential compactness of $\mathcal{V}$ follows from Theorem 2. $\square$

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