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On the mean curvature of a unit vector field

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Abstract. We present an explicit formula for the mean curvature of a unit vector field on a Riemannian manifold, using a special but natural frame. As applications, we treat some known and new examples of minimal unit vector fields. We also give an example of a vector field of constant mean curvature on the Lobachevsky $(n+1)$ space.

Introduction

Let (M, g) be an $n+1$-dimensional Riemannian manifold with metric g. A vector field ξ on it is called *holonomic* if ξ is a field of normals of some family of regular hypersurfaces in M and *non-holonomic* otherwise. The foundation of the classical geometry of unit vector fields was proposed by A. Voss at the end of the nineteenth century. The theory includes the *Gaussian* and *the mean curvature* of a vector field and their generalizations (see [1] for details). Here we will consider a unit vector field from another point of view. Namely, let T_1M be the unit tangent sphere bundle of M endowed with the Sasaki metric [16]. If ξ is a unit vector field on M, then one may consider ξ as a mapping $\xi : M \to T_1M$ so that the image $\xi(M)$ is a submanifold in T_1M with the metric induced from T_1M. H. Gluck and W. Ziller [10] called ξ *a minimal vector field* if $\xi(M)$ is of minimal volume with respect to induced metric. They considered the unit vector field on S^3 tangent to the fibers of a Hopf fibration $S^3 \xrightarrow{S^1} S^2$ and proved that these (Hopf) vector fields are unique ones with global minimal volume. Note that this result is not true for greater dimensions where Hopf vector fields are still critical points for the volume functional but do not

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provide the global minimum among all unit vector fields [14], [15]. The local aspect of the problem was considered first in [8]. The authors have found the necessary and sufficient condition for a unit vector field to generate locally a minimal submanifold in the tangent sphere bundle. In fact, that condition implies that *the mean curvature* of the submanifold $\xi(M)$ is zero. Using that criterion, a number of examples of local minimal vector unit fields have been found (see [3], [4], [11], [12], [17], [18]).

In this paper, we give an *explicit formula* for the mean curvature of $\xi(M)$ using some special but natural normal frame for $\xi(M)$ and give an example of a unit vector field of *constant mean curvature* on a Lobachevsky space. We shall state the main result after some preliminaries.

Let ∇ denote the Levi–Civita connection on M. Then $\nabla_X\xi$ is always orthogonal to ξ and hence, $(\nabla\xi)(X) = \nabla_X\xi : T_pM \to \xi_p^\perp$ is a linear operator at each $p \in M$. We define the adjoint operator $(\nabla\xi)^*(X) : \xi_p^\perp \to T_pM$ by

$$\langle(\nabla\xi)^*X, Y\rangle_g = \langle X, \nabla_Y\xi\rangle_g.$$

Then there is an orthonormal frame $e_0, e_1, \dots, e_n$ in T_pM and an orthonormal frame $f_1, \dots, f_n$ in $\xi_p^\perp$ such that

$$(\nabla\xi)(e_0) = 0, \quad (\nabla\xi)(e_\alpha) = \lambda_\alpha f_\alpha, \quad (\nabla\xi)^*(f_\alpha) = \lambda_\alpha e_\alpha, \qquad \alpha = 1, \dots, n,$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ are the singular values of $\nabla\xi$. As we will see, the vectors

$$\widetilde{n}_{\sigma|} = \frac{1}{\sqrt{1+\lambda_\sigma^2}}\big(-\lambda_\sigma e_\sigma^H + f_\sigma^V\big), \qquad \sigma = 1, \dots, n,$$

where H and V are the horizontal and vertical lifts respectively, form an orthonormal frame in the normal bundle of $\xi(M)$.

Furthermore, we introduce the notation

$$r(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_{\nabla_XY}\xi.$$

Then $R(X,Y)\xi = r(X,Y)\xi - r(Y,X)\xi$, where R is the Riemannian curvature tensor. Now we are able to state our main result.

Theorem 2.5. *Let $H_{\sigma|}$ be the components of the mean curvature vector of $\xi(M)$ with respect to the orthonormal frame $\widetilde{n}_\sigma$. Then*

$$(n+1)H_{\sigma|} = \frac{1}{\sqrt{1+\lambda_\sigma^2}}$$
$$\times \left\{ \langle r(e_0,e_0)\xi, f_\sigma\rangle + \sum_{\alpha=1}^{n} \frac{\langle r(e_\alpha,e_\alpha)\xi, f_\sigma\rangle + \lambda_\sigma\lambda_\alpha\langle R(e_\sigma,e_\alpha)\xi, f_\alpha)\rangle}{1+\lambda_\alpha^2} \right\}.$$

The following very simple example gives a unit vector field of *constant mean curvature*.

Proposition 3.6.1. *Let M be the Lobachevsky 2-plane with the metric*

$$ds^2 = du^2 + e^{2u}dv^2.$$

Let $X_1 = \{1,0\}$ and $X_2 = \{0, e^{-u}\}$. Then $\xi = \cos\omega X_1 + \sin\omega X_2$, where $\omega = au+b$, generates a hypersurface $\xi(M) \subset T_1M$ of constant mean curvature

$$H = \frac{a}{2\sqrt{2+a^2}}.$$

Index convention. Throughout the paper we take $i,j,k,\ldots=0,\ldots,n$ and $\alpha,\beta,\ldots=1,\ldots,n$.

1. Basic concepts from the geometry of the unit tangent sphere bundle

Let $(u^0,\ldots,u^n)$ be a local coordinate system on M and let $\partial/\partial u^i$ be the vectors of a natural frame on M^n. The points of the tangent bundle TM are the pairs $\widetilde{Q} = (Q,\xi)$, where $Q \in M$ and $\xi \in T_QM$. Each point $\widetilde{Q} \in TM$ is uniquely determined by the set of parameters $(u^0,\ldots,u^n;\xi^0,\ldots,\xi^n)$, where $(u^0,\ldots,u^n)$ fix the point Q and $\{\xi^0,\ldots,\xi^n\}$ are the coordinates of ξ with respect to the frame $\{\partial/\partial u^0,\ldots,\partial/\partial u^n\}$. The local coordinates $(u^0,\ldots,u^n;\xi^0,\ldots,\xi^n)$ are called *natural induced coordinates* in the tangent bundle. Each smooth tangent vector field $\xi = \xi(u^0,\ldots,u^n)$ generates a smooth submanifold $\xi(M) \subset TM$ having a parametric representation of the form

$$\begin{cases} u^i = u^i, \\ \xi^i = \xi^i(u^0,\ldots,u^n). \end{cases} \tag{1}$$

Setting $|\xi| = 1$, we get a submanifold in the unit tangent sphere bundle $\xi(M^n) \subset T_1M^n$.

A natural Riemannian metric on the tangent bundle has been defined by S. Sasaki [16]. We describe it in terms of the *connection map.*

The tangent space $T_{\widetilde{Q}}TM$ can be split into *vertical* and *horizontal* parts:

$$T_{\widetilde{Q}}TM^n = H_{\widetilde{Q}}TM^n \oplus V_{\widetilde{Q}}TM^n.$$

The vertical part $V_{\widetilde{Q}}TM$ is tangent to the fiber, while the horizontal part is transversal to it. For $\widetilde{X} \in T_{\widetilde{Q}}TM^n$ we have

$$\widetilde{X} = \widetilde{X}^i \partial/\partial u^i + \widetilde{X}^{n+i} \partial/\partial \xi^i \tag{2}$$

with respect to the natural frame $\{\partial/\partial u^i, \partial/\partial \xi^i\}$ on TM.

Let $\pi : TM \to M$ be the projection map. It is easy to check that the differential $\pi_* : T_{\widetilde{Q}}TM \to T_QM$ of the mapping π acts on $\widetilde{X}$ as follows:

$$\pi_* \widetilde{X} = \widetilde{X}^i \partial/\partial u^i, \tag{3}$$

and is a linear isomorphism between $V_{\widetilde{Q}}TM$ and T_QM.

The *connection map* $K : T_{\widetilde{Q}}TM \to T_QM$ acts on $\widetilde{X}$ by

$$K\widetilde{X} = (\widetilde{X}^{n+i} + \Gamma^i_{jk} \xi^j \widetilde{X}^k) \partial/\partial u^i \tag{4}$$

and it is a linear isomorphism between $H_{\widetilde{Q}}TM$ and T_QM. Moreover, it is easy to see that $V_{\widetilde{Q}}TM = \ker \pi_*$, $H_{\widetilde{Q}}TM = \ker K$. The images $\pi_* \widetilde{X}$ and $K\widetilde{X}$ are called *horizontal* and *vertical* projections of $\widetilde{X}$, respectively.

The *Sasaki metric* on TM is defined by the following scalar product: if $\widetilde{X}, \widetilde{Y} \in T_{\widetilde{Q}}TM$, then

$$\big\langle\langle \widetilde{X}, \widetilde{Y} \rangle\big\rangle_S = \langle \pi_* \widetilde{X}, \pi_* \widetilde{Y} \rangle_g + \langle K\widetilde{X}, K\widetilde{Y} \rangle_g \tag{5}$$

where $\langle \, , \, \rangle_g$ is the scalar product with respect to the metric g on the initial manifold (the base space of tangent bundle). Horizontal and vertical subspaces are mutually orthogonal with respect to Sasaki metric.

The inverse operations of projections (3) and (4) are called *lifts.* Namely, if $X \in T_QM^n$, then

$$X^H = X^i \partial/\partial u^i - \Gamma^i_{jk} \xi^j X^k \partial/\partial \xi^i$$

is in $H_{\tilde{Q}}TM$ and is called the *horizontal lift* of X, and

$$X^V = X^i \partial/\partial \xi^i$$

is in $V_{\tilde{Q}}TM$ and is called the *vertical lift* of X.

Among all lifts of various vectors from $T_Q M$ into $T_{(Q,\xi)}TM$, one can naturally distinguish two of them, namely ξ^H and ξ^V. The vector field ξ^H is the *geodesic flow* vector field, while ξ^V (being normalized) is a *unit normal* vector field of $T_1 M \subset TM$.

In the geometry of the *unit tangent sphere bundle* it appears to be convenient to introduce the notion of *tangential lift* [5]:

$$X^t = X^V - \langle X, \xi \rangle \xi^V. \tag{6}$$

In other words, the tangential lift is the projection of the vertical lift onto the tangent space of $T_1 M$.

We denote by $\tilde{\nabla}$ the Levi–Civita connection of the Sasaki metric on $T_1 M$. In terms of horizontal and tangential lifts we then have [5]:

$$\begin{aligned} \tilde{\nabla}_{X^H} Y^H &= (\nabla_X Y)^H - \frac{1}{2}(R(X,Y)\xi)^t, & \tilde{\nabla}_{X^t} Y^H &= \frac{1}{2}(R(\xi, X)Y)^H, \\ \tilde{\nabla}_{X^H} Y^t &= (\nabla_X Y)^t + \frac{1}{2}(R(\xi_1, Y)X)^H, & \tilde{\nabla}_{X^t} Y^t &= -\langle Y, \xi \rangle X^t. \end{aligned} \tag{7}$$

Remark. It is evident that if $Z \perp \xi$, the vertical and tangential lifts of Z coincide, particulary $(\nabla_X \xi)^t = (\nabla_X \xi)^V$ for any X. We will use this fact throughout the paper without special comments.

2. The mean curvature formula for a unit vector field

2.1. The structure of tangent and normal bundles of $\xi(M)$

Let ξ be the unit tangent vector field on M. We denote by $T\xi(M)$ the tangent bundle of $\xi(M) \subset T_1 M$. The structure of $T\xi(M)$ can be described as follows:

Lemma 2.1. *The vector $\widetilde{X} \in T_{(Q,\xi)}T_1M$ is tangent to $\xi(M)$ at (Q,ξ) if and only if*

$$\widetilde{X} = X^H + (\nabla_X \xi)^V \tag{8}$$

where $X \in T_QM$.

PROOF. Using the local representation (1) of $\xi(M)$, we consider the coordinate frame of $T_{(Q,\xi)}\xi(M)$:

$$\widetilde{e}_i = \left\{0, \dots, 1, 0, \dots, 0;\ \frac{\partial \xi^0}{\partial u^i}, \dots, \frac{\partial \xi^n}{\partial u^i}\right\}.$$

Let $\widetilde{X} \in T_{(Q,\xi)}TM$ be tangent to $\xi(M)$. Then

$$\widetilde{X} = \widetilde{X}^i \widetilde{e}_i.$$

Applying (3) and (4), we obtain

$$\pi_* \widetilde{e}_i = \partial/\partial u^i, \quad K\widetilde{e}_i = \nabla_i \xi.$$

From this we get

$$\pi_* \widetilde{X} = \widetilde{X}^i \partial/\partial u^i, \quad K\widetilde{X} = \nabla_{\pi_* \widetilde{X}} \xi.$$

Setting $X = \pi_* \widetilde{X}$ and taking into account the remark, we get (8). $\square$

To describe the structure of the normal bundle of $\xi(M)$, we use the *adjoint covariant derivative operator*. As ξ is a fixed unit vector field, $\nabla_X \xi$ can be considered as a pointwise linear operator $(\nabla\xi) : T_QM \to \xi^\perp$, where $\xi^\perp$ is the orthogonal complement of ξ in T_QM, acting as

$$(\nabla\xi)(X) = \nabla_X \xi.$$

The matrix of this operator is formed by the covariant derivatives $\nabla_i \xi^k$.

The *adjoint covariant derivative* linear operator $(\nabla\xi)^* : \xi^\perp \to T_QM$ can be defined in a standard way:

$$\big\langle (\nabla\xi)^* X, Y \big\rangle = \big\langle X, (\nabla\xi)(Y) \big\rangle \tag{9}$$

for each $X \in \xi^\perp$. The matrix of $(\nabla\xi)^*$ has the form

$$[(\nabla\xi)^*]^i_j = g^{im} \nabla_m \xi^k g_{kj}.$$

As ∇ is the Riemannian connection for g, we obtain for $(\nabla\xi)^*$ the formally transposed matrix

$$[(\nabla\xi)^*]^i_k = \nabla^i\xi_k.$$

Now the structure of $\xi(M)$ can be described as follows:

Lemma 2.2. *The vector $\widetilde{N} \in T_{(Q,\xi)}T_1M$ is normal to $\xi(M)$ if and only if*

$$\widetilde{N} = -\left[(\nabla\xi)^*N\right]^H + N^V$$

where $N \in T_QM$ and $N \perp \xi$.

The proof follows easily from (5), (8) and (9).

2.2. Second fundamental form of $\xi(M)$ in T_1M

We denote by $\widetilde{\Omega}_{\widetilde{N}}$ the second fundamental form of $\xi(M)$ in T_1M^n with respect to the normal vector field $\widetilde{N}$ defined in Lemma 2.2. Then the following statement holds.

Lemma 2.3. *For $\widetilde{X}, \widetilde{Y}$ being tangent to $\xi(M)$ we have*

$$\widetilde{\Omega}_{\widetilde{N}}(\widetilde{X},\widetilde{Y}) = \frac{1}{2}\big\langle r(X,Y)\xi + r(Y,X)\xi - \nabla_{R(\xi,\nabla_X\xi)Y + R(\xi,\nabla_Y\xi)X}\xi, N\big\rangle,$$

where $r(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_{\nabla_XY}\xi$.

Proof. By definition we have

$$\widetilde{\Omega}_{\widetilde{N}}(\widetilde{X},\tilde{Y}) = \big\langle\big\langle \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, \tilde{N}\big\rangle\big\rangle$$

where $\widetilde{X}, \widetilde{Y} \in T_{(Q,\xi)}\xi(M)$. Using Lemma 2.1, we put $\widetilde{X} = X^H + (\nabla_X\xi)^V$; $\widetilde{Y} = Y^H + (\nabla_Y\xi)^V$. Then applying (7) and (6), we have

$$\begin{aligned}
\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} &= \tilde{\nabla}_{X^H + (\nabla_X\xi)^t}(Y^H + (\nabla_Y\xi)^t) \\
&= \left[\nabla_XY + \frac{1}{2}R(\xi,\nabla_X\xi)Y + \frac{1}{2}R(\xi,\nabla_Y\xi)X\right]^H \\
&\quad + \left[\nabla_X\nabla_Y\xi - \frac{1}{2}R(X,Y)\xi\right]^t \\
&= \left[\nabla_XY + \frac{1}{2}R(\xi,\nabla_X\xi)Y + \frac{1}{2}R(\xi,\nabla_Y\xi)X\right]^H
\end{aligned}$$

$$+\left[\nabla_X\nabla_Y\xi-\frac{1}{2}R(X,Y)\xi\right]^V-\langle\nabla_X\nabla_Y\xi,\xi\rangle\xi^V.$$

Let N be orthogonal to ξ. Then $\widetilde{N}=-\left[(\nabla\xi)^*N\right]^H+N^V$ is normal to $\xi(M)$. Therefore

$$\begin{aligned}\widetilde{\Omega}_{\widetilde{N}}(\widetilde{X},\widetilde{Y})&=-\left\langle\nabla_XY+\frac{1}{2}R(\xi,\nabla_X\xi)Y+\frac{1}{2}R(\xi,\nabla_Y\xi)X,(\nabla\xi)^*N\right\rangle\\&\quad+\left\langle\nabla_X\nabla_Y\xi-\frac{1}{2}R(X,Y)\xi,N\right\rangle\\&=\left\langle\nabla_X\nabla_Y\xi-\frac{1}{2}R(X,Y)\xi-\nabla_{\nabla_XY+\frac{1}{2}R(\xi,\nabla_X\xi)Y+\frac{1}{2}R(\xi,\nabla_Y\xi)X}\xi,N\right\rangle.\end{aligned}\tag{10}$$

To simplify the expression (10), we introduce the following tensor r:

$$r(X,Y)\xi=\nabla_X\nabla_Y\xi-\nabla_{\nabla_XY}\xi.\tag{11}$$

Then for the Riemannian tensor, we get

$$R(X,Y)\xi=r(X,Y)\xi-r(Y,X)\xi$$

and (10) can be rewritten as

$$\begin{aligned}\widetilde{\Omega}_{\widetilde{N}}(\widetilde{X},\widetilde{Y})=\frac{1}{2}\langle r(X,Y)\xi&+r(Y,X)\xi\\&-\nabla_{R(\xi,\nabla_X\xi)Y+R(\xi,\nabla_Y\xi)X}\xi,N\rangle.\end{aligned}\tag{12}$$

□

Next, we determine the components of $\widetilde{\Omega}$ with respect to some special frame.

As $(\nabla\xi):T_QM\to\xi^\perp$ and $(\nabla\xi)^*:\xi^\perp\to T_QM$ are mutually adjoint, then in T_QM and $\xi^\perp$, respectively, there exist orthonormal frames $\{e_0,e_1,\dots,e_n\}$ and $\{f_1,\dots,f_n\}$ such that

$$\begin{cases}(\nabla\xi)e_0=0,\\(\nabla\xi)e_\alpha=\lambda_\alpha f_\alpha,\\(\nabla\xi)^*f_\alpha=\lambda_\alpha e_\alpha,\end{cases}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ is a set of singular values (functions) of the linear operator $\nabla\xi$. Then

$$\begin{cases} \widetilde{e}_0 = e_0^H, \\ \widetilde{e}_\alpha = e_\alpha^H + (\nabla_{e_\alpha}\xi)^V = e_\alpha^H + \lambda_\alpha f_\alpha^V \end{cases} \tag{13}$$

form an orthogonal frame of the tangent space of $T_{(Q,\xi)}\xi(M)$ while

$$\widetilde{n}_\sigma = \frac{1}{\sqrt{1+\lambda_\sigma^2}}\left(-\lambda_\sigma e_\sigma^H + f_\sigma^V\right) \tag{14}$$

form the orthonormal frame in $\xi(M)^\perp$.

Lemma 2.4. *The components of second fundamental form of $\xi(M) \subset T_1M$ with respect to the frames* (13) *and* (14) *are given by*

$$\widetilde{\Omega}_{\sigma|00} = \frac{1}{\sqrt{1+\lambda_\sigma^2}}\big\{\big\langle r(e_0,e_0)\xi, f_\sigma\big\rangle\big\},$$

$$\widetilde{\Omega}_{\sigma|\alpha 0} = \frac{1}{2}\frac{1}{\sqrt{1+\lambda_\sigma^2}}\big\{\big\langle r(e_\alpha,e_0)\xi + r(e_0,e_\alpha)\xi, f_\sigma\big\rangle + \lambda_\sigma\lambda_\alpha\big\langle R(e_\sigma,e_0)\xi, f_\alpha\big\rangle\big\},$$

$$\begin{aligned}\widetilde{\Omega}_{\sigma|\alpha\beta} = \frac{1}{2\sqrt{1+\lambda_\sigma^2}}\big\{&\big\langle r(e_\alpha,e_\beta)\xi + r(e_\beta,e_\alpha)\xi, f_\sigma\big\rangle \\ &+ \lambda_\alpha\lambda_\sigma\big\langle R(e_\sigma,e_\beta)\xi, f_\alpha\big\rangle + \lambda_\beta\lambda_\sigma\big\langle R(e_\sigma,e_\alpha)\xi, f_\beta\big\rangle\big\}.\end{aligned}$$

Proof. Indeed, with respect to (13) and (14) the components of $\tilde{\Omega}$ are

$$\widetilde{\Omega}_{\sigma|ik} = \widetilde{\Omega}_{\widetilde{n}_\sigma}(\widetilde{e}_i, \widetilde{e}_k).$$

Using (12), we have

$$\widetilde{\Omega}_{\sigma|ik} = \frac{1}{2}\frac{1}{\sqrt{1+\lambda_\sigma^2}}\big\langle r(e_i,e_k)\xi + r(e_k,e_i)\xi - \nabla_{R(\xi,\nabla_{e_i}\xi)e_k + R(\xi,\nabla_{e_k}\xi)e_i}\xi, f_\sigma\big\rangle.$$

Setting $i = k = 0$ and applying (13), we get

$$\widetilde{\Omega}_{\sigma|00} = \frac{1}{\sqrt{1+\lambda_\sigma^2}}\big\{\big\langle r(e_0,e_0)\xi, f_\sigma\big\rangle\big\}.$$

Setting $i=\alpha$, $k=0$ and applying (13) again, we obtain

$$\begin{aligned}\widetilde{\Omega}_{\sigma|\alpha 0} &= \frac{1}{2}\frac{1}{\sqrt{1+\lambda_\sigma^2}}\big\{\big\langle r(e_\alpha,e_0)\xi,f_\sigma\big\rangle+\big\langle r(e_0,e_\alpha)\xi,f_\sigma\big\rangle\\ &\quad-\big\langle\nabla_{R(\xi,(\nabla\xi)e_\alpha)e_0}\xi,f_\sigma\big\rangle\big\}\\ &= \frac{1}{2}\frac{1}{\sqrt{1+\lambda_\sigma^2}}\big\{\big\langle r(e_\alpha,e_0)\xi,f_\sigma\big\rangle+\big\langle r(e_0,e_\alpha)\xi,f_\sigma\big\rangle\\ &\quad+\lambda_\sigma\lambda_\alpha\big\langle R(e_\sigma,e_0)\xi,f_\alpha\big\rangle\big\}.\end{aligned}$$

Finally, setting $i=\alpha$, $k=\beta$ applying again (13), we obtain

$$\begin{aligned}\widetilde{\Omega}_{\sigma|\alpha\beta} &= \frac{1}{2}\frac{1}{\sqrt{1+\lambda_\sigma^2}}\,\big\{\big\langle r(e_\alpha,e_\beta)\xi+r(e_\beta,e_\alpha)\xi\\ &\quad-\nabla_{R(\xi,(\nabla\xi)(e_\alpha))e_\beta+R(\xi,(\nabla\xi)(e_\beta))e_\alpha}\xi,f_\sigma\big\rangle\big\}\\ &= \frac{1}{2}\frac{1}{\sqrt{1+\lambda_\sigma^2}}\,\big\{\big\langle r(e_\alpha,e_\beta)\xi+r(e_\beta,e_\alpha)\xi,f_\sigma\big\rangle\\ &\quad-\big\langle\lambda_\alpha R(\xi,f_\alpha)e_\beta+\lambda_\beta R(\xi,f_\beta)e_\alpha,(\nabla\xi)^*(f_\sigma)\big\rangle\big\}\\ &= \frac{1}{2}\frac{1}{\sqrt{1+\lambda_\sigma^2}}\,\big\{\big\langle r(e_\alpha,e_\beta)\xi+r(e_\beta,e_\alpha)\xi,f_\sigma\big\rangle\\ &\quad-\lambda_\alpha\lambda_\sigma\big\langle R(\xi,f_\alpha)e_\beta,e_\sigma\big\rangle-\lambda_\beta\lambda_\sigma\big\langle R(\xi,f_\beta)e_\alpha,e_\sigma\big\rangle\big\}\\ &= \frac{1}{2}\frac{1}{\sqrt{1+\lambda_\sigma^2}}\,\big\{\big\langle r(e_\alpha,e_\beta)\xi,f_\sigma\big\rangle+\big\langle r(e_\beta,e_\alpha)\xi,f_\sigma\big\rangle\\ &\quad+\lambda_\alpha\lambda_\sigma\big\langle R(e_\sigma,e_\beta)\xi,f_\alpha\big\rangle+\lambda_\beta\lambda_\sigma\big\langle R(e_\sigma,e_\alpha)\xi,f_\beta\big\rangle\big\}\,.\end{aligned}$$

So, the lemma is proved. □

2.3. The mean curvature formula

Now we are able to prove the main result.

Theorem 2.5. *The components of the mean curvature vector of $\xi(M)\subset T_1M$ with respect to the frames* (13) *and* (14) *are given by*

$$\begin{aligned}(n+1)H_{\sigma|} &= \frac{1}{\sqrt{1+\lambda_\sigma^2}}\\ &\times\left\{\big\langle r(e_0,e_0)\xi,f_\sigma\big\rangle+\sum_{\alpha=1}^{n}\frac{\big\langle r(e_\alpha,e_\alpha)\xi,f_\sigma\big\rangle+\lambda_\sigma\lambda_\alpha\big\langle R(e_\sigma,e_\alpha)\xi,f_\alpha\big\rangle}{1+\lambda_\alpha^2}\right\}.\end{aligned}\tag{15}$$

PROOF. With respect to the frames (13) and (14) the matrix of the first fundamental form $\widetilde{G}$ of $\xi(M)$ is

$$\widetilde{G}=\begin{pmatrix}1 & 0 & \dots & 0\\ 0 & 1+\lambda_1^2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 1+\lambda_n^2\end{pmatrix}. \tag{16}$$

For the inverse matrix we have

$$\widetilde{G}^{-1}=\begin{pmatrix}1 & 0 & \dots & 0\\ 0 & \frac{1}{1+\lambda_1^2} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \frac{1}{1+\lambda_n^2}\end{pmatrix}. \tag{17}$$

So we have

$$\widetilde{\Omega}_{\sigma|00}=\frac{1}{\sqrt{1+\lambda_\sigma^2}}\big\langle r(e_0,e_0)\xi,f_\sigma\big\rangle,$$

$$\widetilde{\Omega}_{\sigma|\alpha\alpha}=\frac{1}{\sqrt{1+\lambda_\sigma^2}}\big[\big\langle r(e_\alpha,e_\alpha)\xi,f_\sigma\big\rangle+\lambda_\sigma\lambda_\alpha\big\langle R(e_\sigma,e_\alpha)\xi,f_\alpha\big\rangle\big].$$

Taking (17) into account, we have:

$$H_{\sigma|}=\frac{1}{(n+1)}\widetilde{G}^{ii}\widetilde{\Omega}_{\sigma|ii}=\frac{1}{(n+1)\sqrt{1+\lambda_\sigma^2}}$$
$$\times\left\{\big\langle r(e_0,e_0)\xi,f_\sigma\big\rangle+\sum_{\alpha=1}^{n}\frac{\big\langle r(e_\alpha,e_\alpha)\xi,f_\sigma+\lambda_\sigma\lambda_\alpha R(e_\sigma,e_\alpha)\xi,f_\alpha\big\rangle}{1+\lambda_\alpha^2}\right\}.$$

So we get the result. □

2.3.1. Simplified formula for the mean curvature of a unit vector field

It is possible to simplify the formula (15). To do this, we introduce the following notations:

$$E_{i|jk}=\big\langle\nabla_{e_i}e_j,e_k\big\rangle,\quad F_{i|jk}=\big\langle\nabla_{e_i}f_j,f_k\big\rangle,$$

where f_0 is supposed to be zero. Evidently, $E_{i|jk} = -E_{i|kj}$ and $F_{i|jk} = -F_{i|kj}$. Then it is simple to check that

$$\big\langle r(e_i, e_j)\xi, f_k\big\rangle = e_i(\lambda_j)\delta_{jk} + \lambda_j F_{i|jk} - \lambda_k E_{i|jk}.$$

Therefore,

$$\begin{aligned}
&\big\langle r(e_j, e_j)\xi, f_i\big\rangle = e_j(\lambda_j)\delta_{ij} + \lambda_j F_{j|ji} - \lambda_i E_{j|ji},\\
&\big\langle r(e_i, e_j)\xi, f_j\big\rangle = e_i(\lambda_j),\\
&\big\langle r(e_i, e_j)\xi, f_i\big\rangle = e_i(\lambda_j)\delta_{ij} + \lambda_j F_{i|ji} - \lambda_i E_{i|ji}.
\end{aligned}$$

From this it follows that

$$\begin{aligned}
\big\langle R(e_i, e_j)\xi, f_j\big\rangle &= \big\langle r(e_i, e_j)\xi, f_j\big\rangle - \big\langle r(e_j, e_i)\xi, f_j\big\rangle\\
&= e_i(\lambda_j) - e_i(\lambda_j)\delta_{ij} - \lambda_j F_{i|ji} + \lambda_i E_{i|ji}\\
&= e_i(\lambda_j) - e_j(\lambda_j)\delta_{ij} - \lambda_j F_{j|ji}\\
&\quad + \lambda_i E_{j|ji} + (\lambda_i + \lambda_j)(E_{j|ij} - F_{j|ij})\\
&= e_i(\lambda_j) - \big\langle r(e_j, e_j)\xi, f_i\big\rangle - (\lambda_i + \lambda_j)(E_{j|ji} - F_{j|ji}).
\end{aligned}$$

So, we see that

$$\big\langle r(e_j, e_j)\xi, f_i\big\rangle = e_i(\lambda_j) - (\lambda_i + \lambda_j)(E_{j|ji} - F_{j|ji}) - \big\langle R(e_i, e_j)\xi, f_j\big\rangle.$$

Finally, introducing the matrix $G_{i|j}$ with the components

$$G_{i|j} = E_{i|ij} - F_{i|ij},$$

we can rewrite the mean curvature formula as follows

$$(n+1)H_{\sigma|} = \frac{1}{\sqrt{1+\lambda_\sigma^2}} \sum_{i=0}^{n} \frac{e_\sigma(\lambda_i) - (\lambda_i + \lambda_\sigma)G_{i|\sigma} + (\lambda_i\lambda_\sigma - 1)\big\langle R(e_\sigma, e_i)\xi, f_i\big\rangle}{1+\lambda_i^2}, \tag{18}$$

where $\lambda_0 = 0$ and $f_0 = 0$ is supposed.

3. Some special cases and examples

3.1. Normal vector field of a Riemannian foliation

We consider an important special case of a unit *geodesic* vector field ξ such that the orthogonal distribution $\xi^\perp$ is integrable. In other words, suppose that a given Riemannian manifold admits a Riemannian transversally orientable hyperfoliation. Then the following holds.

Theorem 3.1. *Let M^{n+1} admit a Riemannian transversally orientable hyperfoliation. Let ξ be a unit normal vector field of the foliation. Then the components of the mean curvature vector of $\xi(M)$ are*

$$H_{\sigma|} = \frac{1}{(n+1)\sqrt{1+k_\sigma^2}} \sum_{\alpha=1}^{n} \left\{ \frac{-e_\sigma(k_\alpha) + (1-k_\alpha k_\sigma)\big\langle R(\xi, e_\alpha)e_\alpha, e_\sigma\big\rangle}{1+k_\alpha^2} \right\}$$

where e_α determine the principal directions and k_α are the principal curvatures of the fibers.

Remark 3.2. The analogous problem was treated in [3], where the authors considered the *minimality* condition for the vector field. The corresponding conditions in [3] differ from the mean curvature components by a factor. We refer to [6] for applications of this conditions.

Proof. For the given situation, the singular frame is simple. As ξ is geodesic vector field, we have $e_0 = \xi$, while the others are principal vectors of the second fundamental form of the fibers. If we denote the corresponding shape operator by A_ξ, then

$$\nabla_{e_\alpha}\xi = -A_\xi e_\alpha = -k_\alpha e_\alpha.$$

So, neglecting the condition on the λ_α to be *positive* (in fact, we never used this condition in proof of the formula (18)), we may put $f_\alpha = e_\alpha$ and $\lambda_\alpha = -k_\alpha$. Therefore, in (18) we obtain $G_{i|j} = 0$ and the result follows immediately. $\square$

3.2. Strongly normal vector field

A unit vector field ξ is called *normal* if $R(X,Y)\xi = \alpha\xi$ and *strongly normal* if $r(X,Y)\xi = \alpha\xi$ for all $X, Y \in \xi^\perp$. Our result (15) allows to prove easily [11]:

Every unit strongly normal geodesic vector field is minimal.

Indeed, since ξ is geodesic, $\nabla_\xi \xi = 0$ and therefore $e_0 = \xi$. Hence, $r(e_0, e_0)\xi = 0$ and $e_1, \dots, e_n \in \xi^\perp$, $f_1, \dots, f_n \in \xi^\perp$. Evidently, a strongly normal vector field is always normal. So, each term in (15) vanishes.

3.3. Geodesic vector fields on 2-dimensional manifolds

For $\dim M = 2$ the mean curvature of $\xi(M) \subset T_1M$ equals

$$H = \frac{1}{2\sqrt{1+\lambda^2}} \left\{ \left\langle r(e_0, e_0)\xi + \frac{r(e_1, e_1)\xi}{1+\lambda^2}, f_1 \right\rangle \right\} \tag{19}$$

or

$$H = \frac{1}{2\sqrt{1+\lambda^2}} \left\{ -\left\langle \nabla_{e_0} e_0, e_1 \right\rangle \lambda + \frac{e_1(\lambda)}{1+\lambda^2} \right\}.$$

The above formula allows to prove the following statement.

A unit geodesic vector field on a 2-dimensional manifold is minimal if and only if it is strongly normal (see [11]).

Indeed, in this case we can set $e_0 = \xi$, $f_1 = \pm e_1$. So, up to a sign,

$$H = \frac{1}{2(1+\lambda^2)^{3/2}} \left\langle r(e_1, e_1)\xi, e_1 \right\rangle$$

and the statement follows immediately.

In [11], the authors give an example of a geodesic but not strongly normal vector field and hence not minimal. Here we can easily find the mean curvature of that field. Namely, consider the 2-dimensional manifold of non-positive curvature with metric

$$ds^2 = du^2 + e^{2uv} dv^2.$$

Set $\xi = \{1, 0\}$. Then, up to a sign, the singular frame is

$$e_0 = \xi \quad \text{and} \quad e_1 = \{0, e^{-uv}\} = f_1.$$

It is easy to see that

$$\nabla_{e_1} \xi = v e_1.$$

Hence $\lambda = v$ and $e_1(\lambda) = e^{-uv}$. So, the mean curvature of $\xi(M)$ is given by

$$H = \frac{e^{-uv}}{2(1+v^2)^{3/2}}.$$

3.4. Examples of non-geodesic minimal vector fields on some 2-dimensional Riemannian manifolds

Next, we consider a Riemannian 2-manifold M with the metric

$$ds^2 = du^2 + e^{2g(u)}dv^2.$$

As it was shown in [11] for the general situation, the vector field $\partial/\partial u$ is minimal. Here we shall consider the vector field which makes a constant angle with $\partial/\partial u$ along each u-geodesic.

Proposition 3.4.1. *Up to a sign, the mean curvature of the vector field ξ on a 2-dimensional Riemannian manifold with metric $ds^2 = du^2 + e^{2g(u)}dv^2$ which is parallel along each u-geodesic, is*

$$H = \frac{e^{-2g}\omega_{vv}}{2\Big(1 + (e^{-g}\omega_v + g')^2\Big)^{3/2}},$$

where $\omega(v)$ is the angle function of ξ with respect to the direction of u-geodesics.

Proof. Consider the mutually orthogonal unit vector fields $X_1 = \{1, 0\}$ and $X_2 = \{0, e^{-g}\}$. A direct calculation gives

$$\begin{aligned} &\nabla_{X_1}X_1 = 0, && \nabla_{X_1}X_2 = 0, \\ &\nabla_{X_2}X_1 = g'X_2, && \nabla_{X_2}X_2 = -g'X_1. \end{aligned}$$

Let $\omega(u, v)$ be the angle function defining the vector field ξ by

$$\xi = \cos\omega X_1 + \sin\omega X_2.$$

Let η be a unit vector field orthogonal to ξ:

$$\eta = -\sin\omega X_1 + \cos\omega X_2.$$

Then

$$\nabla_{X_1}\xi = X_1(\omega)\eta, \qquad \nabla_{X_2}\xi = -(X_2(\omega) + g')\eta.$$

Now, suppose ξ to be parallel along a u-geodesic, that is, set $X_1(\omega){=}0$. Then the singular frame is: $e_0 = X_1$ and $e_1 = X_2$. The singular function is $\lambda = -(X_2(\omega) + g')$ and we see that, up to a sign, f_1 coincides with η. So

$$H = \frac{e_1(\lambda)}{2(1+\lambda^2)^{3/2}}.$$

For $e_1(\lambda)$ we obtain

$$e_1(\lambda) = X_2(-X_2(\omega) + g') = -X_2(X_2(\omega)) + X_2(g') = -e^{-2g}\omega_{vv}$$

since g does not depend on v. Therefore

$$H = \frac{e^{-2g}\omega_{vv}}{2\Big(1 + (e^{-g}\omega_v + g')^2\Big)^{3/2}},$$

what was claimed. □

From the above formula we conclude:

On a 2-dimensional manifold with metric $ds^2 = du^2 + e^{2g(u)}dv^2$ the unit vector field ξ which is parallel along u-geodesics, is minimal if its angle increment along v-curves is not higher then the linear one.

Particularly, if $\omega = \text{const}$, then ξ is minimal.

3.5. The mean curvature of a general unit vector field on 2-dimensional manifolds

In the case of $\dim M = 2$, the mean curvature of a unit vector field can be expressed in terms of the geodesic curvature of integral curves of the given field and their orthogonal trajectories.

Proposition 3.5.1. *Let ξ and η be unit mutually orthogonal vector fields on a 2-dimensional Riemannian manifold. Denote by k and κ the geodesic curvatures of the integral curves of the field ξ and η, respectively. The mean curvature H of the vector field ξ is given, up to a sign, by*

$$H = \frac{1}{2}\left[\xi\left(\frac{k}{\sqrt{1+k^2+\kappa^2}}\right) - \eta\left(\frac{\kappa}{\sqrt{1+k^2+\kappa^2}}\right)\right].$$

Remark 3.3. The analogous expression can be found in [8] as a condition of minimality of the unit vector field on 2-dimensional manifolds.

Proof. From (19) one can see that after the replacement $\xi \to -\xi$ the mean curvature H just changes its sign. Therefore, we may choose the direction of ξ in such a way that it will be the field of principal normals of the η-curves. The same arguments allow us to consider η as the field of principal normals of the ξ-curves. Denote by ω an angle between ξ and the field e_0 of the singular frame. Then

$$e_0 = \cos\omega\xi + \sin\omega\eta.$$

As $\nabla_{e_0}\xi = 0$, we have

$$\cos\omega\nabla_\xi\xi + \sin\omega\nabla\eta\xi = 0.$$

The Frenet formulas give

$$\nabla_\xi\xi = k\eta, \quad \nabla_\eta\xi = -\kappa\eta.$$

Therefore, we obtain

$$k\cos\omega - \kappa\sin\omega = 0. \tag{20}$$

Denote by e_1 and f_1 the other vectors of the singular frame. It is easy to check that the change of directions of these vectors induces a sign change of H. Therefore, we can always set $f_1 = \eta$ and $e_1 = \pm\sin\omega\xi \mp \cos\omega\eta$ to satisfy the equation $\nabla_{e_1}\xi = \lambda f_1$ with $\lambda \geq 0$. Taking all of this into account, set

$$e_0 = \cos\omega\xi + \sin\omega\eta,$$

$$e_1 = \sin\omega\xi - \cos\omega\eta.$$

Then we have

$$\nabla_{e_0}\xi = \cos\omega\nabla_\xi\xi + \sin\omega\nabla_\eta\xi = 0,$$

$$\nabla_{e_1}\xi = \sin\omega\nabla_\xi\xi - \cos\omega\nabla_\eta\xi = \lambda\eta.$$

From these equations we derive

$$\nabla_\xi\xi = \lambda\sin\omega\,\eta,$$

$$\nabla_\eta\xi = -\lambda\cos\omega\,\eta.$$

Comparing this with the Frenet formulas, we conclude that $k = \lambda \sin\omega$, $\kappa = \lambda\cos\omega$. Therefore,

$$\lambda^2 = k^2 + \kappa^2, \quad \sin\omega = \frac{k}{\lambda}, \quad \cos\omega = \frac{\kappa}{\lambda}. \tag{21}$$

To use the formula (19), we should find $e_1(\lambda)$ and $\langle \nabla_{e_0} e_0, e_1\rangle$. Now, keeping in mind (20), we have

$$e_1(\lambda) = \frac{k}{\lambda}\xi(\lambda) - \frac{\kappa}{\lambda}\eta(\lambda)$$

and

$$\begin{aligned}
\nabla_{e_0} e_0 &= \cos\omega \nabla_\xi(\cos\omega\,\xi + \sin\omega\,\eta) + \sin\omega\nabla_\eta(\cos\omega\,\xi + \sin\omega\,\eta) \\
&= -(\xi(\omega)\cos\omega + \eta(\omega)\sin\omega)e_1 - (k\cos\omega - \kappa\sin\omega)e_1 \\
&= -\big(\xi(\sin\omega) - \eta(\cos\omega)\big)e_1.
\end{aligned}$$

Therefore, using (21), we get

$$-\langle \nabla_{e_0} e_0, e_1\rangle = \xi\left(\frac{k}{\lambda}\right) - \eta\left(\frac{\kappa}{\lambda}\right).$$

Substituting these expressions into (21), we obtain

$$\begin{aligned}
H &= \frac{1}{2}\frac{1}{\sqrt{1+\lambda^2}}\left[\left(\xi\Big(\frac{k}{\lambda}\Big) - \eta\Big(\frac{\kappa}{\lambda}\Big)\right)\lambda + \frac{1}{1+\lambda^2}\left(\frac{k}{\lambda}\xi(\lambda) - \frac{\kappa}{\lambda}\eta(\lambda)\right)\right] \\
&= \frac{1}{2}\frac{1}{(1+\lambda^2)^{3/2}}\left[\big((1+\lambda^2)\,\xi(k) - k\lambda\,\xi(\lambda)\big) - \big((1+\lambda^2)\,\eta(\kappa) - \kappa\lambda\,\eta(\lambda)\big)\right] \\
&= \frac{1}{2}\left[\xi\left(\frac{k}{\sqrt{1+\lambda^2}}\right) - \eta\left(\frac{\kappa}{\sqrt{1+\lambda^2}}\right)\right].
\end{aligned}$$

Taking into account (21), we get what was claimed. $\square$

Corollary. *If ξ is a geodesic vector field then*

$$H = -\frac{1}{2}\frac{\partial}{\partial\sigma}\left(\frac{\kappa}{\sqrt{1+\kappa^2}}\right)$$

where σ is the arc-length parameter of the orthogonal trajectories of the field ξ and κ is their geodesic curvature.

A unit geodesic vector field is said to be *radial* if it is a tangent vector field of geodesics starting at a fixed point. Now we can confirm the following statement [3].

Proposition 3.5.2. *If each radial vector field on a 2-dimensional Riemannian manifold M is minimal, then M has constant curvature.*

Proof. Indeed, if such a vector field is minimal, then its orthogonal trajectories are Gauss circles of constant geodesic curvature, which means that those circles are Darboux ones. Therefore, M is of constant Gaussian curvature (see [2]).

3.6. Some examples of vector fields of constant mean curvature

3.6.1. The example on the Lobachevsky 2-space

Consider the Lobachevsky plane L^2 with the metric

$$ds^2 = du^2 + e^{2u} dv^2.$$

The coordinate lines of L^2 are u-geodesics and their orthogonal trajectories.

Proposition 3.6.1. *The unit vector field on L^2 whose angle function with respect to u-geodesics is $\omega = au + b$ $(a, b = \text{const})$ has constant mean curvature*

$$H = \frac{a}{2\sqrt{2 + a^2}}.$$

Proof. Indeed, consider the field $\xi = \cos\omega X_1 + \sin\omega X_2$ where $\omega = au + b$ and $X_1 = \{1, 0\}$, $X_2 = \{0, e^{-u}\}$. Then

$$\nabla_{X_1} X_1 = 0, \qquad \nabla_{X_1} X_2 = 0,$$
$$\nabla_{X_2} X_1 = X_2, \quad \nabla_{X_2} X_2 = -X_1.$$

Now we define the singular frame for ξ. To do this, we introduce the vector field $\eta = -\sin\omega X_1 + \cos\omega X_2$. Then

$$\nabla_{X_1} \xi = \frac{\partial \omega}{\partial u} \eta = a\eta, \quad \nabla_{X_2} \xi = \eta.$$

Therefore, setting

$$e_0 = \frac{1}{\sqrt{1+a^2}}(X_1 - aX_2), \quad e_1 = \frac{1}{\sqrt{1+a^2}}(aX_1 + X_2),$$

we have

$$\nabla_{e_0}\xi = 0, \quad \nabla_{e_1}\xi = \sqrt{1+a^2}\,\eta.$$

Hence, $f_1 = \eta$ and $\lambda = \sqrt{1+a^2} = \text{const}$. So, $e_1(\lambda) = 0$. Moreover,

$$\nabla_{e_0}e_0 = -\frac{a}{\sqrt{1+a^2}}\,e_1.$$

Substituting this into (19), we have

$$H = \frac{a}{2\sqrt{2+a^2}}.$$

So, the statement is proved. □

3.6.2. The generalized examples on the Lobachevsky $(n+1)$-space

Consider the $(n+1)$-dimensional Lobachevsky space endowed with horospherical coordinates $(u, v^1, \dots, v^n)$. Then

$$ds^2 = du^2 + e^{2u}[(dv^1)^2 + \cdots + (dv^n)^2].$$

Consider the unit vector fields

$$X_0 = \{1, 0, \dots, 0\},\ X_1 = \{0, e^{-u}, \dots, 0\}, \dots,\ X_n = \{0, 0, \dots, e^{-u}\}. \tag{22}$$

It is easy to check that

$$\begin{aligned} &\nabla_{X_0}X_0 = 0, \qquad \nabla_{X_0}X_\alpha = 0,\\ &\nabla_{X_\alpha}X_0 = X_\alpha \quad \nabla_{X_\alpha}X_\alpha = -X_0. \end{aligned}$$

Define the unit vector field ξ as follows:

$$\xi = \cos\theta X_0 + \sin\theta\cos u X_1 + \sin\theta\sin u X_2, \tag{23}$$

where $\theta \in [0, \pi/2]$ is constant.

Proposition 3.6.2. *The unit vector field which is given by* (23) *with respect to the frame* (22) *on Lobachevsky* $(n+1)$*-space with the metric*

$$ds^2 = du^2 + e^{2u}[(dv^1)^2 + \cdots + (dv^n)^2],$$

is a field of constant mean curvature. Namely, we have

$$H_{1|} = \frac{n-2}{n+1}\frac{\sqrt{2}\sin\theta\cos\theta}{1+\cos^2\theta},$$

$$H_{2|} = \frac{n\sqrt{2}\sin\theta}{2(n+1)},$$

$$H_{\sigma|} = 0 \qquad \sigma \geq 3.$$

Proof. With respect to the frame $\{X_0, X_1, \ldots, X_n\}$, the matrix $(\nabla\xi)$ has the form

$$\begin{bmatrix} 0 & -\sin\theta\cos u & -\sin\theta\sin u & 0 & \ldots & 0 \\ -\sin\theta\sin u & \cos\theta & 0 & 0 & \ldots & 0 \\ \sin\theta\cos u & 0 & \cos\theta & 0 & \ldots & 0 \\ 0 & 0 & 0 & \cos\theta & \ldots & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \ldots & \cos\theta \end{bmatrix}.$$

It is easy to find that the matrix $(\nabla\xi)^t(\nabla\xi)$ has the following expression

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

where A is the 3×3 matrix

$$\begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta\sin u & \sin\theta\cos\theta\cos u \\ -\sin\theta\cos\theta\sin u & \cos^2\theta + \sin^2\theta\cos^2 u & \sin^2\theta\sin u\cos u \\ \sin\theta\cos\theta\cos u & sin^2\theta\sin u\cos u & \cos^2\theta + \sin^2\theta\sin^2(u) \end{bmatrix}$$

and B is the diagonal $(n-2)\times(n-2)$ matrix of the form

$$\begin{bmatrix} \cos^2\theta & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \cos^2\theta \end{bmatrix}.$$

The eigenvalues of the matrix $(\nabla\xi)^t(\nabla\xi)$ are

$$\lambda_0^2 = 0, \lambda_1^2 = \lambda_2^2 = 1, \lambda_3 = \dots = \lambda_n^2 = cos^2\theta.$$

Now it is easy to find the vectors of the singular frame. We get

$$\begin{aligned} e_0 &= \cos\theta X_0 + \sin\theta\sin u X_1 - \sin\theta\cos u X_2, \\ e_1 &= \cos u X_1 + \sin u X_2, \\ e_2 &= \sin\theta X_0 - \cos\theta\sin u X_1 + \cos\theta\cos u X_2, \\ e_3 &= X_3, \dots,\ e_n = X_n \end{aligned}$$

and

$$\begin{aligned} f_1 &= -\sin\theta X_0 + \cos\theta\cos u X_1 + \cos\theta\sin u X_2, \\ f_2 &= -\sin u X_1 + \cos u X_2, \\ f_3 &= e_3, \dots,\ f_n = e_n. \end{aligned}$$

So, we have

$$\begin{gathered} \nabla_{e_0}\xi = 0, \quad \nabla_{e_1}\xi = f_1, \quad \nabla_{e_2}\xi = f_2, \\ \nabla_{e_3}\xi = \cos\theta f_3,\ \dots \quad \nabla_{e_n}\xi = \cos\theta f_n. \end{gathered}$$

Straightforward computation gives the following components for the matrix $G_{i|j}$:

$$\begin{bmatrix} 0 & \sin\theta\cos\theta & -\sin\theta & 0 & \dots & 0 \\ -\cos\theta & 0 & -\sin\theta & 0 & \dots & 0 \\ -\cos\theta & -\sin\theta\cos\theta & 0 & 0 & \dots & 0 \\ -\cos\theta & -\sin\theta & -\sin\theta & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\cos\theta & -\sin\theta & -\sin\theta & 0 & \dots & 0 \end{bmatrix}.$$

As all λ_i are constants, we have

$$
\begin{aligned}
H_{1|} &= \frac{1}{(n+1)\sqrt{1+\lambda_1^2}} \sum_{i=0}^{n} \frac{-(\lambda_1+\lambda_i)G_{i|1} + (\lambda_i-\lambda_1)\big\langle R(e_1,e_i)\xi, f_i\big\rangle}{1+\lambda_i^2} \\
&= \frac{1}{(n+1)\sqrt{2}} \left[\sum_{i=0}^{2}(-G_{i|1}) + \sum_{i=3}^{n} \frac{-(1+\lambda_i)G_{i|1} + (\lambda_i - 1)\big\langle \xi, e_1\big\rangle}{1+\cos^2\theta}\right] \\
&= \frac{1}{(n+1)\sqrt{2}} \left[0 + (n-2)\frac{(1+\cos\theta\sin\theta + (\cos\theta-1)\sin\theta}{1+\cos^2\theta}\right] \\
&= \frac{n-2}{n+1}\frac{\sqrt{2}\sin\theta\cos\theta}{1+\cos^2\theta}.
\end{aligned}
$$

Analogously, we get

$$
\begin{aligned}
H_{2|} &= \frac{1}{(n+1)\sqrt{1+\lambda_2^2}} \sum_{i=0}^{n} \frac{-(\lambda_2+\lambda_i)G_{i|2} + (\lambda_i-\lambda_2)\big\langle R(e_2,e_i)\xi, f_i\big\rangle}{1+\lambda_2^2} \\
&= \frac{1}{(n+1)\sqrt{2}} \left[\sum_{i=0}^{2}(-G_{i|2}) + \sum_{i=3}^{n} \frac{-(1+\lambda_i)G_{i|2} + (\lambda_i - 1)\big\langle \xi, e_2\big\rangle}{1+\cos^2\theta}\right] \\
&= \frac{\sqrt{2}}{2(n+1)} \left[2\sin\theta + (n-2)\frac{(1+\cos\theta)\sin\theta + (\cos\theta-1)\sin\theta\cos\theta}{1+\cos^2\theta}\right] \\
&= \frac{\sqrt{2}}{2(n+1)} \left[2\sin\theta + (n-2)\frac{\sin\theta + \sin\theta\cos^2\theta}{1+\cos^2\theta}\right] \\
&= \frac{n\sqrt{2}\sin\theta}{2(n+1)}
\end{aligned}
$$

and $H_{\sigma|} = 0$ for all $\sigma \geq 3$. □

A similar but more complicated computation shows that there exist a family of vector fields of constant mean curvature on the Lobachevsky space. Namely, let ξ be a vector field given by

$$
\xi = \cos\theta X_0 + \sin\theta\cos au X_1 + \sin\theta\sin au X_2, \tag{24}
$$

where a and θ are constants and the frame $X_0, X_1, \dots, X_n$ is chosen as above. Then the following statement is true.

Proposition 3.6.3. *The unit vector field which is given by* (24) *with respect to the frame* (22) *on the Lobachevsky* $(n+1)$*-space with the metric*

$$ds^2 = du^2 + e^{2u}[(dv^1)^2 + \dots + (dv^n)^2],$$

is a field of constant mean curvature. Namely, we have

$$H_{1|} = \frac{\sqrt{2}\sin\theta\cos\theta}{n+1}\left(\frac{1-a^2}{1+\cos^2\theta + a^2\sin^2\theta} + \frac{n-2}{1+\cos^2\theta}\right),$$

$$H_{2|} = \frac{an\sin\theta}{(n+1)\sqrt{1+\cos^2\theta + a^2\sin^2\theta}},$$

$$H_{\sigma|} = 0 \qquad \sigma \geq 3.$$

The proof is based on the fact that the singular values of $(\nabla\xi)$ are the following constants:

$$\lambda_1 = 1, \quad \lambda_2 = \sqrt{\cos^2\theta + a^2\sin^2\theta}, \quad \lambda_3 = \ldots = \lambda_n = \cos\theta.$$

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