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A rigidity theorem for the three dimensional critical point equation

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Abstract. On a compact 3-dimensional manifold M^3, a critical point of the total scalar curvature functional, restricted to the space of metrics with constant scalar curvature of volume 1, satifies the critical point equation (CPE), given by $U_g^*(f) = z_g$. It has been conjectured that a solution (g, f) of CPE is Einstein. In this paper, we prove that, if CPE has two distinct solution functions and Weyl–Schouten tensor vanishes on a certain hypersurface of M^3, (M^3, g) is isometric to a standard 3-sphere.

1. Introduction

Let (M^3, g) be a 3-dimensional compact manifold and $\mathcal{M}_1$ the set of smooth Riemannian structures on M^3 of volume 1. The total scalar curvature functional $\mathcal{S} : \mathcal{M}_1 \to \mathbb{R}$ is defined by

$$\mathcal{S}(g) = \int_{M^3} s_g dv_g$$

where dv_g is the volume form determined by the metric and s_g the scalar curvature of the metric g. It is well known that a critical point of this functional is Einstein. On the other hand, there has been a conjecture (Conjecture A) that a critical point of this functional $\mathcal{S}$ restricted to $\mathcal{C}$ is Einstein [1, Chp 4, F], where $\mathcal{C}$ is the set of constant scalar curvature metrics given by

$$\mathcal{C} = \{g \in \mathcal{M}_1 \mid s_g \text{ constant}\}$$

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This paper is concerned with a study of a sufficient condition that Conjecture A holds.

The Euler–Lagrange equations for a critical point g of $\mathcal{S}$ restricted to $\mathcal{C}$ may be represented as the following critical point equation (CPE, hereafter):

$$U_g^*(f) = z_g \equiv r_g - \frac{s_g}{3} g \tag{1}$$

where f is a function on M^n. Since

$$U_g^*(f) \equiv D_g df - (\Delta_g f)g - f r_g$$

and $\Delta_g f = -\frac{s_g}{2} f$, CPE may also be given as

$$(1+f) z_g = D_g df + \frac{s_g f}{6} g. \tag{2}$$

J. Lafontaine showed that Conjecture A holds if a solution metric g of CPE is conformally flat [7]. The author showed that Conjecture A holds if a solution function f of CPE is greater than or equal to -1 [4]. Furthermore, Conjecture A holds if CPE has two distinct solution functions and certain two surfaces of M^3 are disjoint [5]. M. Obata showed that, if the solution metric of CPE is Einstein, it is isometric to a standard 3-sphere [9]. This paper is partially motivated by considering a generalization of the results of authors mentioned above, Lafontaine [7] and Hwang [5]. More precisely, we prove the following theorem in the present paper:

Main Theorem. *Let CPE have two distinct non-trivial solutions f_1 and f_2 on (M^3, g). Assume that Weyl–Schouten tensor vanishes on Γ where $\Gamma = \varphi^{-1}(0)$ for $\varphi = f_1 - f_2$. Then (M^3, g) is isometric to a 3-sphere.*

Remark 1. (i) Fisher and Marsden showed that Γ in our Main Theorem is an embedded surface of M^3 [2]. It is easy to see that Γ exists in M^3, since we have

$$\int_{M^3} \varphi = -\frac{2}{s_g} \int_{M^3} \Delta_g \varphi = 0$$

where the first equation follows from the equation $\Delta_g \varphi = -\frac{s_g}{2}\varphi$, which may be obtained by taking the trace of the equation (3) in Section 2. For the significance of the surface Γ, see [5].

(ii) Our Main Theorem is a partial answer to the conjecture of [5] which states that (M^3, g) is isometric to a standard 3-sphere if CPE has two distinct non-trivial solutions f_1 and f_2 on (M^3, g).

II. The proof of Main Theorem

This section is devoted to the proof of our Main Theorem. Throughout the present paper, *we assume that there exist two distinct solutions f_1 and f_2 of CPE on* (M^3, g). Then, it is easy to see that $U_g^*(\varphi) = 0$ in virtue of (1), or equivalently we have

$$0 = D_g d\varphi - (\Delta_g \varphi)g - \varphi r_g. \tag{3}$$

Definition. For a given 3-dimensional manifold (M^3, g), let $H = r_g - \frac{s_g}{4}g$. and $d^D H$ be the Wely–Schouten tensor field defined by the differential operator from $C^\infty(S^2(M))$ into $\Lambda^2 M \otimes T^* M$, given by

$$d^D H(x, y, z) = D_x H(y, z) - D_y H(x, z). \tag{4}$$

Remark 2. When $n = 3$, we note that a metric g is conformally flat if and only if Weyl–Schouten tensor $d^D H$ vanishes identically on M^3. Therefore, our metric g is Einstein if Weyl–Schouten tensor $d^D H$ vanishes identically on M^3, in virtue of the result in the introduction of Lafontaine [7]. Our Main Theorem states that our metric g is Einstein, if there are two distinct solutions of CPE and $d^D H$ vanishes on the hypersurface Γ of M^3.

The proof of Main Theorem consists of several lemmas and corollaries. It may be sketched as follows. In the first, we derive three relations involving $|z|^2$, which hold on M^3 or Γ (Lemmas 1 and 2). Lemma 2 implies that z is diagonalized on Γ as in (15). In the second, we prove Lemma 3, in virtue of which we may choose a solution function f_α of CPE such that its gradient df_α is tangent to a connected component Γ_α of Γ. Using this fact and (15), it may be shown that $|z|^2$ and $d^D H$ are related as in (16) on Γ_α^r (Corollary 2). Finally, using (16) and Lemma 4 we first show that $z_g = 0$ on Γ, and then prove that $z_g = 0$ on $M^3 = M_{0,\varphi} \cup \Gamma \cup M_\varphi^0$ (Proof of Main Theorem).

For $\varphi \in Ker\, U_g^*$, let $\Phi = |d\varphi|^2$ and $N_\varphi = \Phi^{-1/2} d\varphi$. For any solution f of CPE, let $W = |df|^2$, $N_f = W^{-1/2} df$, and $h = 1 + f$. In the following lemma, we prove that the equation (5) holds for any solution function f of CPE.

Lemma 1. *The following equations hold on* (M^3, g)*:*

$$8Wh^2|z|^2 = h^4|d^D H|^2 + 3\left|dW + \frac{sf}{3}df\right|^2 \tag{5}$$

$$8\Phi\varphi^2|z|^2 = \varphi^4|d^D H|^2 + 3\left|d\Phi + \frac{s\varphi}{3}d\varphi\right|^2. \tag{6}$$

Sketch of proof. The full detailed proof of (5) is given in [3]. In a given coordinate system $\{e_a\}_{a=1,2,3}$, (4) can be rewritten as

$$d^D H_{cba} = r_{ab;c} - r_{ac;b} - \frac{1}{4}(g_{ab}s_{;c} - g_{ac}s_{;b}) = r_{ab;c} - r_{ac;b}$$

since s is constant. In virtue of the relation

$$hr_{ab} = f_{;ab} + \frac{2+3f}{6}sg_{ab}$$

obtained from (2) and the Ricci identity $f_{;abc} - f_{;acb} = R_{bcla}f^{;l}$ with

$$R_{ijkl} = -\frac{s}{2}(g_{ik}g_{jl} - g_{il}g_{jk}) + (r_{ik}g_{jl} + r_{jl}g_{ik} - r_{il}g_{jk} - r_{jk}g_{il})$$

for $n = 3$, we may conclude that

$$h^4|d^D H|^2 = -s^2f^2W - 2sf\langle df, dW\rangle - 3|dW|^2 + 8|D_g df|^2 W. \tag{7}$$

Since (2) gives

$$h^2|z_g|^2 = |D_g df|^2 - \frac{s^2f^2}{12} \tag{8}$$

the equation (5) follows from (7) and (8). The proof of the equation (6) is similar. □

On Γ the equation (6) is reduced to (9) as in the following lemma:

Lemma 2. *On* $\Gamma = \varphi^{-1}(0)$ *we have*

$$|z|^2 = \frac{3}{2}z(N_\varphi, N_\varphi)^2. \tag{9}$$

Proof. Since the relation

$$d\Phi + \frac{s\varphi}{3}d\varphi = 2D_{d\varphi}d\varphi + \frac{s\varphi}{3}d\varphi = 2\varphi z(d\varphi, \cdot) \tag{10}$$

holds in virtue of (3), on $\varphi^{-1}(c)$ with any constant c we have

$$\left|d\Phi + \frac{s\varphi}{3}d\varphi\right|^2 = 4\varphi^2 \sum_{i=1}^{3} z(d\varphi, e_i)^2 \tag{11}$$

in a given orthonormal basis $\{e_a\}_{i=1,2,3}$ with $e_3 = N_\varphi$. Substitution of (11) into (6) gives

$$8\Phi|z|^2 = 12\sum_{i=1}^{2} z(d\varphi, e_i)^2 + 12\, z(d\varphi, N_\varphi)^2 \quad \text{on } \Gamma. \tag{12}$$

In what follows we claim that the first term of the right-hand side of (12) vanishes. This implies that (12) is reduced to (9), completing the proof of our lemma. In virtue of (3), we have in a neighborhood of Γ

$$\varphi z(d\varphi, X) = \langle D_X d\varphi, d\varphi\rangle = \frac{1}{2}\langle d\Phi, X\rangle \tag{13}$$

for any vector field X tangent to Γ. Taking the Lie derivative of (13) with respect to N_φ on Γ, we have

$$\Phi^{1/2} z(d\varphi, e_i) = \frac{1}{2}\langle D_{N_\varphi} d\Phi, e_i\rangle + \frac{1}{2}\langle d\Phi, D_{N_\varphi} e_i\rangle. \tag{14}$$

On the other hand, we note that

$$\langle D_{N_\varphi} d\Phi, e_i\rangle = \langle D_{e_i} d\Phi, N_\varphi\rangle = e_i\langle d\Phi, N_\varphi\rangle - \langle d\Phi, D_{e_i} N_\varphi\rangle = 0$$

and $d\Phi = \frac{1}{2}\langle D_g d\varphi, d\varphi\rangle = 0$ on Γ. Hence, the right-hand side of (14) vanishes, completing the proof of our claim. □

As a consequence of Lemma 2, we have the following result:

Corollary 1. *For any vector field X tangent to Γ, $z(X, d\varphi) = 0$ on Γ.*

Proof. With respect to a given orthonormal basis $\{e_a\}_{i=1,2,3}$ with $e_3 = N_\varphi$, Lemma 2 implies that on Γ

$$z(e_1, e_1) = z(e_2, e_2) = -\frac{1}{2} z(N_\varphi, N_\varphi) \quad \text{and} \quad z(e_i, e_j) = 0 \quad \text{for } i \neq j \tag{15}$$

since for $z_{ij} = z(e_i, e_j)$ we have

$$0 = |z|^2 - \frac{3}{2} z(N_\varphi, N_\varphi)^2 = \sum_{i,j} z(e_i, e_j)^2 - \frac{3}{2} z(N_\varphi, N_\varphi)^2$$

$$= z_{11}^2 + z_{22}^2 - \frac{1}{2} z_{33}^2 + 2(z_{12}^2 + z_{23}^2 + z_{31}^2)$$

$$= z_{11}^2 + z_{22}^2 - \frac{1}{2}(z_{11} + z_{22})^2 + 2(z_{12}^2 + z_{23}^2 + z_{31}^2)$$

$$= \frac{1}{2}(z_{11} - z_{22})^2 + 2(z_{12}^2 + z_{23}^2 + z_{31}^2)$$

where use of the fact that $z_{33} = -(z_{11} + z_{22})$ is made in the fourth equality. Hence, the proof of the corollary follows from (15). □

Let Γ_α be a connected component of Γ. Then one can find a solution of CPE so that its gradient is tangent to Γ_α, as shown in the following lemma. In fact, the proof of Lemma 3 is given in [5]. However, we include its proof here again for the sake of completeness.

Lemma 3. *There exists a solution f_α of CPE such that $\langle df_\alpha, d\varphi\rangle = 0$ on Γ_α.*

Proof. First we claim that both $\Phi = |d\varphi|^2$ and $\eta = \langle df, d\varphi\rangle$ are constant along Γ_α, where f is f_1 or f_2. The first statement follows from the fact that we have in virtue of (3)

$$\xi(\Phi) = 2\langle D_\xi d\varphi, d\varphi\rangle = -s_g\langle \xi, d\varphi\rangle\varphi + 2\varphi r_g(\xi, d\varphi) = 0$$

for any tangent vector ξ to Γ. The second statement follows from the fact that for any tangent vector field X to Γ_α we have

$$X(\eta) = \langle D_X df, d\varphi\rangle + \langle df, D_X d\varphi\rangle = \langle D_X df, d\varphi\rangle = (1+f)z(X, d\varphi) = 0$$

which are the results of (2) and Corollary 1.

Now, let $f_\alpha = f - \Phi_\alpha^{-1}\eta_\alpha\varphi$, where $\Phi_\alpha = \Phi_{|\Gamma_\alpha}$ and $\eta_\alpha = \eta_{|\Gamma_\alpha}$. Then, clearly f_α is a solution of CPE, since $U_g^*(\varphi) = 0$. Also, along Γ_α we have

$$\langle df_\alpha, d\varphi\rangle = \langle df, d\varphi\rangle - \Phi_\alpha^{-1}\eta_\alpha\langle d\varphi, d\varphi\rangle = \eta_\alpha - \eta_\alpha = 0.$$

This implies that the gradient of f_α is tangent to Γ_α, and hence the proof of Lemma 3 is completed. □

Corollary 2. *Let $\Gamma_\alpha^r = \{x \in \Gamma_\alpha \mid df_\alpha \neq 0\}$. Then we have*

$$6W_\alpha |z|^2 = h_\alpha^2 |d^D H|^2 \quad \text{on } \Gamma_\alpha^r \tag{16}$$

where $W_\alpha = |df_\alpha|^2$ and $h_\alpha = 1 + f_\alpha$.

Proof. We first note that we may take $e_2 = N_{f_\alpha} = |df_\alpha|^{-1/2} df_\alpha$ on Γ_α^r in virtue of Lemma 3 and hence that (15) gives

$$z(N_\varphi, N_\varphi) = -2z(N_{f_\alpha}, N_{f_\alpha}) \quad \text{on } \Gamma_\alpha^r. \tag{17}$$

In the next, we also note that

$$\left| dW_\alpha + \frac{sf_\alpha}{3} df_\alpha \right|^2 = 4h_\alpha^2 \sum_{i=1}^{3} z(df_\alpha, e_i)^2 \tag{18}$$

since (2) implies that $dW_\alpha + \frac{sf_\alpha}{3} df_\alpha = 2Ddf_\alpha + \frac{sf_\alpha}{3} df_\alpha = 2h_\alpha z(df_\alpha, \cdot)$. Hence, in virtue of (15) and (17), (18) becomes

$$\begin{aligned} \left| dW_\alpha + \frac{sf_\alpha}{3} df_\alpha \right|^2 &= 4h_\alpha^2 W_\alpha z(N_f, N_f)^2 \\ &= h_\alpha^2 W_\alpha z(N_\varphi, N_\varphi)^2 = \frac{2}{3} h_\alpha^2 W_\alpha |z|^2 \text{ on } \Gamma_\alpha^r \end{aligned} \tag{19}$$

where the last equality comes from (9). Consequently, substitution of (19) into (5) gives (16). □

In Corollary 2 the behavior of $|z|^2$ on Γ_α^r was studied. In the following final lemma we investigate the behavior of $|z|^2$ on the set $\Gamma_\alpha^c = \Gamma_\alpha \setminus \Gamma_\alpha^r$ when Γ_α^c is not of measure zero. Note that if Γ_α^c is not of measure zero, there exists an open subset Ω of Γ_α^c.

Lemma 4. *If the set Γ_α^c is not of measure zero, $|z|^2$ is constant on any open subset Ω of Γ_α^c.*

Proof. Since $df_\alpha = 0$ on Ω, f_α is constant on Ω and $\langle df_\alpha, \xi \rangle = 0$ on Ω for any tangent vector field $\xi \in T\Omega \subset T\Gamma_\alpha$. Thus, for $p \in \Omega$ and $\eta \in T_p\Omega \subset T_p\Gamma_\alpha$, it follows that $\eta\langle df_\alpha, \xi \rangle(p) = 0$, and hence

$$\langle D_\eta df_\alpha, \xi \rangle_p = \eta \langle df_\alpha, \xi \rangle(p) - \langle df_\alpha, D_\eta \xi \rangle_p = 0.$$

Since $p \in \Omega$ is arbitrary, we may conclude that $D_g df_\alpha(e_i, e_i) = 0$, where $e_i \in T\Omega$ for $i = 1, 2$. Thus, in virtue of (15), we have $h_\alpha z(e_i, e_i) = D_g df_\alpha(e_i, e_i) + \frac{sf_\alpha}{6} = \frac{sf_\alpha}{6}$ and

$$h_\alpha z(N_\varphi, N_\varphi) = -2h_\alpha z(e_i, e_i) = -\frac{sf_\alpha}{3} \tag{20}$$

on Ω. Note that $h_\alpha = 1 + f_\alpha \neq 0$ on Ω in virtue of (20). Therefore on Ω we have

$$|z|^2 = \frac{3}{2} z(N_\varphi, N_\varphi)^2 = \frac{s^2 f_\alpha^2}{6h_\alpha^2} \tag{21}$$

which is a constant. □

Now we are ready to prove our Main Theorem.

Proof of Main Theorem. Since an Einstein solution metric g is isometric to a 3-sphere due to the result in the introduction of Obata [9], it suffices to show that $z_g \equiv 0$ on $M^3 = M_{0,\varphi} \cup \Gamma \cup M_\varphi^0$.

We first show that $z_g = 0$ on each connected component Γ_α of $\Gamma = \bigcup_\alpha \Gamma_\alpha$. It follows from Lemma 3 that there exists a solution function f_α of CPE such that $df_\alpha \in T\Gamma_\alpha$. Since the right-hand side of (16) vanishes for this f_α in virtue of the assumption, we have

$$|z|^2 = 0 \quad \text{on } \Gamma_\alpha^r. \tag{22}$$

Furthermore, if the set $\Gamma_\alpha^c = \Gamma_\alpha \setminus \Gamma_\alpha^r$ is of measure zero, it is clear that $|z|^2 = 0$ on Γ_α^c by the continuity of $|z|^2$. In the case that Γ_α^c is not of measure zero, Lemma 4 gives that $|z|^2$ is constant on any connected component of Γ_α^c. Hence, by the continuity of $|z|^2$, we also have $|z|^2 = 0$ on Γ_α^c even when Γ_α^c is not of measure zero. This proves that regardless of the measure of Γ_α^c we have $z_g = 0$ on $\Gamma_\alpha = \Gamma_\alpha^c \cup \Gamma_\alpha^r$ in virtue of (21). Therefore, $z_g = 0$ on $\Gamma = \bigcup_\alpha \Gamma_\alpha$.

In order to show that $z_g = 0$ on all of M^3, we next prove that $z_g = 0$ on both

$$M_{0,\varphi} = \{x \in M^3 \mid \varphi(x) < 0\} \quad \text{and} \quad M_\varphi^0 = \{x \in M^3 \mid \varphi(x) > 0\}.$$

Let $M'_{0,\varphi}$ be a connected component of $M_{0,\varphi}$ with $\partial M'_{0,\varphi} = \bigcup_\beta \Gamma_\beta$ for some $\{\beta\} \subset \{\alpha\}$. Integration by parts gives

$$\begin{aligned}\int_{M'_{0,\varphi}} \varphi|z|^2 &= \int_{M'_{0,\varphi}} \Big(D_g d\varphi + \frac{s\varphi}{6} g\Big)_{ij} z^{ij} = \int_{M'_{0,\varphi}} (D_g d\varphi)_{ij} z^{ij} \\ &= \int_{M'_{0,\varphi}} div(z(d\varphi,\cdot)) - \int_{M'_{0,\varphi}} (d\varphi)_i (\delta z)^{ik}_k \\ &= \int_{\partial M'_{0,\varphi}} z(d\varphi, N_\varphi) = \sum_\beta \int_{\Gamma_\beta} z(d\varphi, N_\varphi) = 0\end{aligned}$$

where δ is the codifferential. Here, the fourth equality comes from the Stokes theorem and the fact that

$$\delta z_g = \delta\left(r_g - \frac{s_g}{3} g\right) = \delta r_g + d\left(\frac{s_g}{3}\right) = \delta r_g = -\frac{1}{2} d(s_g) = 0.$$

Therefore $z_g = 0$ on each connected component $M'_{0,\varphi}$ of $M_{0,\varphi}$, since $\varphi < 0$ on $M'_{0,\varphi}$. Thus $z_g = 0$ on all of $M_{0,\varphi}$. Applying the similar arguments to M^0_φ, we have $z_g = 0$ on each connected component of M^0_φ, and so on all of M^0_φ. Consequently, we may conclude that $z_g = 0$ on $M^3 = M_{0,\varphi} \cup \Gamma \cup M^0_\varphi$, or equivalently that g is Einstein.

Now that we have proved that $z_g \equiv 0$ on M^3, the proof of our Main Theorem is now completed. $\square$

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