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Single-valued and multi-valued Caristi type operators

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Abstract. New results in connection with some single-valued and multi-valued Caristi type operators are established.

1. Introduction

Caristi's fixed point theorem states that each operator f from a complete metric space (X,d) into itself satisfying the condition: there exists a proper lower semi-continuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that:

$$d(x, f(x)) + \varphi(f(x)) \leq \varphi(x), \quad \text{for each } x \in X \tag{1.1}$$

has at least a fixed point $x^* \in X$, i.e. $x^* = f(x^*)$ (see [5]). There are several extensions and generalizations of this important principle of the nonlinear analysis (see for example [4], [6], etc.). One of the latest, asserts that if (X,d) is a complete metric space, $x_0 \in X$, $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ is a proper lower semi-continuous function and $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $\int_0^\infty \frac{ds}{1+h(s)} = +\infty$, then each single-valued operator f from X to itself satisfying the condition:

$$\text{for each } x \in X, \quad \frac{d(x, f(x))}{1 + h(d(x_0, x))} + \varphi(f(x)) \leq \varphi(x), \tag{1.2}$$

has at least a fixed point (see [15]). For the multi-valued case, if F is an operator of the complete metric space X into the space of all nonempty

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subsets of X and there exists a proper lower semi-continuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that

$$(1.3) \quad \text{for each } x \in X, \text{ there is } y \in F(x) \text{ so that } d(x,y) + \varphi(y) \leq \varphi(x),$$

then the multi-valued map F has at least a fixed point $x^* \in X$, i.e. $x^* \in F(x^*)$ (see for example [1]). Moreover, if F satisfies the stronger condition:

$$(1.4) \quad \text{for each } x \in X \text{ and each } y \in F(x) \text{ we have } d(x,y) + \varphi(y) \leq \varphi(x),$$

then the multi-valued map F has at least a strict fixed point $x^* \in X$, i.e. $\{x^*\} = F(x^*)$ (see [1]). On the other hand, if F is a multi-valued operator with nonempty closed values and $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ is a proper lower semi-continuous function such that the following condition holds:

$$(1.5) \qquad \text{for each } x \in X, \quad \inf\{d(x,y) + \varphi(y) : y \in F(x)\} \leq \varphi(x),$$

then F has at least a fixed point (see [9]). In this framework, let us remark that if we replace (1.5) by a weaker condition (see (1.6) below), then the conjecture stated by J.-P. Penot in [11] as follows: *Let (X,d) be a complete metric space, $\varphi : X \to \mathbb{R}_+$ be a lower semicontinuous function and F be a multivalued operator of X into the family of all nonempty closed subsets of X satisfying the following condition*:

$$(1.6) \qquad \inf\{d(x,y) : y \in F(x)\} + \inf\{\varphi(y) : y \in F(x)\} \leq \varphi(x),$$

for each $x \in X$. Then F has at least a fixed point. is false (see [9] for a counterexample). It is easy to see that (1.4) $\Rightarrow$ (1.3) $\Rightarrow$ (1.5) and (1.5) $\Rightarrow$ (1.3) provided that F has nonempty compact values.

The purpose of this paper is to present several new results in connection with the above mentioned single-valued and multi-valued Caristi type operators in complete metric spaces.

2. Main results

Let (X,d) be a metric space and $\mathcal{P}(X)$ the space of all subsets of X. We denote by $P(X)$ the space of all nonempty subsets of X and by $P_p(X)$ the set of all nonempty subsets of X having the property "p", where "p" could be: *cl* = *closed*, *b* = *bounded*, *cp* = *compact*, *cv* =

convex (for normed spaces X), etc. We consider the following (generalized) functionals:

$$D : P(X) \times P(X) \to \mathbb{R}_+,\ D(A,B) = \inf\{d(a,b) \mid a \in A,\ b \in B\}$$

$$\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ \delta(A,B) = \sup\{d(a,b) \mid a \in A,\ b \in B\}$$

$$\rho : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ \rho(A,B) = \sup\{D(a,B) \mid a \in A\}$$

$$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ H(A,B) = \max\{\rho(A,B), \rho(B,A)\}$$

It is well-known that if (X,d) is a complete metric space, then $(P_{b,cl}(X), H)$ is also a complete metric space. We denote by $\mathrm{Fix}(F)$ the fixed points set of F and by $S\,\mathrm{Fix}(F)$ the strict fixed points set of F. Let (X,d) be a metric space and $F : X \to P(X)$ be a multi-valued map.

Definition 2.1. A proper function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ is called:

(i) a weak entropy of F if the condition (1.3) holds;

(ii) an entropy of F if the condition (1.4) holds.

The map $F : X \to P(X)$ is said to be *weakly dissipative* iff there exists a weak entropy of F and F is said to be *dissipative* iff there is an entropy of it.

Definition 2.2. Let (X,d) be a metric space and $F : X \to P(X)$ be a multi-valued operator. Then F is said to be:

(i) a-contraction iff there exists $a \in [0,1[$ such that

$$H(F(x), F(y)) \le ad(x,y), \quad \text{for each } x, y \in X;$$

(ii) *Reich type operator* iff there exist $a, b, c \in \mathbb{R}_+$, with $a+b+c < 1$ such that

$$H(F(x), F(y)) \le ad(x,y) + bD(x, F(x)) + cD(y, F(y)),$$

for each $x, y \in X$;

(iii) δ-Reich type operator iff there exist $a, b, c \in \mathbb{R}_+$, with $a+b+c < 1$ such that

$$\delta(F(x), F(y)) \le ad(x,y) + b\delta(x, F(x)) + c\delta(y, F(y)),$$

for each $x, y \in X$.

Let us remark now, that if f is a (single-valued) a-contraction in a complete metric space X, then f satisfies condition (1.1) with $\varphi(x) = (1-a)^{-1}\, d(x, f(x))$, for each $x \in X$, so that part of the Banach contraction principle which says about the existence of a fixed point can be obtained by Caristi's theorem. For the multi-valued case we have the following result:

Theorem 2.1. *Let* (X, d) *be a metric space and* $F : X \to P_{cl}(X)$ *be an* a*-contraction. Then:*

a) *F satisfies the condition* (1.5) *with* $\varphi(x) = (1-a)^{-1}\; D(x, F(x))$, *for each* $x \in X$.

b) *If, in addition* $F(x) \in P_{cp}(X)$, *for each* $x \in X$, *then F is weakly dissipative with a weak entropy given by the formula* $\varphi(x) = (1-a)^{-1} D(x, F(x))$, *for each* $x \in X$.

PROOF. a) is Corrolary 1 in [9] and b) follows immediately from a) and the conditions (1.3) $\Longleftrightarrow$ (1.5). □

Remark 2.1. It is an open question if a multi-valued a-contraction is dissipative.

First main result of this paper is

Theorem 2.2. *Let* (X, d) *be a metric space and* $F : X \to P_{cl}(X)$ *be a Reich type multi-valued map. Then there exists* $f : X \to X$ *a selection of F satisfying the Caristi type condition* (1.1).

PROOF. Let $\varepsilon > 0$ such that $a < \varepsilon < 1 - b - c$. We denote by $U_x = \{y \in F(x) \mid \varepsilon d(x, y) \le (1 - b - c)D(x, F(x))\}$, for each $x \in X$. Obviously, for each $x \in X$, the set U_x is nonempty (otherwise, if $x \in X$ is not a fixed point of F and we suppose that for each $y \in F(x)$ we have $\varepsilon d(x, y) > (1 - b - c)D(x, F(x))$, then we reach the contradiction $\varepsilon D(x, F(x)) \ge (1 - b - c)D(x, F(x))$; if $x \in X$ is a fixed point of F, then clearly $U_x \neq \emptyset$). We can choose a single-valued mapping $f : X \to X$ such that $f(x) \in U_x$, i.e. $f(x) \in F(x)$ and $\varepsilon d(x, f(x)) \le (1 - b - c)D(x, F(x))$, for each $x \in X$. Then we have successively:

$$\begin{aligned} D(f(x), F(f(x))) &\le H(F(x), F(f(x))) \\ &\le ad(x, f(x)) + bD(x, F(x)) + cD(f(x), F(f(x))) \end{aligned}$$

and hence

$$(1 - c)D(f(x), F(f(x))) - bD(x, F(x)) \le ad(x, f(x)).$$

In view of this we obtain:

$$\begin{aligned} d(x,f(x)) &= (\varepsilon-a)^{-1}\big[\varepsilon d(x,f(x)) - ad(x,f(x))\big] \\ &\le (\varepsilon-a)^{-1}\big[(1-b-c)D(x,F(x)) \\ &\quad - (1-c)D(f(x),F(f(x))) + bD(x,F(x))\big] \\ &= (1-c)/(\varepsilon-a)\big[D(x,F(x)) - D(f(x),F(f(x)))\big]. \end{aligned}$$

If we define $\varphi : X \to \mathbb{R}_+$ by $\varphi(x) = (1-c)/(\varepsilon-a)D(x,F(x))$, for each $x \in X$, then it is easy to see that

$$d(x,f(x)) \le \varphi(x) - \varphi(f(x)), \quad \text{for each } x \in X.$$

Moreover, for each $x \in X$ and each $y \in F(x)$ we have:

$$\begin{aligned} |\varphi(x) - \varphi(y)| &= (1-c)/(\varepsilon-a)\big|D(x,F(x)) - D(y,F(y))\big| \\ &\le (1-c)/(\varepsilon-a)\big|d(x,y) + H(F(x),F(y))\big| \\ &\le (1-c)/(\varepsilon-a)\big[d(x,y) + ad(x,y) + bD(x,F(x)) + cD(y,F(y))\big] \\ &= (1-c)(1+a)/(\varepsilon-a)d(x,y) + b(1-c)/(\varepsilon-a)D(x,F(x)) \\ &\quad + c(1-c)/(\varepsilon-a)D(y,F(y)), \end{aligned}$$

proving the fact that the selection f is a kind of single-valued Reich type operator. □

Remark 2.2. If the multi-valued operator $F : X \to P_{cl}(X)$ is an upper semicontinuous Reich type operator, then φ is a lower semicontinuous entropy of f (because the map $x \mapsto D(x,F(x))$ is lower semicontinuous).

Remark 2.3. If in Theorem 2.2 we take $b = c = 0$, then we obtain Theorem 5 in [10]. Moreover, we get that a multivalued a-contraction $(0 \le a < 1)$ is weakly dissipative.

Theorem 2.3. *Let (X,d) be a metric space and $F : X \to P(X)$ be a δ-Reich type operator. Then the multi-valued operator F is dissipative.*

Proof. Let $\varepsilon > 0$ such that $a < \varepsilon < 1-b-c$. Let $x \in X$ and $y \in F(x)$. It is not difficult to see that

$$\varepsilon d(x,y) \le (1-b-c)\delta(x,F(x)).$$

Using the fact that $y \in F(x)$ and the condition from hypothesis we have

$$\delta(y, F(y)) \leq \delta(F(x), F(y)) \leq ad(x,y) + b\delta(x, F(x)) + c\delta(y, F(y)).$$

It follows that

$$-ad(x,y) \leq b\delta(x, F(x)) - (1-c)\delta(y, F(y)).$$

So, we have

$$\begin{aligned} d(x,y) &= (\varepsilon - a)^{-1}[\varepsilon d(x,y) - ad(x,y)] \\ &\leq (\varepsilon - a)^{-1}\big[(1-b-c)\delta(x,F(x)) + b\delta(x,F(x)) - (1-c)\delta(y,F(y))\big] \\ &= (1-c)/(\varepsilon - a)\big[\delta(x,F(x)) - \delta(y,F(y))\big]. \end{aligned}$$

We define $\varphi(x) : X \to \mathbb{R}_+ \cup \{+\infty\}$ as follows: $\varphi(x) = (1-c)/(\varepsilon - a)\delta(x, F(x))$, for each $x \in X$ and we get

$$d(x,y) + \varphi(y) \leq \varphi(x), \quad \text{for each } x \in X \text{ and for all } y \in F(x),$$

i.e. the multi-valued operator F is dissipative. □

An extension of Proposition 1 in [9] is as follows:

Theorem 2.4. *Let (X, d) be a complete metric space, $x_0 \in X$ be arbitrarily, $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ a proper lower semi-continuous function and $h : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous non-decreasing function such that $\int_0^\infty \frac{ds}{1+h(s)} = +\infty$. Let $F : X \to P_{cl}(X)$ be a multi-valued operator such that:*

$$\inf\left\{\frac{d(x,y)}{1+h(d(x_0,x))} + \varphi(y) : y \in F(x)\right\} \leq \varphi(x),$$

for each $x \in X$. Then F has at least a fixed point.

Proof. We shall prove that for each $x \in X$ there exists $f(x) \in F(x)$ such that:

$$\frac{d(x, f(x))}{1+h(d(x_0,x))} + 2\varphi(f(x)) \leq 2\varphi(x).$$

If $D(x, F(x)) = 0$ then $x \in F(x)$ and we put $x = f(x)$. If $D(x, F(x)) > 0$ then we get successively:

$$\begin{aligned} &\frac{D(x, F(x))}{1+h(d(x_0,x))} + \inf\left\{\frac{d(x,y)}{1+h(d(x_0,x))} + 2\varphi(y) : y \in F(x)\right\} \\ &\quad\leq 2\inf\left\{\frac{d(x,y)}{1+h(d(x_0,x))} + \varphi(y) : y \in F(x)\right\} \leq 2\varphi(x), \end{aligned}$$

for each $x \in X$. It follows that:

$$\inf\left\{\frac{d(x,y)}{1+h(d(x_0,x))}+2\varphi(y) : y \in F(x)\right\} < 2\varphi(x)$$

and hence there exists $f(x) \in F(x)$ such that:

$$\frac{d(x,f(x))}{1+h(d(x_0,x))}+2\varphi(f(x)) \leq 2\varphi(x).$$

If we define $\psi(t) = 2\varphi(t)$ we get that f satisfies the hypothesis of Lemma 1.2 in [15] and hence there exists $x^* \in X$ such that $x^* = f(x^*) \in F(x^*)$. □

In what follows we shall study the data dependence of the fixed points set of multi-valued operators which satisfy the Caristi type condition (1.3) and the data dependence of the strict fixed points set of multi-valued operators which satisfy the Caristi type condition (1.4), by giving some results.

Theorem 2.5. *Let (X,d) be a complete metric space and $F_1, F_2 : X \to P(X)$ be two multi-valued operators. We suppose that:*

(i) *there exist two lower semi-continuous functions $\varphi_1, \varphi_2 : X \to \mathbb{R}_+$ such that for all $x \in X$, there exists $y \in F_i(x)$ so that*

$$d(x,y) \leq \varphi_i(x) - \varphi_i(y), \quad i \in \{1,2\};$$

(ii) *there exists $c_i \in \,]0,+\infty[$ such that*

$$\varphi_i(x) \leq c_i d(x,y), \quad \text{for each } x \in X \text{ and for all } y \in F_i(x),\ i \in \{1,2\};$$

(iii) *there exists $\eta > 0$ such that*

$$H(F_1(x),F_2(x)) \leq \eta, \quad \text{for all } x \in X.$$

Then

$$H(\mathrm{Fix}(F_1),\mathrm{Fix}(F_2)) \leq \eta \max\{c_1,c_2\}.$$

Proof. From the condition (i) we have that $\mathrm{Fix}(F_i) \neq \emptyset$, $i \in \{1,2\}$. Let $\varepsilon \in \,]0,1[$ and $x_0 \in \mathrm{Fix}(F_1)$. It follows, from Ekeland variational principle (see for example [8]), that there exists $x^* \in X$ such that

$$\varepsilon d(x_0,x^*) \leq \varphi_2(x_0) - \varphi_2(x^*)$$

and

$$\varphi_2(x^*) - \varphi_2(x) < \varepsilon d(x, x^*), \quad \text{for each } x \in X \setminus \{x^*\}.$$

For $x^* \in X$, there exists $y \in F_2(x^*)$ so that

$$d(x^*, y) \le \varphi_2(x^*) - \varphi_2(y).$$

If we suppose that $y \neq x^*$, then we reach the contradiction

$$d(x^*, y) \le \varphi_2(x^*) - \varphi_2(y) < \varepsilon d(y, x^*).$$

So $y = x^*$ and therefore $x^* \in F_2(x^*)$, i.e. $x^* \in \mathrm{Fix}(F_2)$. Let $q \in \mathbb{R}$, $q > 1$. Then, there exists $x_1 \in F_2(x_0)$ such that

$$d(x_0, x_1) \le qH(F_1(x_0), F_2(x_0)).$$

Taking into account the conditions (ii) and (iii) we are able to write

$$\begin{aligned}\varepsilon d(x_0, x^*) &\le \varphi_2(x_0) - \varphi_2(x^*) = \varphi_2(x_0) \le c_2 d(x_0, x_1)\\ &\le c_2 q H(F_1(x_0), F_2(x_0)) \le c_2 q \eta.\end{aligned}$$

Hence

$$d(x_0, x^*) \le \eta c_2 q/\varepsilon.$$

Analogously, for all $y_0 \in \mathrm{Fix}(F_2)$, there exists $y^* \in \mathrm{Fix}(F_1)$ such that

$$d(y_0, y^*) \le \eta c_1 q/\varepsilon.$$

Using the last two inequalities, we obtain

$$H(\mathrm{Fix}(F_1), \mathrm{Fix}(F_2)) \le \eta q \varepsilon^{-1} \max\{c_1, c_2\}.$$

From this, letting $q \searrow 1$ and $\varepsilon \nearrow 1$, the conclusion follows. □

Remark 2.4. In the condition (ii) of the Theorem 2.5 it is sufficient to ask that $\varphi_i(x) = 0$, for all $x \in \mathrm{Fix}(F_i)$ and the existence of $c_i \in]0, +\infty[$ such that

$$\varphi_i(x) \le c_i d(x, y),$$

for each $x \in \mathrm{Fix}(F_j)$ and for all $y \in F_i(x)$, $i, j \in \{1, 2\}$, $i \neq j$.

Theorem 2.6. *Let (X,d) be a complete metric space and $F : X \to P(X)$ be a multi-valued operator. We suppose that:*

(i) *there exists $\varphi : X \to \mathbb{R}_+$ a lower semicontinuous function such that*

$$d(x,y) \leq \varphi(x) - \varphi(y), \quad \text{for each } x \in X \text{ and for all } y \in F(x);$$

(ii) *there exists $c \in \,]0,+\infty\,[$, such that*

$$\varphi(x) \leq cd(x,y), \quad \text{for each } x \in X \text{ and for all } y \in F(x).$$

Then

$$\mathrm{Fix}(F) = S\,\mathrm{Fix}(F) \neq \emptyset.$$

Proof. From the condition (i) we have that $S\,\mathrm{Fix}(F) \neq \emptyset$. Let $x^* \in \mathrm{Fix}(F)$ and $y \in F(x^*)$. It follows that

$$d(x^*,y) \leq \varphi(x^*) - \varphi(y) = -\varphi(y) \leq 0.$$

Hence $d(x^*,y) = 0$ and therefore $y = x^*$. So $F(x^*) = \{x^*\}$, i.e. $x^* \in S\,\mathrm{Fix}(F)$ and thus we are able to write that $\mathrm{Fix}(F) \subseteq S\,\mathrm{Fix}(F)$. □

Remark 2.5. In the condition (ii) of the Theorem 2.6 it is sufficient to ask that $\varphi(x) = 0$, for all $x \in \mathrm{Fix}(F)$.

Example 2.1. Let $F : [0,1] \to P([0,1])$, $F(x) = [x/3, x/2]$, for each $x \in [0,1]$ and $\varphi : X \to \mathbb{R}_+$, $\varphi(x) = kx$, for each $x \in [0,1]$, where $k \in \mathbb{R}$, $k \geq 1$. It is not difficult to see that $|x-y| \leq \varphi(x) - \varphi(y)$, for each $x \in [0,1]$ and for all $y \in F(x)$ and there exists $c = 2k > 0$ such that $\varphi(x) \leq c|x-y|$ for each $x \in [0,1]$ and for all $y \in F(x)$. From Theorem 2.6 we have $\mathrm{Fix}(F) = S\,\mathrm{Fix}(F) \neq \emptyset$.

Theorem 2.7. *Let (X,d) be a complete metric space and $F_1, F_2 : X \to P(X)$ be two multi-valued operators. We suppose that:*

(i) *there exist two lower semicontinuous functions $\varphi_1, \varphi_2 : X \to \mathbb{R}_+$ such that*

$$d(x,y) \leq \varphi_i(x) - \varphi_i(y),$$

for each $x \in X$ and for all $y \in F_i(x)$, $i \in \{1,2\}$;

(ii) *there exists* $c_i \in]0, +\infty[$ *such that*

$$\varphi_i(x) \leq c_i d(x, y), \quad \text{for each } x \in X \text{ and for all } y \in F_i(x),\ i \in \{1, 2\};$$

(iii) *there exists* $\eta > 0$ *such that*

$$H(F_1(x), F_2(x)) \leq \eta, \quad \text{for all } x \in X.$$

Then

$$H(\operatorname{Fix}(F_1), \operatorname{Fix}(F_2)) = H(S\operatorname{Fix}(F_1), S\operatorname{Fix}(F_2)) \leq \eta \max\{c_1, c_2\}.$$

Proof. From the Theorem 2.6 we have $\operatorname{Fix}(F_i) = S\operatorname{Fix}(F_i) \neq \emptyset$, $i \in \{1, 2\}$ and applying Theorem 2.5 the conclusion follows. □

We mention that the Remark 2.4 can be made for this theorem, too.

Example 2.2. Let $F_1, F_2 : [0, 1] \to P([0, 1])$, $F_1(x) = [x/3, x/2]$, for each $x \in [0, 1]$ and $F_2(x) = [(x + 1)/2, (x + 2)/3]$, for each $x \in [0, 1]$. Let $\varphi_1, \varphi_2 : [0, 1] \to \mathbb{R}_+$, $\varphi_1(x) = x$, for each $x \in [0, 1]$ and $\varphi_2(x) = 1 - x$, for each $x \in [0, 1]$. By an easy calculation we get that $|x - y| \leq \varphi_i(x) - \varphi_i(y)$, for each $x \in [0, 1]$ and for all $y \in F_i(x)$, $i \in \{1, 2\}$ and there exist $c_1 = 2$ and $c_2 = 2$ such that $\varphi_i(x) \leq c_i|x - y|$, for each $x \in [0, 1]$ and for all $y \in F_i(x)$, $i \in \{1, 2\}$. Also, there exists $\eta = 2/3 > 0$ so that $H(F_1(x), F_2(x)) \leq \eta$, for all $x \in [0, 1]$. Then, from Theorem 2.7 we have $H(\operatorname{Fix}(F_1), \operatorname{Fix}(F_2)) = H(S\operatorname{Fix}(F_1), S\operatorname{Fix}(F_2)) \leq 4/3$.

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