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On functions additive with respect to algorithms

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Abstract. In this paper we prove that for an arbitrary interval filling sequence there exist two algorithms such that the additivity of a function with respect to them implies its linearity. In contrast to some known results cited in Section 2 of this paper, we will prove the linearity of the function without requiring any special properties for the interval filling sequence and any regularity properties for the function.

1. Introduction

Let Λ be the set of the strictly decreasing sequences $\lambda = (\lambda_n)$ of positive real numbers for which $L(\lambda) := \sum_{n=1}^{\infty} \lambda_n < +\infty$. A sequence $(\lambda_n) \in \Lambda$ is called an *interval filling sequence* if, for any $x \in [0, L(\lambda)]$, there exists a sequence (δ_n) such that $\delta_n \in \{0, 1\}$ for all $n \in \mathbb{N}$ (the set of all positive integers) and $x = \sum_{n=1}^{\infty} \delta_n \lambda_n$. This concept has been introduced in [1]. The set of the interval filling sequences will be denoted by IF.

For a number $x \in \,]0, L(\lambda)[$ there can be more than one sequences $\delta = (\delta_n) \in \{0, 1\}^{\mathbb{N}}$ such that $x = \sum_{n=1}^{\infty} \delta_n \lambda_n$. For example, if $\lambda_n = q^{-n}$ where $1 < q < \frac{1+\sqrt{5}}{2}$ then for every $x \in \,]0, L(\lambda)[$ the cardinality of the set of such representations of x is continuum [9].

An *algorithm* (with respect to $\lambda = (\lambda_n) \in IF$) is defined in [3] as a sequence of functions $\alpha_n : [0, L(\lambda)] \to \{0, 1\}$ $(n \in \mathbb{N})$ for which

$$x = \sum_{n=1}^{\infty} \alpha_n(x) \lambda_n \qquad (x \in [0, L(\lambda)]).$$

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We denote the set of algorithms (with respect to $\lambda = (\lambda_n) \in IF$) by $\mathcal{A}(\lambda)$. Obviously, $\mathcal{A}(\lambda) \neq \emptyset$ for all $\lambda \in IF$, namely it was proved in [1] and [2] that, if $\lambda = (\lambda_n) \in IF$, $x \in [0, L(\lambda)]$ and

$$\varepsilon_n(x) = \begin{cases} 0 & \text{if } x < \sum_{i=1}^{n-1} \varepsilon_i(x)\lambda_i + \lambda_n, \\ \\ 1 & \text{if } x \geq \sum_{i=1}^{n-1} \varepsilon_i(x)\lambda_i + \lambda_n, \end{cases}$$

or

$$\varepsilon_n^*(x) = \begin{cases} 0 & \text{if } x \leq \sum_{i=1}^{n-1} \varepsilon_i^*(x)\lambda_i + \lambda_n, \\ \\ 1 & \text{if } x > \sum_{i=1}^{n-1} \varepsilon_i^*(x)\lambda_i + \lambda_n, \end{cases}$$

or

$$\varepsilon_n'(x) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n-1} \varepsilon_i'(x)\lambda_i + \sum_{i=n+1}^{\infty} \lambda_i < x, \\ \\ 0 & \text{if } \sum_{i=1}^{n-1} \varepsilon_i'(x)\lambda_i + \sum_{i=n+1}^{\infty} \lambda_i \geq x, \end{cases}$$

then $\varepsilon = (\varepsilon_n)$, $\varepsilon^* = (\varepsilon_n^*)$, $\varepsilon' = (\varepsilon_n') \in \mathcal{A}(\lambda)$. The algorithms ε, ε^* and ε' are called *regular (or greedy), quasiregular* and *antiregular (or lazy)* algorithms, respectively.

If $\lambda = (\lambda_n) \in IF$, $\mathcal{A}_0 \subset \mathcal{A}(\lambda)$, $\mathcal{A}_0 \neq \emptyset$, $F : [0, L(\lambda)] \to \mathbb{R}$ and

$$F(x) = \sum_{n=1}^{\infty} \alpha_n(x) F(\lambda_n) \qquad (x \in [0, L(\lambda)])$$

for all $(\alpha_n) \in \mathcal{A}_0$ then F will be called an $\mathcal{A}_0$-*additive* function (with respect to λ) [3]. If

$$F\left(\sum_{n=1}^{\infty} \delta_n \lambda_n\right) = \sum_{n=1}^{\infty} \delta_n F(\lambda_n) \qquad \left(\delta = (\delta_n) \in \{0,1\}^{\mathbb{N}}\right)$$

i.e. F is additive with respect to any algorithm then f is called *completely additive* [1].

2. Known results

Theorem 2.1 ([4]). *If* $\lambda = (\lambda_n) \in IF$ *and* $F : [0, L(\lambda)] \to \mathbb{R}$ *is a completely additive function (with respect to* λ*) then there exists* $c \in \mathbb{R}$*, such that* $F(x) = c \cdot x$ *for any* $x \in [0, L(\lambda)]$ *(i.e. briefly:* F *is linear).*

Theorem 2.2 ([2]). *If* $\lambda = (\lambda_n) \in IF$ *and* $F : [0, L(\lambda)] \to \mathbb{R}$ *is* $\{\varepsilon\}$*-additive then* F *is right continuous.*

Remark 2.3 ([2]). There exist $\lambda = (\lambda_n) \in IF$ and $F : [0, L(\lambda)] \to \mathbb{R}$ such that F is ε-additive but F is non-continuous at the points of a countably infinite dense set. (At the so-called finite points, i.e. the points x for which $\{n \in \mathbb{N} \mid \varepsilon_n(x) = 1\}$ is a finite set.)

Theorem 2.4 ([2]). *If* $\lambda = (\lambda_n) \in IF$ *and* $F : [0, L(\lambda)] \to \mathbb{R}$ *is* $\{\varepsilon, \varepsilon^*\}$*-additive then* F *is continuous.*

Remark 2.5 ([5]). There exist $\lambda = (\lambda_n) \in IF$ and $F : [0, L(\lambda)] \to \mathbb{R}$ such that F is $\{\varepsilon, \varepsilon^*\}$-additive (i.e. continuous) but is non-differentiable at any point $x \in [0, L(\lambda)]$.

Theorem 2.6 ([6]). *Let* $F : [0, L(\lambda)] \to \mathbb{R}$ *be a so-called smooth interval filling sequence and let* F *be* $\{\varepsilon, \varepsilon^*\}$*-additive (i.e. continuous). If* F *is differentiable on a set of positive measure or* $F(x) > 0$ *for* $x > 0$ *then* F *is linear.*

Theorem 2.7 ([7]). *If* $\lambda = (\lambda_n) \in IF$ *and* $F : [0, L(\lambda)] \to \mathbb{R}$ *is* $\{\varepsilon, \varepsilon^*\}$*-additive (i.e. continuous) and* F *is differentiable at a finite point then* F *is linear.*

Theorem 2.8 ([8]). *Let* $\lambda = (\lambda_n) \in IF$ *and* $\lambda_n \geq \lambda_{n+1} + \lambda_{n+2}$ *for* $n \in \mathbb{N}$*. If* $F : [0, L(\lambda)] \to \mathbb{R}$ *is an* $\{\varepsilon, \varepsilon^*, \varepsilon'\}$*-additive function then* F *is linear.*

3. Sufficiency of two algorithms

Definition 3.1. Let $\lambda = (\lambda_n)$ be an interval filling sequence. For $x \in [0, L(\lambda)]$ and $n \in \mathbb{N}$ let

$$\varepsilon_n^M(x) = \begin{cases} \varepsilon_n^*(x) & \text{if } \ x = \lambda_m \text{ for an } m \in \mathbb{N}, \\ \varepsilon_n(x) & \text{otherwise.} \end{cases}$$

It is obvious that $\varepsilon^M = (\varepsilon_n^M)$ is an algorithm, it will be called the *mixed regular* algorithm.

Theorem 3.2. *If $\lambda = (\lambda_n) \in IF$ and $F : [0, L(\lambda)] \to \mathbb{R}$ is additive with respect to the mixed regular algorithm then F is continuous.*

Proof. We will prove that F is $\{\varepsilon, \varepsilon^*\}$-additive and the continuity will follow from Theorem 2.4. The $\{\varepsilon\}$-additivity of F is obvious. If $\{n \in \mathbb{N} \mid \varepsilon_n(x) = 1\}$ is an infinite set then $(\varepsilon_n^*(x))$ coincides with $(\varepsilon_n(x))$. The case when $x = \lambda_m$ for an $m \in \mathbb{N}$ is also trivial. Thus we have to deal only with the quasiregular representations of those numbers x for which $\{n \in \mathbb{N} \mid \varepsilon_n(x) = 1\}$ has finitely many, but at least two elements. Let us denote the maximum of this set by k. Then

$$\begin{aligned} F(x) &= F\left(\sum_{n=1}^{\infty} \varepsilon_n^M(x)\lambda_n\right) = \sum_{n=1}^{\infty} \varepsilon_n^M(x)F(\lambda_n) = \sum_{n=1}^{\infty} \varepsilon_n(x)F(\lambda_n) \\ &= \sum_{n=1}^{k} \varepsilon_n(x)F(\lambda_n) = \sum_{n=1}^{k-1} \varepsilon_n(x)F(\lambda_n) + F(\lambda_k) = \sum_{n=1}^{k-1} \varepsilon_n(x)F(\lambda_n) \\ &\quad + F\left(\sum_{n=1}^{\infty} \varepsilon_n^M(\lambda_k)\lambda_n\right) = \sum_{n=1}^{k-1} \varepsilon_n(x)F(\lambda_n) + \sum_{n=1}^{\infty} \varepsilon_n^M(\lambda_k)F(\lambda_n) \\ &= \sum_{n=1}^{k-1} \varepsilon_n^*(x)F(\lambda_n) + \sum_{n=k+1}^{\infty} \varepsilon_n^*(\lambda_k)F(\lambda_n) = \sum_{n=1}^{\infty} \varepsilon_n^*(x)F(\lambda_n), \end{aligned}$$

so F is $\{\varepsilon^*\}$-additive and this completes our proof. □

To prove our main result we will need the following two lemmas.

Lemma 3.3. *Let $\lambda = (\lambda_n) \in IF$ and let $F : [0, L(\lambda)] \to \mathbb{R}$ be a continuous function. If*

$$F\left(\sum_{n=1}^{\infty} \alpha_n\lambda_n\right) = \sum_{n=1}^{\infty} \alpha_n F(\lambda_n)$$

whenever $(\alpha_n) \in \{0,1\}^{\mathbb{N}}$ and $\{n \in \mathbb{N} \mid \alpha_n = 1\}$ is finite then F is linear.

Proof. We will prove the complete additivity of F and the linearity will follow from Theorem 2.1. Let $(\delta_n) \in \{0,1\}^{\mathbb{N}}$. Then

$$\begin{aligned} F\left(\sum_{n=1}^{\infty} \delta_n\lambda_n\right) &= F\left(\lim_{k\to\infty} \sum_{n=1}^{k} \delta_n\lambda_n\right) = \lim_{k\to\infty} F\left(\sum_{n=1}^{k} \delta_n\lambda_n\right) \\ &= \lim_{k\to\infty} \left(\sum_{n=1}^{k} \delta_n F(\lambda_n)\right) = \sum_{n=1}^{\infty} \delta_n F(\lambda_n), \end{aligned}$$

so F is completely additive. $\square$

Lemma 3.4. *If $B^{(i)}$ is a countably infinite set for each $i \in \mathbb{N}$ then there exist pairwise disjoint countably infinite sets $C^{(i)}$ $(i \in \mathbb{N})$ such that*

$$C^{(i)} \subset B^{(i)} \qquad (i \in \mathbb{N}).$$

Proof. Let $\sigma = (\sigma_1, \sigma_2) : \mathbb{N} \to \mathbb{N}^2$ be a bijection. We define a sequence (c_k) by recursion. Let $c_1 \in B_{\sigma_1(1)}$ and if $k > 1$ then let $c_k \in B_{\sigma_1(k)} \setminus \{c_n \mid n \in \mathbb{N},\ n < k\}$. Now

$$C^{(i)} := \{c_k \mid k \in \mathbb{N},\ \sigma_1(k) = i\} \qquad (i \in \mathbb{N}).$$

These sets $C^{(i)}$ $(i \in \mathbb{N})$ are obviously disjoint and it follows from the definition that $C^{(i)} \subset B^{(i)}$. Since $H_i = \{(i, n) \mid n \in \mathbb{N}\} \subset \mathbb{N}^2$ is an infinite set for every $i \in \mathbb{N}$, $\sigma^{-1}(H_i) \subset \mathbb{N}$ is also infinite. And if $k \in \sigma^{-1}(H_i)$ then $c_k \in C^{(i)}$, so $C^{(i)}$ is infinite. $\square$

Now we are ready to prove our main result:

Theorem 3.5. *Let $\lambda = (\lambda_n)$ be an arbitrary interval filling sequence. There exist two algorithms μ, ν such that if a function $F : [0, L(\lambda)] \to \mathbb{R}$ is $\{\mu, \nu\}$-additive then F is linear.*

Proof. Let $\mu = \varepsilon^M$. By Theorem 3.2, the μ-additivity of F implies its continuity, so, by Lemma 3.3, the proof of the theorem will be complete if we show that there exists an algorithm ν such that if F is $\{\mu, \nu\}$- additive then

$$F\left(\sum_{n=1}^{\infty} \alpha_n \lambda_n\right) = \sum_{n=1}^{\infty} \alpha_n F(\lambda_n)$$

whenever $(\alpha_n) \in \{0, 1\}^{\mathbb{N}}$ and $\{n \in \mathbb{N} \mid \alpha_n = 1\}$ is finite.

There exist countably many $0, 1$-sequences $\alpha = (\alpha_n)$ for which $\{n \in \mathbb{N} \mid \alpha_n = 1\}$ is finite. Hence there exists a sequence $(\alpha^{(i)})$ of *all* these sequences (i.e. $(\alpha_n^{(i)}) \in \{0, 1\}^{\mathbb{N}}$ for every $i \in \mathbb{N}$). Let us denote $\max\{n \in \mathbb{N} \mid \alpha_n^{(i)} = 1\}$ by $m(i)$. We define another sequence $(\beta^{(i)})$ of $0, 1$-sequences by the following formula:

$$\beta_n^{(i)} = \begin{cases} \alpha_n^{(i)} & \text{if } n < m(i), \\ 0 & \text{if } n = m(i), \\ \varepsilon_n^*(\lambda_{m(i)}) & \text{if } n > m(i). \end{cases}$$

Then $\sum_{n=1}^{m(i)} \alpha_n^{(i)} \lambda_n = \sum_{n=1}^{\infty} \beta_n^{(i)} \lambda_n$, denote this sum by $x^{(i)}$. Let

$$B^{(i)} = \Big\{ x \in \mathbb{R} \;\Big|\; x = \sum_{n=1}^{N} \beta_n^{(i)} \lambda_n, N \in \mathbb{N} \Big\}.$$

If $x \in B^{(i)}$ then let us denote by $N(x,i)$ the minimal integer N for which $x = \sum\limits_{n=1}^{N} \beta_n^{(i)} \lambda_n$. The sets $B^{(i)}$ satisfy the conditions of Lemma 3.4, so there exist pairwise disjoint infinite subsets $C^{(i)} \subset B^{(i)}$ $(i \in \mathbb{N})$. At this point we are able to define our second algorithm:

$$\nu_n(x) := \begin{cases} \beta_n^{(i)} & \text{if there is an } i \in \mathbb{N} \text{ such that } x \in C^{(i)} \text{ and } n \le N(x,i), \\ 0 & \text{if there is an } i \in \mathbb{N} \text{ such that } x \in C^{(i)} \text{ and } n > N(x,i), \\ \mu_n(x) & \text{if } \ x \notin \bigcup_{k=1}^{\infty} C^{(k)}. \end{cases}$$

The definition of ν is correct because of the disjoint property of sets $C^{(i)}$. Now let i be an arbitrary positive integer and let F be $\{\mu, \nu\}$-additive. Then

$$\begin{aligned} F\Big(\sum_{n=1}^{m(i)} \alpha_n^{(i)} \lambda_n \Big) &= F\Big(\sum_{n=1}^{\infty} \beta_n^{(i)} \lambda_n \Big) = F(x^{(i)}) = \lim_{\substack{x \to x^{(i)} \\ x \in C^{(i)}}} F(x) \\ &= \lim_{\substack{x \to x^{(i)} \\ x \in C^{(i)}}} F\Big(\sum_{n=1}^{\infty} \nu_n(x) \lambda_n \Big) = \lim_{\substack{x \to x^{(i)} \\ x \in C^{(i)}}} \Big(\sum_{n=1}^{\infty} \nu_n(x) F(\lambda_n) \Big) \\ &= \lim_{\substack{x \to x^{(i)} \\ x \in C^{(i)}}} \Big(\sum_{n=1}^{N(x,i)} \beta_n^{(i)} F(\lambda_n) \Big) = \sum_{n=1}^{\infty} \beta_n^{(i)} F(\lambda_n) \\ &= \sum_{n=1}^{m(i)-1} \alpha_n^{(i)} F(\lambda_n) + \sum_{n=m(i)+1}^{\infty} \varepsilon_n^{*}(\lambda_{m(i)}) F(\lambda_n) \\ &= \sum_{n=1}^{m(i)-1} \alpha_n^{(i)} F(\lambda_n) + F(\lambda_{m(i)}) = \sum_{n=1}^{m(i)} \alpha_n^{(i)} F(\lambda_n), \end{aligned}$$

which was to be proved. $\square$

Remark 3.6. Note that $\mu(x) = \nu(x)$ for all but countably many points $x \in [0, L(\lambda)]$, so these two algorithms “almost coincide”. Moreover, it is easy to prove that for $(\lambda_n) = (\frac{1}{2^n})$ the additivity of F with respect to the mixed regular algorithm implies the linearity. It is an open problem to characterize those interval filling sequences $\lambda \in IF$ for which there exists *one* algorithm α such that if a function $F : [0, L(\lambda)] \to \mathbb{R}$ is $\{\alpha\}$- additive then F is linear.

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