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On weakly Ricci symmetric spaces

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Abstract. First we prove the existence of a weakly Ricci symmetric space by an example. Next we study some properties of a weakly Ricci symmetric space. Among others it is proved that a conformally flat weakly Ricci symmetric space is of almost constant curvature and if the Ricci tensor of the space is cyclic then the sum of the associated 1-forms is zero. Finally it is proved that a conformally flat weakly Ricci symmetric space-time is the Robertson–Walker space-time.

1. Preliminaries

The notions of weakly symmetric and weakly Ricci symmetric spaces were introduced by L. Tamássy and T. Q. Binh in [1], [2]. A non-flat Riemannian space V^n $(n > 2)$ is called weakly symmetric if the curvature tensor R_{hijk} statisfies the condition:

$$R_{hijk,\ell} = a_\ell R_{hijk} + b_h R_{\ell ijk} + c_i R_{h\ell jk} + d_j R_{hi\ell k} + e_k R_{hij\ell} \tag{1}$$

where a, b, c, d, e are 1-forms (non-zero simultaneously) " , " in (1) denotes covariant differentiation with respect to the metric tensor. a, b, c, d, e are called the associated 1-forms of the space. This space is denoted by $(WS)_n$. Such a space have been studied by M. Prvanović [3], T. Q. Binh [4], U. C. De and S. Bandyopadhyay [5] and others.

A non-flat Riemannian space is called weakly Ricci symmetric and denoted by $(WRS)_n$ if the Ricci tensor is non-zero and satisfies the condition:

$$R_{ij,k} = a_k R_{ij} + b_i R_{kj} + c_j R_{ik}. \tag{2}$$

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where a, b, c, are 1-forms (non-zero simultaneously). If in (1) a_ℓ is replaced by $2a_\ell$ and e_k is replaced by a_k then the space is called a generalized pseudo symmetric space intorduced by Chaki [6], and if in (2) a_k is replaced by $2a_k$ then the space is called a generalized pseudo Ricci symmetric space introduced by Chaki and Koley [7]. So the defining conditions of $(WS)_n$ and $(WRS)_n$ are little weaker than that of generalized pseudo symmetric and generalized pseudo Ricci symmetric spaces respectively. But weakly symmetric and weakly Ricci symmetric spaces are different from that of pseudo symmetric and Ricci pseudo symmetric spaces in the sense of R. Deszcz [8].

At the study of a $(WRS)_n$ the vector

$$\lambda_i = b_i - c_i$$

plays an important role.

In Section 2 we give a concrete example of a $(WRS)_n$. In Sections 3 and 4 we study $(WRS)_n$ of definite metric and with $\lambda \neq 0$. In Section 3 A) we show that the scalar curvature R does not vanish and $R_{ij} = R\theta_i\theta_j$ where θ is a unit vector, and B) we study the orthogonality of $R_{,h}$ and θ_h. In Section 4 we consider conformally flat $(WRS)_n$ $(n > 3)$ and show that the space is of almost constant curvature. In Section 5 we study $(WRS)_n$ of cyclic Ricci tensor. Finally we prove that a conformally flat weakly Ricci symmetric space-time is the Robertson–Walker space-time.

2. Example of $(WRS)_n$

Let each Latin index run over $1, 2, \dots, n$ and each Greek index over $2, 3, \dots, n-1$. We define the metric g in R^n $(n \geq 4)$ by the formula (see [9])

$$ds^2 = \varphi(dx^1)^2 + k_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1dx^n \tag{3}$$

where $[k_{\alpha\beta}]$ is a symmetric and non-singular matrix consisting of constant entries, and φ is a function independent of x^n.

In the metric considered, the only non-vanishing components of Christoffel symbols and Ricci tensor R_{ij} are (see [9])

$$\{{}_1{}^\beta{}_1\} = (-1/2)k^{\beta\alpha}\varphi_{.\alpha}\,, \quad \{{}_1{}^n{}_1\} = (1/2)\varphi_{.1}\,, \quad \{{}_1{}^n{}_\alpha\} = (1/2)\varphi_{.\alpha}$$

and

$$R_{11} = (1/2)k^{\alpha\beta}\varphi_{.\alpha\beta} \tag{4}$$

where (.) denotes partial differentiation and $[k^{\alpha\beta}]$ is the inverse matrix of $[k_{\alpha\beta}]$.

Here we assume that $k_{\alpha\beta} = \delta_{\alpha\beta}$ and $\varphi = \delta_{\alpha\beta}\ x^\alpha x^\beta e^{2x^1}$. In this case φ reduces to

$$\varphi = \sum_{\alpha=2}^{n-1} x^\alpha x^\alpha e^{2x^1}.$$

Hence $\varphi_{.\alpha\beta} = 2\delta_{\alpha\beta}e^{2x^1}$. Then it follows from (4) that the only non-zero components of R_{ij} and $R_{ij,k}$ are R_{11} and $R_{11,1}$ respectively, where

$$R_{11} = (1/2)\sum_{\alpha=1}^{n-1} k^{\alpha\alpha}\varphi_{.\alpha\alpha} = (1/2)(n-2)2.\,e^{2x^1} = (n-2)e^{2x^1} \tag{5}$$

$$R_{11,1} = 2(n-2)e^{2x^1}.$$

So neither the Ricci tensor nor its covariant derivative vanish.

We claim that R^n $(n > 3)$ with the given metric g is a $(WRS)_n$.

To verify the relation (2) it is sufficient to check the followings:

(A) $\quad R_{11,1} = a_1R_{11} + b_1R_{11} + c_1R_{11},$

(B) $\quad R_{11,k} = a_kR_{11} + b_1R_{k1} + c_1R_{k1}, \quad k \neq 1$

(C) $\quad R_{1j,1} = a_1R_{1j} + b_1R_{1j} + c_jR_{11}, \quad j \neq 1$

(D) $\quad R_{i1,1} = a_1R_{i1} + b_iR_{11} + c_1R_{i1}, \quad i \neq 1$

(E) $\quad R_{ij,1} = a_1R_{ij} + b_iR_{1j} + c_jR_{i1}, \quad i, j \neq 1$

(F) $\quad R_{ij,k} = a_kR_{ij} + b_iR_{kj} + c_jR_{ik}, \quad i, j \neq 1.$

In virtue of (5), (A) holds iff

$$2 = a_1(x) + b_1(x) + c_1(x), \tag{6}$$

(B) holds iff

$$0 = a_k(n-2)e^{2x^1}, \quad k \neq 1 \iff a_2 = \cdots = a_n = 0. \tag{7}$$

Similarly (C) and (D) yield

$$b_2 = \dots\dots\dots = b_n = 0,$$
$$c_2 = \dots\dots\dots = c_n = 0. \tag{8}$$

These mean that (2) can be satisfied by a number of $a(x)$, $b(x)$, $c(x)$, namely by those which fulfil (6), (7), (8).

Thus we have the following

Theorem 1. *R^n with the metric g, defined by (3) forms a weakly Ricci symmetric space.*

Remark. Here the metric defined by (3) is indefinite and $g^{11} = 0$. Therefore $R = g^{ij} R_{ij} = g^{11} R_{11} = 0$.

3. $(WRS)_n$ of definite metric

We assume that $\mathrm{Det}\,|g_{ij}| \neq 0$ (the space is definite) and $\lambda \neq 0$.

A) $R_{ij,h} - R_{ji,h} = 0$ yields

$$\lambda_j R_{ih} = \lambda_i R_{hj}, \quad \lambda_i = b_i - c_i. \tag{9}$$

Transvecting this by g^{ih} we get

$$\lambda_j R = \lambda^h R_{hj}. \tag{10}$$

Transvecting (9) again by λ^i we obtain

$$\lambda^i \lambda_j R_{ih} = \lambda^i \lambda_i R_{hj} = \|\lambda\|^2 R_{hj}, \quad \text{and hence}$$

$$R_{hj} = \frac{1}{\|\lambda\|^2} \lambda_j \lambda^i R_{ih} \overset{(10)}{=} \frac{\lambda_j \lambda_h}{\|\lambda\|^2} R = R \theta_j \theta_h, \tag{11}$$

where $\theta_h = \frac{\lambda_h}{\|\lambda\|^2}$ is a unit vector. From (11) we conclude that $R \neq 0$, for $R = 0$ implies $R_{hj} = 0$ which is inadmissible by the definition of $(WRS)_n$.

B) From (2) we obtain

$$R_{ij,k} - R_{ik,j} = T_k R_{ij} - T_j R_{ik}, \quad \text{where } T_j = a_j - c_j.$$

Then from (11) we have

$$R_{ij,k} - R_{ik,j} = R(T_k\theta_i\theta_j - T_j\theta_i\theta_k). \tag{12}$$

It follows from the 2nd Bianchi identity that $R^j{}_{k,j} = (1/2)R_{,k}$ ([10, p. 82]). Transvecting the equation (12) by g^{ij} and using the above relation we obtain

$$R_{,k} = 2R\big(T_k - (\theta^j T_j)\theta_k\big). \tag{13}$$

Now transvecting again by θ^k, we have

$$\theta^k(R_{,k}) = 2R\big(\theta^k T_k - \theta^j T_j\big) = 0. \tag{14}$$

Again transvecting (13) with T^k we get

$$T^k R_{,k} = 2R(T - P^2), \quad \text{where } T = T^k T_k,\ P = \theta^j T_j. \tag{15}$$

Thus from (11), (14) and (15) we have the following

Theorem 2. *In a $(WRS)_n$ of definite metric, if $b_i - c_i \neq 0$, then*

A) *the scalar curvature R is non-zero and the Ricci tensor R_{ij} is of the form*

$$R_{ij} = R\theta_i\theta_j$$

B) $R_{,k}$ is orthogonal to θ_k but it is not orthogonal to T_k except $P^2 = T$.

4. Conformally flat $(WRS)_n$ $(n > 3)$ of definite metric

For a conformally flat Riemannian space V^n we have ([10, p. 92])

$$R_{hijk} = \frac{1}{n-2}(g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}) - \frac{R}{(n-1)(n-2)}(g_{hk}\,g_{ij} - g_{hj}\,g_{ik}).$$

Let us suppose now that this V^n is a $(WRS)_n$, and $\lambda \neq 0$. Then in virtue of (11) we have

$$R_{hijk} = a(g_{hk}\,g_{ij} - g_{hj}\,g_{ik}) + b(g_{hj}\theta_i\theta_k - g_{hk}\theta_i\theta_j - g_{ij}\theta_h\theta_k + g_{ik}\theta_h\theta_j), \tag{16}$$

where $a = -\frac{R}{(n-1)(n-2)}$ and $b = -\frac{R}{(n-2)}$.

D. Smaranda [11] calls a Riemannian space whose curvature tensor satisfies (16), a space of almost constant curvature. Thus we can state the following:

Theorem 3. *In a conformally flat* $(WRS)_n$ $(n > 3)$ *of definite metric, if* $\lambda \neq 0$, *then the space is of almost constant curvature.*

Remark. The questions similar to that of Theorem 2 and Theorem 3 under different conditions were investigated by Chaki and Koley [7].

5. $(WRS)_n$ with cyclic Ricci tensor

A Riemannian space is said to have cyclic Ricci tensor if the Ricci tensor satisfies

$$R_{ij,k} + R_{jk,i} + R_{ki,j} = 0. \tag{17}$$

Using (2) in (17) we obtain

$$\theta_k R_{ij} + \theta_i R_{kj} + \theta_j R_{ik} = 0, \tag{18}$$

where $\theta_i = a_i + b_i + c_i$.

Now we sate

Walker's Lemma [12]: *If* a_{ij}, b_i *are numbers satisfying* $a_{ij} = a_{ji}$, $a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0$ *for* $i, j, k = 1, 2, \dots, n$, *then either all the* a_{ij} *are zero or all the* b_i *are zero.*

Hence by the above lemma we get from (18) that either $\theta_i = 0$ or $R_{ij} = 0$. But by definition of $(WRS)_n$, $R_{ij} \neq 0$. Therefore $\theta_i = 0$. Thus we can state the following.

Theorem 4. *If a* $(WRS)_n$ *satisfies cyclic Ricci tensor, then the sum of the associated 1-forms is zero.*

6. Results concerning the warped product

In an earlier paper [13] the first author and B. K. De proved that in a conformally flat generalized pseudo Ricci symmetric space the vector $\lambda_i = b_i - c_i$ is a proper concircular vector field. A weakly Ricci symmetric space is almost the same as a generalized pseudo Ricci symmetric space. Thus it is easy to see that a conformally flat $(WRS)_n$ admits a concircular vector field. K. Yano [14] proved that in order that a Riemannian space admits a concircular vector filed, it is necessary and sufficient that there exists a coordinate system with respect to which the fundamental quadratic differential form may be written in the form

$$ds^2 = (dx^1)^2 + f(x^1)\, \overset{*}{g}_{\alpha\beta}\, dx^\alpha dx^\beta,$$

where $\overset{*}{g}_{\alpha\beta} = \overset{*}{g}_{\alpha\beta}(x^\gamma)$ are the functions of x^γ only $(\alpha, \beta, \gamma = 2, 3, \dots, n)$ and f is a function of x^1 only.

Similarly we can prove that a Lorentzian space with the metric of signature $(- + + +)$ admits a concircular vector field if and only if there exists a coordinate system with respect to which the fundamental quadratic differential form may be written in the form

$$ds^2 = -(dx^1)^2 + f(x^1)\, \overset{*}{g}_{jk}\, dx^j dx^k.$$

In this section we consider a $(WRS)_n$ space-time. By a space-time, we will mean a 4-dimensional space endowed with Lorentz metric of signature $(- + + +)$. Since a conformally flat $(WRS)_n$ admits a concircular vector field, therefore the conformally flat $(WRS)_n$ can be expressed as the warped product $I \times_f \overset{*}{M}$ where I is an open interval of R and $(M, \overset{*}{g})$ is a 3-dimensional Riemannian space.

In a conformally flat space we have [10]

$$R_{ij,k} - R_{ik,j} = \frac{1}{2(n-1)}(R_{,k}\, g_{ij} - R_{,j}\, g_{ik}). \tag{19}$$

A. Gebarowski [15] proved that a warped product $I \times_f \overset{*}{M}$ satisfies (19) if and only if $\overset{*}{M}$ is an Einstein space. Thus if a $(WRS)_n$ space-time is conformally flat, it must be a warped product $I \times_f \overset{*}{M}$, where $\overset{*}{M}$ is a 3-dimensional Einstein space. It is known ([10]) that a 3-dimensional Einstein space is a space of constant curvature. Hence a conformally flat $(WRS)_n$ space-time is the warped product $I \times_f \overset{*}{M}$, where $\overset{*}{M}$ is a space of constant curvature. But such a warped product is Robertson–Walker space-time [16].

Thus we have the following

Theorem 5. *A conformally flat weakly Ricci symmetric space-time is the Robertson–Walker space-time.*

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References

[1] L. Tamássy and T. Q. Binh, On weakly symmetric and weakly projective symmetric Riemannian manifolds, *Coll. Math. Soc. J. Bolyai* **50** (1989), 663–670.

[2] L. Tamássy and T. Q. Binh, On weakly symmetries of Einstein and Sasakian manifolds, *Tensor. N. S.* **53** (1993), 140–148.

[3] M. Prvanović, On weakly symmetric Riemannian manifolds, *Publ. Math. Debrecen* **46** (1995), 19–25.

[4] T. Q. Binh, On weakly symmetric Riemannian spaces, *Publ. Math. Debrecen* **42** (1993), 103–107.

[5] U. C. De and S. Bandyopadhyay, On weakly symmetric Riemannian spaces, *Publ. Math. Debrecen* **54** (1999), 377-381.

[6] M. C. Chaki, On generalized pseudo symmetric manifolds, *Publ. Math. Debrecen* **45** (1994), 305–312.

[7] M. C. Chaki and S. Koley, On generalized pseudo Ricci symmetric manifolds, *Periodica Mathematica Hungarica* **28** (1994), 123–129.

[8] R. Deszcz, On pseudo-symmetric spaces, *Bull. Soc. Math. Belg.* Serie A, **44** (1992), 1–34.

[9] W. Roter, On conformally symmetric Ricci-recurrent spaces, *Colloquium Mathematicum* **31** (1974), 87–96.

[10] L. P. Eisenhart, Riemannian Geometry, *Princeton University Press*, 1949.

[11] D. Smaranda, Pseudo Riemannian recurrent manifolds with almost constant curvature, The XVIII Int. Conf. On Geometry and Topology, Oradea, 1989, 175–180, Preprint 88–2, Univ. "Babeş–Bolyai", Cluj Napoca, 1988.

[12] A. G. Walker, On Ruse's spaces of recurrent curvature, *Proc. of London Math. Soc.* **52** (1950), 36–54.

[13] U. C. De and B. K. De, On conformally flat generalized pseudo Ricci symmetric manifolds, *Soochow J. Math.* **23** (1997), 381–389.

[14] K. Yano, On the torse-forming directions in Riemannian spaces, *Proc. Imp. Academy, Tokyo* **20** (1944), 340–345.

[15] A. Gebarowski, Nearly conformally symmetric warped product manifolds, *Bulletin of the Institute of Mathematics, Academia Sinica,* **20** (1992), 359–371.

[16] B. O'Neill, Semi-Riemannian Geometry with Applicaions to Relativity, *Academic Press*, *New York*, 1983.

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