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# The conormal type function for CR manifolds

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**Abstract.** We introduce the total type function $\Phi(p, \delta)$, and its microlocalization, the conormal type function $\Phi(p, \xi, \delta)$, for a CR manifold $M$. These functions generalize important aspects of the classical Levi form, as well as higher Levi forms. We show how these type functions have direct applications to important questions about CR manifolds.

## 1. Introduction

In this paper we consider a smooth locally embeddable CR manifold $M$ of general type $(n, k)$; i.e., one which can be generically embedded in $\mathbb{C}^{n+k}$ with real codimension $k$. As a general reference for CR manifolds, the reader can consult [HN1]. We introduce here the *total type function* and the *conormal type function* for $M$.

The total type function $\Phi(p, \delta)$ is defined for $p \in M$ and $0 \le \delta \le \delta_0$. It is jointly continuous in both variables. Roughly speaking, it measures the maximal distance from the point $p$ to the centers of analytic discs of "radius" at most $\delta$ with boundaries on $M$. When $M$ is a hypersurface we may also consider $\Phi_+$ and $\Phi_-$ corresponding to the two sides of $M$.

The conormal type function $\Phi(p, \xi, \delta)$ is defined for $p \in M$, $0 \le \delta \le \delta_0$, and for all characteristic cotangent directions $\xi$ at $p$. Roughly speaking, it is the total type function "microlocalized" in the $\xi$-direction. It is also jointly continuous in all of its variables.

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These concepts are most naturally defined in a given coordinate system, with respect to a specific embedding. However the basic properties of interest are biholomorphic invariant, see §3 and §8.

Actually this paper is a natural continuation of [HT1], which should be regarded as providing the necessary preparatory study concerning analytic discs whose boundaries lie on $M$. It is also a direct continuation of our previous work [DH1], [DH2], in which we discussed only the case of a hypersurface $M^3$ in $\mathbb{C}^2$. There the situation simplifies considerably. The present paper is also related to [DH3], [BDN], [D] and [HN2]. A somewhat different point of view is taken in the papers of Baouendi and Rothschild [BR1], [BR2], [BR3], Baouendi, Rothschild and Trépreau [BRT], Hanges and Trèves [HaT], Trépreau [Tr1], [Tr2], [Tr3], and Tumanov [Tu1], [Tu2].

The conormal type function is designed to mimic and generalize certain aspects of the intrinsic Levi form of $M$ (see [HN1]) as is explained in §9. This provides a subtle tool which enables us to study, in an intrinsic way, points of infinite type as well as points of finite type. From this point of view it is very natural to define points of positive type and points of pseudoconcave type, see §6 and §9. Through a point of positive type there cannot pass the germ of a maximal dimensional complex analytic variety which lies on $M$, see §4. On the other hand, the notion of a point of pseudoconcave type, which uses the conormal type function, leads to a natural generalization of the concept of a pseudoconcave CR manifold, which uses the Levi form, (see [HN1]). This is illustrated by the applications in §6. These type functions give a natural instrument to measure the direction, as well as the distance, of holomorphic and CR extension of CR functions, see §5, §10 and §11. It should be mentioned that we are able to associate, to a point on $M$, a finite codimensional Banach submanifold $\Sigma$ of the infinite dimensional Banach space of all local analytic discs near the point whose boundaries lie on $M$, see §10 and §11. The authors feel that this Banach submanifold $\Sigma$ merits further study.

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## 2. The total type function

Let $M$ be a CR manifold of type $(n,k)$ which is locally generically embedded in $\mathbb{C}^{n+k}$. This means that CR-$\dim_{\mathbb{C}} M = n$ and $\operatorname{codim}_{\mathbb{R}} M = k$. Consider a point $p$ on $M$. The total type function $\Phi(p,\delta)$ is designed to measure the maximum distance that the center of an analytic disc can have from $p$, provided that the boundary of the disc lies on $M$, and that the disc has radius less than or equal to $\delta$, when measured in a Hölder norm $\alpha$, $0<\alpha<1$.

Let us assume first that $p$ is the origin, and the local defining equations for $M$ are

$$M_{\mathrm{loc}} : \begin{cases} y = h(x,w) \\ h(0,0)=0, \quad dh(0,0)=0, \end{cases} \tag{2.1}$$

where $h$ is a smooth map: $\mathbb{R}^k \times \mathbb{C}^n \to \mathbb{R}^k$ defined near the origin. The *total type function of* $M$ at the origin is defined by

$$\Phi(0,\delta) = \sup |y(0)[w]|, \tag{2.2}$$

where

$$y(0)[w] = \frac{1}{2\pi}\int_0^{2\pi} h(x(e^{i\theta})[w], w(e^{i\theta}))\, d\theta. \tag{2.3}$$

Here the supremum is taken over all $w(\cdot) \in [\mathcal{O}(D)\cap C^\alpha(\overline{D})]^n$ with $w(0)=0$ and $|w|_\alpha \le \delta$, where $D$ is the unit disc in $\mathbb{C}$, and $|y(0)[w]|$ denotes the Euclidean norm of the vector $y(0)[w]$ in $\mathbb{R}^k$. Here $|w|_\alpha$ indicates the $\alpha$ norm over $S^1$, which is equivalent to the $\alpha$ norm $|w|_\alpha^{\overline{D}}$ over $\overline{D}$ (see the Appendix). Note that $y(0)[w]$ is the center of the "lifted analytic disc", with boundary on $M$, and $x(e^{i\theta})[w]$ is the unique solution of the Bishop equation (A.8) corresponding to the parameters $c=0$, $w = w(\zeta) = (w_1(\zeta),\dots,w_n(\zeta)) \in [\mathcal{O}(D)\cap C^\alpha(\overline{D})]^n$ (see the discussion of analytic discs and the Bishop equation in the Appendix, and also [HT1]). In fact each lifted disc is of the form $g(\zeta) = (z(\zeta),\, w(\zeta))$, and the imaginary part of its first component $z(\zeta) = (z_1(\zeta),\dots,z_k(\zeta))$ has mean value given by (2.3). Here the bracket $[w]$ denotes the functional dependence on the parameter disc $w(\cdot)$.

Next consider a point $p$ on $M$ near the origin. In a small neighborhood of the origin we can fix a smooth mapping $\Omega_p : M_{\mathrm{loc}} \to AU(\mathbb{C}^{n+k})$, where

$AU(\mathbb{C}^{n+k})$ is the space of affine unitary transformations, such that for each $p \in M_{\text{loc}}$, $\Omega_p : \mathbb{C}^{n+k} \to \mathbb{C}^{n+k}$ arranges that

$$\Omega_p(p) = (0,0), \quad \Omega_{p*}(T_pM) = \{y = 0\}, \quad \Omega_{p*}(H_pM) = \{z = 0\},$$

where $(z, w) \in \mathbb{C}^k \times \mathbb{C}^n$ and $z = x + iy$. Using the standard inner product on $\mathbb{C}^k \times \mathbb{C}^n$, we have a natural choice of normal space at $p$, with

$$N_pM = \Omega_{p*}^{-1}(\{x = 0,\ w = 0\}).$$

Then for each $p \in M_{\text{loc}}$ we can express $\Omega_p M$ locally as a graph over its tangent space $\{y = 0\}$ at the origin:

$$\Omega_p M : \begin{cases} y = h^p(x, w) \\ h^p(0,0) = 0, \quad dh^p(0,0) = 0, \end{cases} \tag{2.4}$$

where $h^p$ is a smooth map: $\mathbb{R}^k \times \mathbb{C}^n \to \mathbb{R}^k$, defined near the origin.

Then the *total type function*, at the point $p \in M$, is defined by

$$\Phi(p, \delta) = \sup |y^p(0)[w]|, \tag{2.5}$$

where $y^p(0)[w]$ is given by (2.3), in which $h$ is replaced by $h^p$, and $x$ is replaced by $x^p$. Here $x^p$ denotes the unique solution to the Bishop equation for $h^p$, corresponding to the same parameters as before.

Actually the definitions given above depend on the local embedding of $M$ near $p$, and the choice of $\Omega$. We shall show, however, below that the essential properties of the total type function, which are useful for applications, are invariant. We first list some properties of $\Phi(p, \delta)$ that follow immediately from the definition:

$$\begin{aligned} &\Phi(p, 0) = 0, \quad \Phi(p, \delta) \geq 0 \\ &\Phi(p, \delta) \text{ is monotonically decreasing as } \delta \searrow 0. \end{aligned} \tag{2.6}$$

Using the stability results of [HT1, Section 5] concerning the continuous dependence of solutions to the Bishop equation with respect to perturbations of the functions $h$, it follows that the total type function $\Phi(p, \delta)$ is jointly continuous with respect to the variables $(p, \delta)$ for $(p, \delta) \in M_{\text{loc}} \times [0, \delta_0]$, for appropriate $\delta_0 > 0$.

The supremum in (2.5) can be achieved:

**Proposition 1.** *Let $(p, \delta)$ be fixed. Then there is a $w_0(\cdot) \in [\mathcal{O}(D) \cap C^\alpha(\overline{D})]^n$, with $w_0(0) = 0$ and $|w_0|_\alpha \le \delta$, such that*

$$\Phi(p, \delta) = |y(0)[w_0]|. \tag{2.7}$$

Proof. Let $\Phi(p, \delta) = m$. We set

$$B_{\alpha,\delta} = \{w(\cdot) \in [\mathcal{O}(D) \cap C^\alpha(\overline{D})]^n, \quad w(0) = 0, \quad |w|_\alpha \le \delta\}, \tag{2.8}$$

and recall that we are always taking parameters with $c = 0$. By definition of the supremum there is a sequence $\{w_j(\cdot)\}$, $j = 1, 2, \dots,$ in $B_{\alpha,\delta}$ such that $|y(0)[w_j]| \to m$ as $j \to \infty$. We need to extract a subsequence which converges in some norm with respect to which the solution of the Bishop equation depends continuously on the data. Choose $\alpha'$ with $0 < \alpha' < \alpha$. It is well known that bounded sets in $C^\alpha$ are precompact in $C^{\alpha'}$. Thus we may pass to a subsequence, which we indicate for simplicity again by $\{w_j\}$, that converges in $C^{\alpha'}$ to an element $w_0$.

We claim that $w_0 \in B_{\alpha,\delta}$: Since the convergence is in particular uniform, we have that $w_0(0) = 0$ and $w_0$ is holomorphic in $D$. To see that $|w_0|_\alpha \le \delta$, we set

$$\delta_j^1 = \sup_{\zeta \in S^1} |w_j(\zeta)|, \qquad \delta_j^2 = \sup_{\zeta,\eta \in S^1} \frac{|w_j(\zeta) - w_j(\eta)|}{|\zeta - \eta|^\alpha},$$

with $|w_j|_\alpha = \delta_j^1 + \delta_j^2 \le \delta$. Clearly $\delta_j^1 \to \delta_0^1 = \sup_{\zeta \in S^1} |w_0(\zeta)|$. For any $\varepsilon > 0$ there is a $j_0$ such that for $j > j_0$ we have $\delta_j^2 \le \delta - \delta_0^1 + \varepsilon$. Consequently, for $j > j_0$ and fixed $\zeta$ and $\eta$ we get

$$|w_j(\zeta) - w_j(\eta)| \le \delta_j^2 |\zeta - \eta|^\alpha \le (\delta - \delta_0^1 + \varepsilon)|\zeta - \eta|^\alpha.$$

Passing to the limit as $j \to \infty$ we obtain

$$|w_0(\zeta) - w_0(\eta)| \le (\delta - \delta_0^1 + \varepsilon)|\zeta - \eta|^\alpha$$

and from this

$$|w_0(\zeta) - w_0(\eta)| \le (\delta - \delta_0^1)|\zeta - \eta|^\alpha.$$

Since $\zeta$ and $\eta$ were arbitrary, we obtain that

$$\delta_0^2 = \sup_{\zeta,\eta} \frac{|w_0(\zeta) - w_0(\eta)|}{|\zeta - \eta|^\alpha} \le \delta - \delta_0^1,$$

which gives that $w_0 \in C^\alpha$ and $|w_0|_\alpha \leq \delta$.

To conclude we use the continuous dependence of solutions to the Bishop equation on the data, with respect to the $\alpha'$-norm, obtaining that $|y(0)[w_j]| \to |y(0)[w_0]|$ as $j \to \infty$. Hence $|y(0)[w_0]| = m$ as desired. □

### 3. Biholomorphic invariance of the total type function

Now we prove that the essential properties of the function $\Phi(p,\delta)$ are invariant under local biholomorphic mappings. Let $M \subset \mathbb{C}^k \times \mathbb{C}^n$ and $\tilde{M} \subset \mathbb{C}^k \times \mathbb{C}^n$ be locally embedded CR manifolds, each having type $(n,k)$. Assume that $F$ is a biholomorphic mapping of a neighborhood $U$ of $M$ onto a neighborhood $\tilde{U}$ of $\tilde{M}$, such that $F|_M : M \to \tilde{M}$. We set $\tilde{p} = F(p)$.

**Theorem 1.** *For $K \subset M$ a compact neighborhood of $p_0$, there exist positive constants $a$, $b$, $A$, $B$, $\delta_0$ such that*

$$a\Phi(p, b\delta) \leq \tilde{\Phi}(\tilde{p}, \delta) \leq A\Phi(p, B\delta),$$

*for all $(p,\delta) \in K \times [0, \delta_0]$.*

*Remark.* It follows from (2.6) that for the function $\Phi$ there are two mutually exclusive possibilities: either $\Phi(p,\delta) > 0$ for sufficiently small $\delta > 0$, or else $\Phi(p,\delta) \equiv 0$ for sufficiently small $\delta > 0$. The point of the above theorem is that these are biholomorphically invariant conditions, independent of the choice of the local embedding.

Proof. We fix affine unitary transformations $\Omega$, $\tilde{\Omega}$ for $M$, $\tilde{M}$, respectively, as in Section 2. For each $p \in M$ we have a mapping $\Lambda^p = (\lambda_1^p, \lambda_2^p)$, with $\lambda_1^p$ having $k$ components and $\lambda_2^p$ having $n$ components, given by

$$\Lambda^p = \tilde{\Omega}_{\tilde{p}} \circ F \circ \Omega_p^{-1}, \quad \tilde{p} = F(p), \tag{3.1}$$

which maps a neighborhood of $\Omega_p M$ biholomorphically onto a neighborhood of $\tilde{\Omega}_{\tilde{p}}\tilde{M}$ and with

$$\Lambda^p|_{\Omega_p M} : \Omega_p M \to \tilde{\Omega}_{\tilde{p}}\tilde{M}. \tag{3.2}$$

It is clear that $\Lambda^p$ depends smoothly on $p$. It follows from the nature of $\Omega_p$, $\tilde{\Omega}_{\tilde{p}}$ that $\Lambda^p$ has the form

$$\Lambda^p(z,w) = \begin{cases} a(p)z + O(|(z,w)|^2) \\ c(p)z + d(p)w + O(|(z,w)|^2), \end{cases} \tag{3.3}$$

with $a(p)$ a real $k \times k$ matrix, $c(p)$ a complex $n \times k$ matrix, and $d(p)$ complex $n \times n$. These matrices depend smoothly on $p$.

Take an analytic disc $w = w(\zeta)$, $w \in [\mathcal{O}(D) \cap C^\alpha(\overline{D})]^n$, $w(0) = 0$, $|w|_\alpha \le \delta$ and consider

$$y^p(0)[w] = \frac{1}{2\pi} \int_0^{2\pi} h^p(x^p(e^{i\theta})[w], w(e^{i\theta}))\, d\theta, \tag{3.4}$$

where $x^p = x^p(e^{i\theta})[w]$ is the solution to the Bishop equation (A.8) for $\Omega_p M$, corresponding to the data $(0, w(\cdot))$. The total type function $\Phi(p, \delta)$ is defined in terms of the function $w(\cdot) \mapsto y^p(0)[w]$, so we must take a closer look at this functional and the associated discs $\zeta \mapsto (z^p(\zeta)[w], w(\zeta))$.

Next we consider, one at a time, the lifts to $\tilde{\Omega}_{\tilde{p}} \tilde{M}$ which correspond to taking as data:

$$\begin{aligned}
&(3.5) \qquad \Re\lambda_1^p(z^p(0)[w], 0), \quad && \lambda_2^p(z^p(\cdot)[w], w(\cdot)),\\
&(3.6) \qquad 0, && \lambda_2^p(z^p(\cdot)[w], w(\cdot)),\\
&(3.7) \qquad 0, && \lambda_2^p(z^p(\cdot)[w], w(\cdot)) - \lambda_2^p(z^p(0)[w], 0).
\end{aligned}$$

For the sake of simplicity, we drop the index $p$ in $z^p$, and also write $z(\zeta)$ instead of $z^p(\zeta)[w]$.

The lift corresponding to the first set of data (3.5) (i.e., the image of $\zeta \mapsto (z(\zeta), w(\zeta))$ by $\Lambda^p$) is

$$\zeta \mapsto (\lambda_1^p(z(\zeta), w(\zeta)), \lambda_2^p(z(\zeta), w(\zeta))).$$

It follows from (3.3) that there exist constants $c_1$, $c_2$ which depend only on the compact set $K \subset M$, and do not depend on $\delta$ or the disc $w = w(\zeta)$, such that

$$c_1 |y(0)| \le |\Im \lambda_1^p(iy(0), 0)| \le c_2 |y(0)|. \tag{3.8}$$

The second set of data (3.6) gives the following family of analytic discs

$$\begin{gathered}
\zeta \mapsto (z^\flat(\zeta),\ \lambda_2^p(z^\flat(\zeta),\ w(\zeta))),\\
z^\flat(\zeta) = x^\flat(\zeta) + iy^\flat(\zeta), \quad x^\flat(0) = 0.
\end{gathered}$$

Using the results of [HT1] concerning the dependence of the solution of the Bishop equation upon the data, we get that there exists a constant $c_3 = c_3(K)$, such that

$$|\lambda_1^p(z(0), 0) - iy^\flat(0)| \leq c_3 |\Re\lambda_1^p(z(0), 0)|. \tag{3.9}$$

Since $a(p)$ is real, by using (3.3) again, and $z(0) = iy(0)$, we get immediately that there exists a constant $c_4 = c_4(K)$ such that

$$|\Re\lambda_1^p(z(0), 0)| \leq c_4 |y(0)|^2. \tag{3.10}$$

Thirdly, the data (3.7) gives the following family of analytic discs

$$\begin{gathered}\zeta \mapsto (z^\sharp(\zeta),\ \lambda_2^p(z(\zeta),\ w(\zeta)) - \lambda_2^p(z(0), 0)),\\ z^\sharp(\zeta) = x^\sharp(\zeta) + iy^\sharp(\zeta), \quad x^\sharp(0) = 0.\end{gathered}$$

The difference between the parameter discs (3.6) and (3.7) is $\lambda_2^p(z(0), 0)$, and it obviously, as a constant, has $\alpha$-norm less than or equal to const $|y(0)|$, so using the properties of dependence of analytic discs on parameters again, we obtain that there exist constants $c_5 = c_5(K)$, $\tilde{c}_5 = \tilde{c}_5(K)$, such that

$$\sup_{|\zeta|\leq 1} |x^\sharp(\zeta) - x^\flat(\zeta)| \leq \tilde{c}_5 |\lambda_2^p(z(0), 0)| \leq c_5 |y(0)|. \tag{3.11}$$

By writing down the integrals for $y^\sharp(0)$, $y^\flat(0)$, respectively, then making use of (3.11), of the inequality $|\lambda_2(z(0), 0)| \leq \text{const}\,|y(0)|$, and of the fact that $h^p(0, 0) = 0$ and $dh^p(0, 0) = 0$, we get immediately that there exists $c_6 = c_6(K)$ such that

$$|y^\sharp(0) - y^\flat(0)| \leq c_6 \delta |y(0)|. \tag{3.12}$$

Combining (3.8), (3.9), (3.10), and (3.12) we see that there exist constants $c_7 = c_7(K)$, $c_8 = c_8(K)$, $\delta_0 = \delta_0(K)$, such that on $K$

$$c_7 |y(0)| \leq |y^\sharp(0)| \leq c_8 |y(0)| \tag{3.13}$$

for $0 \leq \delta \leq \delta_0$.

To complete the proof of the theorem we need to take a closer look at the family of analytic discs given by the data (3.7). Notice that this family of discs is allowed in the definition of the type function $\tilde{\Phi}$ at $F(p)$, because the value of the second component of (3.7) at 0 is zero.

We estimate the "radius" of the parameter disc (3.7), i.e. $|\lambda_2^p(z(\cdot), w(\cdot)) - \lambda_2^p(z(0), 0)|_\alpha$ in terms of $|w|_\alpha$. Using the property that the Hilbert transform (the operator $T$) is continuous in the $|\cdot|_\alpha$ norm, if $(z(\cdot), w(\cdot))$ is the lifted disc that corresponds to the parameter disc $(0, w(\cdot))$, then $|z|_\alpha \leq \text{const}\,|w|_\alpha$. So in particular there exists a constant $B = B(K)$, such that

$$|\lambda_2^p(z(\cdot), w(\cdot)) - \lambda_2^p(z(0), 0)|_\alpha \leq B|w|_\alpha. \tag{3.14}$$

This, together with (3.13), gives immediately

$$c_7\Phi(p, \delta) \leq \tilde{\Phi}(F(p), B\delta).$$

The other inequality is obtained by reversing the roles of $M$ and $\tilde{M}$. The proof is complete. □

## 4. Geometric meaning of the total type function

According to the remark following Theorem 1 the total type function enables us to discern two mutually exclusive possibilities at a point $p \in M$. In this section we investigate their geometric meaning.

First we consider the case where $\Phi(p, \delta) \equiv 0$.

**Theorem 2.** *The total type function $\Phi(p, \delta) \equiv 0$ for sufficiently small $\delta > 0$ if and only if there is a germ of a complex $n$-dimensional submanifold of $\mathbb{C}^{n+k}$ which lies on $M$ and passes through $p$.*

*Remark.* An open neighbourhood $U$ of $p \in M$ is foliated by such complex $n$-dimensional leaves if and only if $\Phi(q, \delta) \equiv 0$ for $(q, \delta)$ belonging to some neighborhood of $U \times \{0\}$ in $M \times \mathbb{R}^+$.

Proof of the theorem. The theorem is obvious in one direction; namely, if there is such a complex $n$-dimensional submanifold on $M$ passing through $p$, then $\Phi(p, \delta) \equiv 0$ for sufficiently small $\delta > 0$. This follows easily from the uniqueness of solutions to the Bishop equation, since in this case every lifted disc with sufficiently small parameters of the form $(0, w(\cdot))$ with $w(0) = 0$, must lie entirely on $M$ and have its center at the origin.

In order to prove the theorem in the opposite direction, we fix $p \in M$ and assume that $\Phi(p, \delta) \equiv 0$ for $0 \leq \delta \leq \delta_0$. Then for any analytic disc $w \in [\mathcal{O}(D) \cap C^\alpha(\overline{D})]^n$, $|w|_\alpha < \delta$, $w(0) = 0$, the solution $x(\cdot)[w]$ to the Bishop equation (A.8) with data $(0, w)$ satisfies (see (A.2))

$$E\{h(x(\cdot)[w], w(\cdot))\} = 0. \tag{4.1}$$

We fix $w$ for a moment, and take any function $q \in [\mathcal{O}(D) \cap C^\alpha(\overline{D})]^n$, $q(0) = 0$, and consider $w(\cdot) + tq(\cdot)$, where $t \in \mathbb{R}$ is sufficiently small. The solution $x(\cdot)[w + tq]$ can be written in the form

$$x(\cdot)[w + tq] = x(\cdot)[w] + t\ell(\cdot)[q] + o(t). \tag{4.2}$$

Moreover, the $\mathbb{R}$ linear functional $\ell(\cdot)[q]$ satisfies the linear Bishop equation

$$\ell(\cdot)[q] = -T\{h_x \ell(\cdot)[q] + h_w q + h_{\overline{w}}\overline{q}\}, \tag{4.3}$$

where the partial differentials $h_x$, $h_w$, $h_{\overline{w}}$ in (4.3) are taken at $(x(\cdot)[w], w(\cdot))$. Here (4.3) is an inhomogeneous real linear $k \times k$ system of equations, $h_x$ is a real $k \times k$ matrix, and $h_w$, $h_{\overline{w}}$ are $k \times n$ complex matrices.

Associated to (4.3) there is the $k \times k$ complex linear system

$$\eta = -T\{h_x \eta(\cdot)[q] + h_w q\}, \tag{4.4}$$

where $\eta = \eta(\cdot)[q]$ depends $\mathbb{C}$-linearly on $q$. If $\eta$ is a solution of (4.4), then the unique solution of (4.3) is given by

$$\ell(\cdot)[q] = \eta(\cdot)[q] + \overline{\eta(\cdot)[q]}. \tag{4.5}$$

Since $q(0) = 0$ the relation (4.1) must hold with $w$ replaced by $w + tq$. Differentiating the resulting relation with respect to $t$, and setting $t = 0$, we obtain an $\mathbb{R}$-linear system whose $\mathbb{C}$-linear form, as above, is

$$E\{h_x \eta(\cdot)[q] + h_w q\} = 0. \tag{4.6}$$

We shall need the following

**Lemma 1.** *The unique solution of* (4.4) *which satisfies* (4.6) *is given by*

$$\eta = i[I - ih_x]^{-1} h_w q. \tag{4.7}$$

Proof of the lemma. For the proof of this lemma we closely follow, in part, arguments by Tumanov [Tu1, p. 134].

As $h$ vanishes to the second order at the origin, so that $h_x$ is small, the Neuman series of the matrix system

$$G = I + T\{Gh_x\}$$

converges to a unique $k \times k$ fundamental solution $G$. Note that $E\{G\} = I$, where $I$ is the $k \times k$ identity matrix. Let $\varphi = h_x \eta + h_w q$. Then by the assumption of the lemma and using property (A.7), we have

$$E\{\varphi\} = 0, \quad \eta = -T\varphi, \quad \varphi = T\eta. \tag{4.8}$$

From the definition of $G$ and using again (A.7), we get

$$TG = -Gh_x + c, \quad c = E\{Gh_x\}, \tag{4.9}$$

and the constant matrix $c$ is arbitrarily small if $|w|_\alpha$ is sufficiently small. We have

$$\begin{aligned} Gh_w q &= G(\varphi - h_x \eta) = G\varphi + (-TG + c)T\varphi \\ &= G\varphi - (TG)(T\varphi) + cT\varphi. \end{aligned} \tag{4.10}$$

Applying the operator $T$ to both sides of (4.10), using $E\{\varphi\} = 0$ and (4.9), (A.4), (A.7), we obtain

$$\begin{aligned} T\{Gh_w q\} &= GT\varphi + (TG)\varphi - c\varphi = GT\varphi - Gh_x\varphi \\ &= -G\eta - Gh_x(h_x\eta + h_w q) \\ &= -G(I + h_x^2)\eta - Gh_x h_w q. \end{aligned} \tag{4.11}$$

It follows from (4.10), (4.8), (A.3), and (A.5) that $E\{Gh_w q\} = 0$ for arbitrary $q$. But the latter gives that $Gh_w$ is the boundary value of a $k \times n$ matrix with holomorphic entries.

As an immediate consequence we get that $T\{Gh_w q\} = -iGh_w q$ because both $Gh_w$ and $q$ are the boundary values of holomorphic functions. Insertion of this on the left in (4.11) allows us to evaluate $\eta$, and we find (4.7). The lemma is proved. $\square$

Now we continue with the proof of Theorem 2. Let us write the solution $x(\cdot)[w]$ in a slightly different form: we can write $w(\zeta) = a_1\zeta + a_2\zeta^2 + \dots$, where $a_j$ are column vectors of length $n$, and then $x(e^{i\theta})[w] \equiv x(a; e^{i\theta}) \equiv x(a_1, a_2, \dots; e^{i\theta})$.

Fix $a$ and consider small $b$ of the form $b = (0, \dots, 0, b_m, 0, \dots)$. Note that $b_m$ is a column vector of length $n$. By Lemma 1 we obtain

$$x(a+b; e^{i\theta}) = x(a; e^{i\theta}) + i[I - ih_x]^{-1} h_w b_m e^{im\theta} \\ - i[I + ih_x]^{-1} h_{\overline{w}} \bar{b}_m e^{-im\theta} + o(|b_m|).$$

Therefore the $k \times n$ Jacobian matrix

$$\text{(4.12)} \qquad \left.\frac{\partial(x_1, \dots, x_k)}{\partial(\bar{b}^1_m, \dots, \bar{b}^n_m)}\right|_{b_m=0} \equiv \left.\frac{\partial x}{\partial \bar{b}_m}\right|_{b_m=0} \\ = -i[I + ih_x]^{-1} h_{\overline{w}} e^{-im\theta} \equiv \psi(a; e^{i\theta}) e^{-im\theta},$$

in which the $k \times n$ matrix $\psi$ does not depend on $m$.

We have that

$$\int_0^{2\pi} \left[x(a; e^{i\theta}) + ih(x(a; e^{i\theta}), a_1 e^{i\theta} + a_2 e^{2i\theta} + \dots)\right] e^{il\theta} d\theta = 0 \\ \text{for } l = 1, 2, \dots,$$

because the vector function $e^{i\theta} \to x(a; e^{i\theta}) + ih(x(a; e^{i\theta}),\ a_1 e^{i\theta} + a_2 e^{2i\theta} + \dots)$ is the boundary value of a holomorphic function in $D$. Computing the partial differential of the above expression with respect to $\overline{a}_m$ we obtain

$$\text{(4.13)} \qquad \int_0^{2\pi} [\psi + ih_x\psi + ih_{\overline{w}}] e^{i(l-m)\theta}\, d\theta = 0 \qquad \text{for} \quad l, m = 1, 2, \dots.$$

Thus the integrand in (4.13) is identically zero, which means, in particular that the mapping

$$a_m \to x(a; 1) + ih(x(a; 1), a_1 + a_2 + \dots)$$

is holomorphic. We may take $m = 1$, $a_2 = a_3 = \cdots = 0$, and we arrive at the fact that

$$a_1 \to (x(a_1, 0, \ldots; 1) + ih(x(a_1, 0, \ldots; 1), a_1)$$

is a complex $n$-dimensional manifold in $\mathbb{C}^{k+n}$ which lies on $M$ and passes through the point under consideration. This completes the proof of Theorem 2. $\square$

The essential point of the following Corollary is that if a CR manifold of CR dimension $n$ contains the germ of a complex variety of dimension $n$, then the variety must be smooth.

**Corollary 1.** *The total type function $\Phi(p, \delta) > 0$ for sufficiently small $\delta > 0$ if and only if there does not exist the germ of a complex $n$-dimensional subvariety of $\mathbb{C}^{n+k}$ which lies on $M$ and passes through $p$.*

PROOF. We obtain the implication in one direction as a consequence of Theorem 2. Namely, if there does not exist the germ of a complex $n$-dimensional subvariety of $\mathbb{C}^{n+k}$ which lies on $M$ and passes through $p$, then we must have that $\Phi(p, \delta) > 0$ since the only other alternative is that $\Phi(p, \delta) \equiv 0$, which by Theorem 2 implies that $M$ contains the germ of a complex $n$-dimensional submanifold of $\mathbb{C}^{n+k}$ which lies on $M$ and passes through $p$, giving a contradiction.

In order to prove the implication in the opposite direction, we shall assume that $M$ does contain the germ of such a variety $V$ passing through $p$, and show that $V$ must be a complex $n$-dimensional submanifold. Then by Theorem 2 we obtain that $\Phi(p, \delta)$ cannot be $> 0$ for arbitrarily small $\delta > 0$.

Without loss of generality we may assume that $p$ is the origin and that $M$ is locally given as in (2.1). Let $\mathcal{D}$ be the Levi distribution on $M$; i.e., the distribution of real $2n$-planes on $M$ which assigns to each point $p$ on $M$ the holomorphic tangent space to $M$ at $p$. Let $\tilde{\mathcal{D}}$ be a smooth extension of the Levi distribution to an ambient neighborhood of $M$. Consider the decomposition $V = V_{\text{reg}} \cup V_{\text{sing}}$ into its regular and singular parts; we have $V_{\text{sing}} \subset \overline{V}_{\text{reg}}$, and that $0 \in V_{\text{sing}}$, since otherwise there is nothing to prove.

We introduce $W = \{(0, w) \in \mathbb{C}^k \times \mathbb{C}^n \mid (z, w) \in V_{\text{reg}}\}$, which is an open set in $\mathbb{C}^n$ whose closure contains the origin. Consider any real curve $\gamma$ which is $C^1$ in $W \cup \{0\}$ and set $\Gamma = \mathbb{C}^k \times \gamma$. The intersection $\tilde{\mathcal{D}} \cap \Gamma$ provides us with a $C^1$ distribution of real lines on the manifold $\Gamma$, which therefore has unique integral curves. There is only one such integral curve passing through 0.

The intersection $V \cap \Gamma = (V_{\text{reg}} \cap \Gamma) \cup \{0\}$ consists of a finite number of real integral curves, each of which approaches the origin in a $C^1$ manner. By the uniqueness mentioned above there can be only one such curve. As $\gamma$ was arbitrary it follows that $V_{\text{reg}}$ is the graph of a single holomorphic function, and hence that $V$ is a complex manifold.

This argument was inspired by a similar argument in [Pin]. $\square$

## 5. The conormal type function for a hypersurface

In this section we consider the special case where the CR manifold $M$ is a hypersurface; i.e., is of type $(n, 1)$. This case merits special attention due to the fact that domains in $\mathbb{C}^{n+1}$ are bounded by such hypersurfaces. Thus we consider (2.1) or (2.4) with $k = 1$ and choose a local orientation so that $y > h(x, w)$ corresponds to the + side, and $y < h(x, w)$ corresponds to the – side of the hypersurface.

This enables us to define the *conormal type functions* $\Phi_+(p, \delta)$ and $\Phi_-(p, \delta)$ by

$$\begin{aligned} \Phi_+(p, \delta) &= \sup\{y(0)[w]\} \\ \Phi_-(p, \delta) &= -\inf\{y(0)[w]\} \end{aligned} \tag{5.1}$$

where the supremum and infimum are taken just as before; i.e., with $c = 0$ and over all $w(\cdot) \in [\mathcal{O}(D) \cap C^\alpha(\overline{D})]^n$ with $w(0) = 0$ and $|w|_\alpha \leq \delta$.

Note that the conormal type function $\Phi_\pm(p, \delta)$ is a more refined concept than the total type function $\Phi(p, \delta)$. The function $\Phi_+(p, \delta)$ for example measures the "maximum distance", in the positively oriented direction, from the point $p$ to the centers of analytic discs of radius at most $\delta$ with boundaries on $M$.

As in (2.6) these functions satisfy:

$$\begin{aligned} &\Phi_\pm(p, 0) = 0, \quad \Phi_\pm(p, \delta) \geq 0 \\ &\Phi_\pm(p, \delta) \text{ is monotonically decreasing as } \delta \searrow 0. \end{aligned} \tag{5.2}$$

We have that $\Phi_\pm(p, \delta)$ is jointly continuous with respect to the variables $(p, \delta)$ for $(p, \delta) \in M_{\text{loc}} \times [0, \delta_0]$. Likewise the remark following Theorem 1 applies separately to either $\Phi_+(p,\delta)$ or $\Phi_-(p,\delta)$. In fact $\Phi_+(p,\delta)$ and $\Phi_-(p,\delta)$ each satisfy an estimate as in Theorem 1, as is easily seen; hence the essential properties of $\Phi_+$ and $\Phi_-$ are invariant under local biholomorphic mappings.

The relation between the total type function and the conormal type function for a hypersurface is simply

$$\Phi(p,\delta) = \max\{\Phi_+(p,\delta), \Phi_-(p,\delta)\},$$

see Proposition 3 and the remark following it.

**Proposition 2.**

(a) $\Phi_-(q,\delta) \equiv 0$ *for all points* $q$ *sufficiently near* $p$, *and for all sufficiently small* $\delta > 0$, *if and only if* $M$ *is weakly pseudoconvex at* $p$ *(i.e., the region* $\{y > h(x,w)\}$ *is weakly pseudoconvex near the origin).*

(b) $\Phi_-(q,\delta) \equiv 0$ *for all points* $q$ *sufficiently near* $p$, *and* $\Phi_+(p,\delta) > 0$; *for all sufficiently small* $\delta > 0$, *if and only if* $M$ *is weakly pseudoconvex at* $p$ *and* $M$ *does not contain the germ at* $p$ *of any complex* $n$*-dimensional subvariety.*

*Remarks.* We give examples below which show that the asymmetry between $p$ and $q$ in part (b) of the proposition cannot be removed:

1. The reverse implication in (b) fails if we change the statement about $\Phi_+$ to read that $\Phi_+(q,\delta) > 0$ for all $q$ sufficiently near $p$ and for all sufficiently small $\delta > 0$.

2. The forward implication in (b) fails if we change the statement about $\Phi_-$ to read that $\Phi_-(p,\delta) \equiv 0$ for all sufficiently small $\delta > 0$.

Proof of the proposition. It suffices to observe that if, for fixed $p$, $\Phi_-(p,\delta) \equiv 0$ for all sufficiently small $\delta > 0$, then the Levi form of $M$ at $p$ is positive semidefinite. Thus both implications in part (a) are immediate consequences of well known classical results about pseudoconvex domains (see [Lv], [Lw], [B], [W], [HT1]). In view of part (a), part (b) is a consequence of Corollary 1. $\square$

**Examples**

In this subsection we give examples concerning relations between the type functions and holomorphic extension in the case of hypersurfaces, as explained in the remark after Proposition 2.

*Example 1.* In $\mathbb{C}^2$, with coordinates $(z, w)$, we consider the hypersurface defined by $y = h(x, w)$, where

$$h(x,w) = \begin{cases} 0 & \text{if } \varepsilon^2|w|^2 - x^2 \leq 0 \\ \exp\left(-\dfrac{1}{\varepsilon^2|w|^2 - x^2}\right) & \text{if } \varepsilon^2|w|^2 - x^2 > 0. \end{cases}$$

Fix an arbitrarily small $\varepsilon > 0$, and take a sufficiently small neighborhood $U$ of the origin in the $(x, w)$ space. Let $M$ be the piece of hypersurface for which $(x, w) \in U$. For each $\varepsilon > 0$ there is a neighborhood $U$ such that $M$ is weakly pseudoconvex. It is easy to see that $\Phi_+(p, \delta) > 0$ for all sufficiently small $\delta > 0$. Let $\Sigma$ be the subset of $M$ which corresponds to $(x, w) \in U \cap \{\varepsilon^2|w|^2 - x^2 \geq 0\}$. Note that we may regard $\Sigma$ as being an exceptional set of arbitrarily small measure on $M$. By (b) $M$ does not contain the germ at 0 of any complex 1-dimensional subvariety. But for $q \in M \setminus \Sigma$ we clearly have $\Phi_+(q, \delta) \equiv 0$ for sufficiently small $\delta > 0$, since in that region $M$ is flat.

*Example 2.* Next we construct an example, also in $\mathbb{C}^2$, such that $\Phi_-(p, \delta) \equiv 0$ and $\Phi_+(p, \delta) > 0$, for all sufficiently small $\delta > 0$, but for which $M$ is not weakly pseudoconvex at $p$. A similar example was considered in [BT2].

With $(z, w) = (x + iy, u + iv)$ let $M$ be the hypersurface in $\mathbb{C}^2$ given by the equation

$$y = h(w) \equiv v^4 - c\,u^4, \tag{5.3}$$

where $0 \leq c \leq 1$ is a constant. We note that $M$ is not pseudoconvex for any $0 < c \leq 1$, and is convex for $c = 0$. In terms of the variable $w$, the equation takes the form

$$\begin{aligned} y = h(w) = \frac{1+c}{4}\Bigg[&\frac{1-c}{4(1+c)}w^4 - w^3\overline{w} \\ &+ \frac{3}{2}\cdot\frac{1-c}{1+c}w^2\overline{w}^2 - w\overline{w}^3 + \frac{1-c}{4(1+c)}\overline{w}^4\Bigg]. \end{aligned} \tag{5.4}$$

Since we consider the centers of analytic discs obtained by lifting the discs $w = w(\zeta)$, $w(0) = 0$, we note that the hypersurface $M$ has exactly the same properties, with respect to the centers of discs, as the following hypersurface $\tilde{M}$:

$$\text{(5.5)} \qquad \tilde{M}: \qquad y = -w^3\overline{w} + bw^2\overline{w}^2 - w\overline{w}^3, \quad \text{where } b = \frac{3}{2}\cdot\frac{1-c}{1+c}.$$

We note that $0 \le b \le \frac{3}{2}$ if $0 \le c \le 1$. It is easy to check that $\tilde{M}$ is pseudoconvex if $\frac{3}{2} \le b$ (if we allow all positive $b$'s), which in terms of $c$ can happen only if $c = 0$.

Take any function $w(\zeta) = u(\zeta) + iv(\zeta)$, $w(0) = 0$, $w \not\equiv 0$, holomorphic in $D$ and continuous on $\overline{D}$. Then

$$w^4(\zeta) = u^4(\zeta) - 6u^2(\zeta)v^2(\zeta) + v^4(\zeta) + 4i\big[u^3(\zeta)v(\zeta) - u(\zeta)v^3(\zeta)\big]$$

is also holomorphic and vanishes at 0. So, in particular, we have

$$\text{(5.6)} \qquad \frac{1}{2\pi}\int_0^{2\pi} \big[u^4(e^{i\theta}) - 6u^2(e^{i\theta})v^2(e^{i\theta}) + v^4(e^{i\theta})\big]\, d\theta = 0.$$

To simplify the notation we drop $e^{i\theta}$ in the integrals below. Using (5.6) and the Hölder inequality we obtain

$$\text{(5.7)} \qquad \begin{aligned} \int_0^{2\pi} \big(u^4 + v^4\big)\, d\theta &= \int_0^{2\pi} 6u^2v^2\, d\theta \\ &\le 6\left[\int_0^{2\pi} u^4\, d\theta\right]^{1/2} \left[\int_0^{2\pi} v^4\, d\theta\right]^{1/2}. \end{aligned}$$

The inequality in (5.7) cannot be improved in the class of functions $u$ and $v$ which are considered. This can be seen (after some computations) if we take the following family of functions $F_R$, $R \ge 1$, and let $R \to \infty$. Define

$$F_R(z) = u_R(z) + iv_R(z) = Re^{-i\pi/8}\left[\frac{1 + i\left(\frac{z(1+ia_R)+(i+a_R)}{z(i+a_R)+(1+ia_R)}\right)^{1/2}}{1 - i\left(\frac{z(1+ia_R)+(i+a_R)}{z(i+a_R)+(1+ia_R)}\right)^{1/2}}\right]^{1/4} - 1,$$

where

$$a_R = \frac{1 + 2R^4 - R^8}{-1 + 2R^4 + R^8}.$$

$F_R$ maps the unit disc onto a sector of the disc of radius $R$ centered at $-1$, with the opening equal to $\pi/4$, and with the real axis as the axis of symmetry. Also we have

$$F(0) = 0, \quad F(1) = R - 1, \quad F(-1) = -1.$$

This kind of example was suggested to the authors by Chris Bishop. (Similar examples are also considered in [Pic].)

Denote

$$\alpha = \int_0^{2\pi} u^4 \, d\theta, \quad \beta = \int_0^{2\pi} v^4 \, d\theta.$$

Then (5.7) implies $\alpha + \beta \leq 6\sqrt{\alpha\beta}$, which gives

$$17 - 12\sqrt{2} \leq \frac{\alpha}{\beta} \leq 17 + 12\sqrt{2}. \tag{5.8}$$

Now we consider the integral

$$\begin{aligned} I &= \frac{1}{2\pi} \int_0^{2\pi} h(w(e^{i\theta}))\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} [v^4(e^{i\theta}) - cu^4(e^{i\theta})]\, d\theta \\ &= \frac{\beta}{2\pi} \left(1 - c\frac{\alpha}{\beta}\right), \end{aligned} \tag{5.9}$$

which gives the $y$-component of the center of the analytic disc that corresponds to the parameters $(0, w(\cdot))$. Using (5.8) and (5.9), we obtain

$$\frac{\beta}{2\pi}\big[1 - c(17 + 12\sqrt{2}\,)\big] \leq I \leq \frac{\beta}{2\pi}\big[1 - c(17 - 12\sqrt{2}\,)\big]. \tag{5.10}$$

We have two consequences of (5.9) and (5.10):

(1) If $0 \leq c \leq \dfrac{1}{17 + 12\sqrt{2}}$, then $I \geq 0$; i.e., $\Phi_-(0, \delta) \equiv 0$.
Note that $c = \dfrac{1}{17 + 12\sqrt{2}}$ corresponds to $b = \sqrt{2}$.

(2) The property that the inequality in (5.7) cannot be improved gives that $I$ can be positive or negative if $\dfrac{1}{17 + 12\sqrt{2}} < c \leq 1$.

*Conclusion.* The function $\Phi_+(0, \delta) > 0$ for all sufficiently small $\delta > 0$. This follows from the form of the equation of the hypersurface (5.3).

In so far as $\Phi_-$ is concerned we have:

1° If $0 < c \leq \dfrac{1}{17+12\sqrt{2}}$, then $\Phi_-(0,\delta) \equiv 0$ but $M$ is not pseudoconvex.

2° If $\dfrac{1}{17+12\sqrt{2}} < c \leq 1$, then $\Phi_+(0,\delta) > 0$ and $\Phi_-(0,\delta) > 0$ for small $\delta > 0$.

Note that 1° establishes the claim made in point 2 of the remarks. Moreover, the behavior of $\Phi_+$ and $\Phi_-$ agrees with the holomorphic extension of CR functions from $M$, which is given by the sector property considered in [BT2]. Thus for this example, at the origin, we have (see below): CR functions extend to both sides if and only if 0 is a point of pseudoconcave type. CR functions extend to one side, and not to the other side, if and only if 0 is a point of positive type, which is not a point of pseudoconcave type, the side being determined by the conormal type function.

## 6. Points of positive and pseudoconcave type for a hypersurface

Now that we have the total type function and the conormal type functions available, for a hypersurface, we are able to make the following definitions: (1) a point $p$ on $M$ will be called a point of *positive type* iff $\Phi(p,\delta) > 0$ for sufficiently small $\delta > 0$. We shall say $M$ has positive type provided each point of $M$ is of positive type. (2) a point $p$ on $M$ will be called a point of *pseudoconcave type* iff $\Phi_+(p,\delta) > 0$ and $\Phi_-(p,\delta) > 0$ for sufficiently small $\delta > 0$. We shall say $M$ has pseudoconcave type provided each point of $M$ is of pseudoconcave type.

In terms of our previous discussion we have the following interpretations:

(i) $p$ is a point of positive type iff there does not exist the germ of any complex $n$-dimensional submanifold (equivalently $n$-dimensional subvariety) of $\mathbb{C}^{n+1}$ which lies on $M$ and passes through $p$.

(ii) If $p$ is a point of positive type, but not of pseudoconcave type, then continuous CR functions (or CR distributions) defined in a neighborhood of $p$ on $M$ all have holomorphic extensions to the side of $M$ where the conormal type function is positive, with continuous (distribution) boundary values. See Section 10 and the remark just below.

(iii) At a point $p$ of pseudoconcave type we have the local holomorphic extension of germs of CR functions (or CR distributions) to a full neighbourhood of $p$ in $\mathbb{C}^{n+1}$.

*Remark.* CR distributions on a hypersurface of pseudoconcave type are smooth. This follows from classical results about the extension of holomorphic functions, by using the Mayer–Vietoris sequence for distributions. The same Mayer–Vietoris sequence for distributions allows one to demonstrate the claim about holomorphic extension of distribution CR functions made in (ii) (see [AHLM], [HM], [NV]).

We now pass to some applications of these notions (see [HN2]).

**Theorem 3.** *Let $M$ be a connected CR hypersurface of positive type and let $g$, $g \not\equiv 0$, be a continuous CR function on $M$. Then*

(a) *The zero locus $Z_g = \{p \in M \mid g(p) = 0\}$ does not disconnect $M$.*

(b) *If $f$ is a continuous CR function defined on $M \setminus Z_g$ and if locally $f = O(g^{-k})$ for some $k \geq 1$, then $f$ has a CR meromorphic extension $\tilde{f}$ to all of $M$; i.e., $\tilde{f}$ is locally equal to the ratio of two continuous CR functions.*

*Remark.* Thus, according to (i) above, the zero locus of a continuous CR function cannot disconnect $M$, provided that the hypersurface $M$ does not contain the germ of any maximal dimensional complex submanifold. It worth pointing out that there are examples of Stensones [St] as follows: Let $D$ be a strongly pseudoconvex domain in $\mathbb{C}^{n+1}$ with a smooth $2n+1$-dimensional boundary $M$. It is possible to construct a function $G \in C(\overline{D}) \cap \mathcal{O}(D)$, whose boundary values $g$ provide a continuous CR function on $M$ such that $Z_g$ has Hausdorff dimension equal to $2n+1$.

The next result is an analogue of the classical *Riemann theorem on removable singularities*.

**Theorem 4.** *Let $M$ be a connected CR hypersurface of pseudoconcave type and let $g$, $g \not\equiv 0$, be a CR function on $M$. If $f$ is a bounded (locally on $M$) CR function on $M \setminus Z_g$, then $f$ uniquely extends to be a CR function on $M$.*

The following is a generalization of the classical *Jacobian theorem* for holomorphic maps.

**Theorem 5.** *Let $M_1$, $M_2$ be CR hypersurfaces of pseudoconcave type. Suppose that $F : M_1 \to M_2$ is a CR mapping which is a local homeomorphism. Then the real Jacobian of $F$ does not vanish at each point of $M_1$, $F$ is a local diffeomorphism of $M_1$ onto $F(M_1)$, and each local inverse is CR on its domain of definition in $M_2$.*

In [HN2] very simple proofs of the above theorems were found under stronger hypotheses involving the Levi form, which utilize the generalization [RS] of Radó's theorem to CR hypersurfaces. But by using the properties of the total and conormal type functions of a CR hypersurface, which we have developed, and the improved version of the Radó's theorem in [C], we are able to apply verbatim the same proofs.

## 7. The conormal type function in higher codimension

Now we turn to the definition of the conormal type function in the case where $M$ is of type $(n, k)$.

The duality pairing between the real tangent and cotangent bundles of $\mathbb{C}^{k+n}$ will be denoted by $\langle \eta, v \rangle$, with $\eta$ a real cotangent vector in $\mathbb{C}^{k+n}$ and $v$ a real tangent vector in $\mathbb{C}^{k+n}$. Since $M$ is always assumed to be locally CR embedded in some $\mathbb{C}^{k+n}$, it will cause no confusion to use the same letter $J$ to indicate both the intrinsic partial complex structure on $M$, associated to the CR structure, and the extrinsic multiplication by $\sqrt{-1}$ in $\mathbb{C}^{k+n}$. In other words we have $J : HM \to HM$, $J^2 = -I$, where $HM = TM \cap JTM$ denotes the holomorphic tangent bundle to $M$. The *characteristic bundle* $H^0M$ is the annihilator of $HM$ in $T^*M$; it is a rank $k$ subbundle of $T^*M$. The formal transpose ${}^tJ : H^0M \to N^*M$ satisfies $({}^tJ)^2 = -I$; actually ${}^tJ = -J$. Here $N^*M$ is the conormal bundle of $M$.

Let $M$ be represented as in (2.1) or (2.4). The *conormal type function*

$$\Phi : H^0M_{\text{loc}} \times [0, \delta_0] \to \mathbb{R}^+$$

is defined by

$$\Phi(p, \xi, \delta) = \sup \langle {}^tJ\xi,\ y^p(0)[w] \rangle, \tag{7.1}$$

in which

$$(p, \xi) \in H^0M, \quad \xi \in H^0_pM, \quad \mathbb{R}^+ = \{t \geq 0\}.$$

Here, just as before,

$$y^p(0)[w] = \frac{1}{2\pi}\int_0^{2\pi} h^p(x^p(e^{i\theta})[w], w(e^{i\theta}))\,d\theta, \tag{7.2}$$

with $c = 0$, the supremum being taken over all $w(\cdot) \in [\mathcal{O}(D) \cap C^\alpha(\overline{D})]^n$ with $w(0) = 0$ and $|w|_\alpha \leq \delta$, and $\delta_0$ is appropriately small.

The function $\Phi(p, \xi, \delta)$ is jointly continuous as a function of $(p, \xi, \delta)$. This follows, as before, from the stability of solutions to the Bishop equation. For each fixed $(p, \xi)$, as a function of $\delta$, we have

$$\begin{aligned} &\Phi(p,\xi,0) = 0, \quad \Phi(p,\xi,\delta) \geq 0 \\ &\Phi(p,\xi,\delta) \text{ is monotonically decreasing as } \delta \searrow 0. \end{aligned} \tag{7.3}$$

Hence for each fixed $(p, \xi)$, there are two mutually exclusive possibilities: either $\Phi(p, \xi, \delta) > 0$ for sufficiently small $\delta > 0$, or else $\Phi(p, \xi, \delta) \equiv 0$ for sufficiently small $\delta > 0$.

Note that for $\lambda > 0$

$$\Phi(p, \lambda\xi, \delta) = \lambda\Phi(p, \xi, \delta). \tag{7.4}$$

There is a simple relationship between the total type function and the conormal type function of $M$:

**Proposition 3.**

(a) $\Phi(p, \delta) = \max_{|\xi|=1} \Phi(p, \xi, \delta)$.

(b) *For any basis $\{\xi_j\}_{j=0}^k$ of $H_p^0 M$, there exist positive constants $c$ and $C$ such that*

$$c \max_{1\leq j\leq k}\{\Phi(p, \pm\xi_j, \delta)\} \leq \Phi(p, \delta) \leq C \max_{1\leq j\leq k}\{\Phi(p, \pm\xi_j, \delta)\}. \tag{7.5}$$

(c) *Let $\xi_1, \dots, \xi_k \in H_p^0 M$, and let $t_j \geq 0$, $\sum_{j=1}^k t_j = 1$. Then*

$$\Phi\Bigl(p, \sum_{j=1}^k t_j\xi_j, \delta\Bigr) \leq \sum_{j=1}^k t_j \Phi(p, \xi_j, \delta). \tag{7.6}$$

*In particular, if $\Phi(p, \xi_j, \delta) \equiv 0$, $j = 1, \dots, k$, for all sufficiently small $\delta > 0$, then $\Phi(p, \xi, \delta) \equiv 0$ for all $\xi$ from the simplex spanned by $\xi_1, \dots, \xi_k$.*

PROOF. One inequality in (a) is easy, namely since $|{}^tJ\xi| = |\xi| = 1$, it follows that

$$\langle {}^tJ\xi, y(0)[w]\rangle \leq |{}^tJ\xi|\,|y(0)[w]| = |y(0)[w]|.$$

Taking first the supremum of both sides, and then the maximum on the left, we obtain $0 \leq \max_{|\xi|=1} \Phi(p,\xi,\delta) \leq \Phi(p,\delta)$.

For the opposite inequality in (a), consider the parameter disc $w_0(\cdot)$ from Proposition 1; i.e., $|y(0)[w_0]| = \Phi(p,\delta)$. Note that the vector $y(0)[w_0]$ lies in the normal space $N_pM$. If it is the zero vector then there is nothing to prove. If on the other hand, $y(0)[w_0] \neq 0$ then there is a linear functional $\eta \in N_p^*M$, with $|\eta| = 1$, such that $\langle \eta, y(0)[w_0]\rangle = |y(0)[w_0]|$. It then suffices to take $\xi = ({}^tJ)^{-1}\eta = J\eta$.

Part (b) is an immediate consequence of the equivalence of the Euclidean and maximum norms in $\mathbb{R}^k$.

Part (c) follows from the estimates:

$$\begin{aligned}\Phi\left(p, \sum_{j=1}^k t_j\xi_j, \delta\right) &= \sup \left\langle \sum_{j=1}^k t_j {}^tJ\xi_j, y^p(0)[w] \right\rangle \\ &\leq \sum_{j=1}^k t_j \sup\langle {}^tJ\xi_j, y^p(0)[w]\rangle = \sum_{j=1}^k t_j \Phi(p,\xi_j,\delta).\end{aligned}$$

This proves the proposition. □

Note that when the codimension is equal to one, because of (7.4), we have in effect that $\xi$ can be replaced by $\{+,-\}$ and therefore (a) becomes

$$\Phi(p,\delta) = \max\{\Phi_+(p,\delta), \Phi_-(p,\delta)\}.$$

The supremum in (7.1) can be achieved:

**Proposition 4.** *Let $(p,\xi,\delta)$ be fixed. Then there is a $w_0(\cdot) \in [\mathcal{O}(D) \cap C^\alpha(\overline{D})]^n$, with $w_0(0) = 0$ and $|w_0|_\alpha \leq \delta$, such that*

$$\Phi(p,\xi,\delta) = \langle {}^tJ\xi, y^p(0)[w_0]\rangle. \tag{7.7}$$

The proof of Proposition 4 is exactly the same as of Proposition 1.

## 8. Biholomorphic invariance of the conormal type function

Next we investigate the behavior of the conormal type function under biholomorphic mappings. Let $M \subset \mathbb{C}^k \times \mathbb{C}^n$ and $\tilde{M} \subset \mathbb{C}^k \times \mathbb{C}^n$ be locally embedded CR manifolds, each having type $(n, k)$. Assume that $F$ is a biholomorphic mapping of a neighborhood $U$ of $M$ onto a neighborhood $\tilde{U}$ of $\tilde{M}$, such that $F|_M : M \to \tilde{M}$, and set $\tilde{p} = F(p)$. Likewise $\xi \in H_p^0 M$ and $\tilde{\xi} \in H_{\tilde{p}}^0 \tilde{M}$ are related by $\xi = F^* \tilde{\xi}$.

The biholomorphic mapping $F$ will slightly distort the condition we have imposed on the parameter disc corresponding to the lifted disc. As is usual in microlocal analysis, this must be compensated for by introducing conical neighborhood. We shall utilize the intersection of cones, having opening $\varepsilon$ about $\tilde{\xi} \neq 0$, with the unit sphere in the space of characteristic directions at a point $\tilde{p}$:

$$\Gamma_\varepsilon(\tilde{\xi}) \equiv \{\sigma \in H_{\tilde{p}}^0 \tilde{M} \mid |\sigma| = 1,\ \angle(\sigma, \tilde{\xi}) \leq \varepsilon\}. \tag{8.1}$$

**Theorem 6.** *For $K \subset M$ a compact neighborhood of $p_0 \in M$, and any $\varepsilon > 0$, there exist positive constants $A = A(\varepsilon)$, $B$ and $\delta_0$ such that*

$$\Phi(p, \xi, \delta) \leq A(\varepsilon) \max_{\sigma \in \Gamma_\varepsilon(\tilde{\xi})} \tilde{\Phi}(\tilde{p}, \sigma, B\delta), \tag{8.2}$$

*for all $p \in K$, $\xi \in H_p^0 M$ with $|\xi| = 1$ and $0 \leq \delta \leq \delta_0$.*

Proof. As in the beginning of the proof of Theorem 1, we fix affine unitary transformations $\Omega$, $\tilde{\Omega}$ for $M$, $\tilde{M}$, respectively. For each $p \in M$ we have a mapping $\Lambda^p = (\lambda_1^p, \lambda_2^p)$, with $\lambda_1^p$ having $k$ components and $\lambda_2^p$ having $n$ components, given by (3.1), which maps a neighborhood of $\Omega_p M$ biholomorphically onto a neighborhood of $\tilde{\Omega}_{\tilde{p}} \tilde{M}$ and such that (3.2) holds. The mapping $\Lambda^p$ depends smoothly on $p$ and is of the form (3.3).

Take an analytic disc $w = w(\zeta)$, $w \in [\mathcal{O}(D) \cap C^\alpha(\overline{D})]^n$, $w(0) = 0$, $|w|_\alpha \leq \delta$ and consider (3.4), where $x^p = x^p(e^{i\theta})[w]$ is the solution to the Bishop equation (A.8) for $\Omega_p M$, corresponding to the data $(0, w(\cdot))$.

Take any $\xi \in H_0^0 M$ with $|\xi| = 1$. To estimate $\Phi(0, \xi, \delta)$ it is enough to estimate the expression under the supremum on the right-hand side of (7.1). Notice that ${}^t J\xi = a_1 dy_1 + \cdots + a_k dy_k$ for some coefficients $a_1, \ldots, a_k$ with $a_1^2 + \cdots + a_k^2 = 1$. The center of a disc is of the form

$(y_1(0)[w], \ldots, y_k(0)[w])$. After application of the biholomorphic mapping $\Lambda$, we get ${}^tJ\tilde{\xi} = [(\Lambda^p)^*]^{-1}\,{}^tJ\xi = \tilde{a}_1 d\tilde{y}_1 + \ldots + \tilde{a}_k d\tilde{y}_k$. Note that $|{}^tJ\tilde{\xi}|$ is not necessarily 1, as biholomorphic maps are not isometries, but there are positive constants $c_1$ and $C_1$ such that $c_1 \leq |{}^tJ\tilde{\xi}| \leq C_1$.

Next we consider, as in the proof of Theorem 1, the lifts to $\tilde{\Omega}_{\tilde{p}}\tilde{M}$ which correspond to taking as data (3.5)–(3.7). Actually here we consider only (3.5) and (3.7). The center of the disc that corresponds to (3.5) is of the form $(\lambda_1^p(z(0), w(0)), \lambda_2^p(z(0), w(0)))$. Since the parameters for this disc do not correspond to the ones allowed in the definition of the type function, we consider the lift of the parameter disc (3.7). From the proof of Theorem 1 (see (3.8)–(3.12)), we get an estimate for the distance between the $\tilde{y}$ component of the centers of the above two discs:

$$|\Im\lambda_1^p(z(0), w(0)) - y^\sharp(0)| \leq C_2\delta|y^\sharp(0)|. \tag{8.3}$$

Now we start to estimate the conormal type function $\Phi(p, \xi, \delta)$. Note that to obtain (8.2) it is enough to prove the estimate if $\Phi(p, \xi, \delta) > 0$ for all $\delta > 0$ sufficiently small, since otherwise the estimate is obvious. By the argument from the proof of Proposition 1, for each $\delta > 0$ sufficiently small, we can choose a $w_\delta$ such that $\Phi(p, \xi, \delta) = \langle {}^tJ\xi, y^p(0)[w_\delta]\rangle > 0$. Then we have

$$\langle {}^tJ\xi, y^p(0)[w_\delta]\rangle = \langle {}^tJ\tilde{\xi}, \Lambda_*^p(y^p(0)[w_\delta], 0)\rangle. \tag{8.4}$$

It follows from the form (3.3) of $\Lambda^p$ that there is a constant $C_3$ so that

$$|(\lambda_1^p)_*(y^p(0)[w_\delta], 0) - \Im\{\lambda_1^p(iy^p(0)[w_\delta], 0)\}| \leq C_3|y^p(0)[w_\delta]|^2. \tag{8.5}$$

Using (8.4) and (8.5) we obtain by elementary geometry, that there is a constant $C_4(\varepsilon)$ such that

$$\begin{aligned} &\left\langle {}^tJ\tilde{\xi}, \Lambda_*^p(y^p(0)[w_\delta], 0)\right\rangle \\ &\leq C_4(\varepsilon) \sup_{\sigma \in \Gamma_{\varepsilon/2}(\tilde{\xi})} \left\langle {}^tJ\sigma, \Im\{\lambda_1^p(iy^p(0)[w_\delta], 0)\}\right\rangle, \end{aligned} \tag{8.6}$$

uniformly with respect to $\delta$.

Finally, if on the right in (8.6) we replace the disc that corresponds to the parameters (3.5) by the disc that corresponds to the parameters (3.7), and use (8.3), we obtain that there is a constant $C_5(\varepsilon)$, so that

$$\begin{aligned}(8.7) \qquad & \sup_{\sigma\in\Gamma_{\varepsilon/2}(\sigma)} \left\langle {}^tJ\sigma, \Im\{\lambda_1^p(iy^p(0)[w_\delta],0)\}\right\rangle \\ & \le C_5(\varepsilon) \sup_{\sigma\in\Gamma_{\varepsilon}(\sigma)} \left\langle {}^tJ\sigma, (y^{\#}(0)[w_\delta],0)\right\rangle,\end{aligned}$$

uniformly with respect to $\delta$.

To complete the proof, we use exactly the same argument as at the very end of the proof of Theorem 1. So we see that the "radius" of the parameter disc (3.7) can be estimated by const $|w|_\alpha$. This, together with (8.7) gives the estimate of the theorem. $\square$

Next, in order to gain some control over the behavior of centers of discs as $\delta \to 0$, we decompose the unit sphere $S^{k-1} = \{\xi \in H_p^0 M \mid |\xi| = 1\}$ as follows: let

$$\begin{aligned}\mathcal{P} &= \{\xi \in S^{k-1} \mid \Phi(p,\xi,\delta) > 0 \text{ for all sufficiently small } \delta > 0,\\ \mathcal{Z} &= \{\xi \in S^{k-1} \mid \text{there exists } \delta(\xi) > 0 \text{ such that } \Phi(p,\xi,\delta) \equiv 0\\ &\qquad \text{for all } 0 < \delta \le \delta(\xi)\}.\end{aligned}$$

This gives a disjoint decomposition $S^{k-1} = \mathcal{P} \cup \mathcal{Z}$.

If $A$ is a subset of $S^{k-1}$, we will say that $A$ is *convex on the sphere*, provided that $A$ is the intersection of $S^{k-1}$ with a convex cone in $H_p^0 M$ with vertex at the origin.

**Theorem 7.** *Fix a point $p \in M$, where $M$ is of type $(n,k)$. Then*

(a) *$\mathcal{Z}$ is an $F_\sigma$, is convex on the sphere, and hence is path connected, unless $\mathcal{Z}$ consists of two antipodal points. If $\mathcal{Z}$ is not the whole sphere, then $\overline{\mathcal{Z}}$ is contained in a closed hemisphere.*

(b) *$\mathcal{P}$ is a $G_\delta$, and is either void, or else contains an open hemisphere.*

(c) $\mathrm{Int}(\mathcal{Z})$ *is biholomorphically invariant.*

(d) *$\overline{\mathcal{P}}$ is biholomorphically invariant.*

(e) *If* $\mathrm{Int}(\mathcal{Z}) \neq \emptyset$ *then* $\overline{\mathcal{Z}} = \overline{\mathrm{Int}(\mathcal{Z})}$.

(f) $\mathrm{mes}_{k-1}(\overline{\mathcal{P}} \setminus \mathrm{Int}(\mathcal{P})) = \mathrm{mes}_{k-1}(\overline{\mathcal{Z}} \setminus \mathrm{Int}(\mathcal{Z})) = 0$.

Proof. To prove (a) we choose a sequence $\{\varepsilon_j\} \searrow 0$ and write $\mathcal{Z} = \bigcup_j \mathcal{Z}_j$, where

$$\mathcal{Z}_j = \{\xi \in S^{k-1} \mid \text{such that } \Phi(p, \xi, \delta) \equiv 0 \text{ for all } 0 < \delta \leq \varepsilon_j\}.$$

We note that the set $\mathcal{Z}_j$ is closed and consequently $\mathcal{Z}$ is an $F_\sigma$. It follows from (7.6) that each $\mathcal{Z}_j$ is convex, and hence $\mathcal{Z}$ is convex since it is a nested increasing union of the $\mathcal{Z}_j$. Suppose $\mathcal{Z}$ is not the whole sphere, then the cone subtended by $\mathcal{Z}$ has a supporting hyperplane, and $\overline{\mathcal{Z}}$ must be contained in one of the closed half spaces it determines, which intersects $S^{k-1}$ in a closed hemisphere.

For (b) we observe that $\mathcal{P}$ is a $G_\delta$, since its complement is an $F_\sigma$. If $\mathcal{P} \neq \emptyset$ then $\mathcal{Z}$ is not the whole sphere, so $\mathcal{P}$ contains an open hemisphere.

Part (c) is an immediate consequence of (8.2).

For part (d) it suffices to observe that $\overline{\mathcal{P}} = S^{k-1} \setminus \mathrm{Int}(\mathcal{Z})$.

To demonstrate (e), assume the contrary; i.e., that $\overline{\mathcal{Z}} \not\subset \overline{\mathrm{Int}(\mathcal{Z})}$. Then there exists an $x \in \mathcal{Z}$, and an open neighborhood $U$ of $x$ on $S^{k-1}$ such that $U \cap \overline{\mathrm{Int}(\mathcal{Z})} = \emptyset$. We form the cone $C$ in $H^0_p M$ having vertex at $x$ and with base $\mathrm{Int}(\mathcal{Z})$, and consider the infinite positive cone $\hat{C}$, with vertex at the origin, which is subtended by $C$. Set $\Sigma = \hat{C} \cap S^{k-1}$. Then we have that $\mathrm{Int}(\Sigma) \subset \mathrm{Int}(\mathcal{Z})$, as $\mathcal{Z}$ is convex on the sphere, and clearly $\mathrm{Int}(\Sigma) \cap U \neq \emptyset$, which gives a contradiction.

In order not to overburden the exposition with a measure theoretic argument, we leave the proof of (f) to the reader. $\square$

## 9. Relation to the Levi form and finite type

First consider the conormal type function of a hypersurface $M$ in $\mathbb{C}^{n+1}$ with a local defining equation (2.1), and set $\rho = h - y$. The classical Levi form of $M$ at the origin is then the Hermitian form in $n$ variables defined by $\sum_{j,k=1}^{n+1} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(0)\zeta_j\overline{\zeta}_k$ restricted to the holomorphic tangent space $\sum_{j=1}^{n+1} \frac{\partial \rho}{\partial z_j}(0)\zeta_j = 0$, where $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{n+1}) \in \mathbb{C}^{n+1}$. Here we choose $\Omega = \{\rho < 0\}$, so that $\Omega$ corresponds to the $+$ side of $M$. It is then staightforward to verify the following:

*If the classical Levi form at* 0 *has at least one positive (negative) eigenvalue, then* $\Phi_+(0, \delta) \geq \mathrm{const}\, \delta^2$ $(\Phi_-(0, \delta) \geq \mathrm{const}\, \delta^2)$ *for sufficiently* *small $\delta > 0$. Conversely, if $\Phi_-(0, \delta) \equiv 0$ ($\Phi_+(0, \delta) \equiv 0$) for all sufficiently small $\delta > 0$, then the classical Levi form at 0 is positive semidefinite (negative semidefinite).*

Of course if the classical Levi form at 0 has all zero eigenvalues, it can still happen that $\Phi_+$, or $\Phi_-$, or both, are positive.

Next suppose that $\Omega$ is weakly pseudoconvex near 0 (so $\Phi_-(0, \delta) \equiv 0$ for sufficiently small $\delta > 0$). If $M$ has finite D'Angelo type (i.e., finite singular 1-type, see [DA]) at 0, then $\Phi_+(0, \delta) > 0$ for sufficiently small $\delta > 0$, so that in our terminology 0 is a point of positive type. Of course, as we have seen in §5, in this situation 0 can have infinite D'Angelo type and yet be a point of positive type in our sense. Also it can happen that a point $p \in M$ is of positive type and a singular complex curve lies in $M$.

In fact we may drop the assumption that $\Omega$ is weakly pseudoconvex, and we have the following: If the point $p$ on $M$ is a point of finite regular 1-type $m$, then $p$ is a point of positive type, and for the total type function there is an estimate of the form $\Phi(p, \delta) \geq \text{const}\, \delta^m$, as $\delta \to 0$. The proof is the same as in [DH1, Proposition 5.1].

In higher codimension it is possible to give an intrinsic characterization of the Levi form of $M$ at $p$ (see [HN1]): Consider a general $M$ of type $(n, k)$ in $\mathbb{C}^{n+k}$. Given $\xi \in H_p^0 M$ and $X, Y \in H_p M$, let us choose $\tilde{\xi} \in \Gamma_{\text{loc}}(M, H^0 M)$ and $\tilde{X}, \tilde{Y} \in \Gamma_{\text{loc}}(M, HM)$ such that $\tilde{\xi}(p) = \xi$, $\tilde{X}(p) = X$, $\tilde{Y}(p) = Y$. We then have that

$$d\tilde{\xi}(X, Y) = -\xi([\tilde{X}, \tilde{Y}]), \tag{9.1}$$

and hence the two sides of (9.1) depend only on $\xi$, $X$, $Y$. In this way we associate to $\xi \in H_p^0 M$ a quadratic form

$$L(p, \xi, X) = \xi([J\tilde{X}, \tilde{X}]) = d\tilde{\xi}(X, JX)$$

on $H_p M$. This form is Hermitian with respect to the complex structure tensor $J$ of $H_p M$. Indeed we have

$$L(p, \xi, JX) = L(p, \xi, X) \qquad \forall X \in H_p M.$$

This Hermitian form $L(p, \xi, \cdot)$ is called the *intrinsic Levi form* of $M$ at $p$ and the characteristic direction $\xi$.

Suppose that for some $\xi \in H_p^0 M$ the intrinsic Levi form has at least one positive eigenvalue. Then the conormal type function $\Phi(p, \xi, \delta) > 0$

for all $\delta > 0$ sufficiently small; in fact we have an estimate of the form $\Phi(p, \xi, \delta) \geq \mathrm{const}\, \delta^2$. This estimate is an immediate consequence of Theorem 9.1 in [HT1]. Conversely, if for fixed $p$, $\Phi(p, -\xi, \delta) \equiv 0$ for sufficiently small $\delta > 0$, then the intrinsic Levi form $L(p, \xi, \cdot)$ is positive semidefinite.

Now let us impose the possitivity condition for all $\xi \neq 0$: Recall that $M$ is said to be pseudoconcave at $p$ (see [HN1] for details) provided that at $p$ the intrinsic Levi form has at least one negative (same as positive, upon replacing $\xi$ by $-\xi$) eigenvalue in each characteristic direction $\xi$; i.e., for all $\xi \in H^0_p M$, $\xi \neq 0$. In this case we have for the conormal type function, that

$$(9.2) \qquad \begin{cases} \Phi(p, \xi, \delta) > 0 \quad \text{for all } \xi \in H^0_p M,\ \xi \neq 0, \\ \text{and for all sufficiently small } \delta > 0. \end{cases}$$

Hence, in higher codimension, we may take (9.2) to be the definition of $p$ being a point of *pseudoconcave type* on $M$. Thus we obtain a generalization of the notion of a pseudoconcave CR manifold; namely we generalize to CR manifolds $M$ of *pseudoconcave type*, by requiring that (9.2) hold at every point $p \in M$. Likewise, for the total type function, if $\Phi(p, \delta) > 0$ for all sufficiently small $\delta > 0$, then $p$ will be called a point *positive type*.

Finally we explain the connection with "higher Levi forms": Choose any local basis $Z_1, \dots, Z_n$ near $p$ for the space of smooth complex vector fields of type $(1, 0)$; so that $\overline{Z}_1, \dots, \overline{Z}_n$ give a basis for those of type $(0, 1)$. Then consider all possible Lie brackets of the $Z_j$'s and $\overline{Z}_j$'s, and evaluate them at the point $p$.

**Proposition 5.** *Assume that, after some finite number of commutators, one obtains a vector which is not in the span of $Z_1, \dots, Z_n, \overline{Z}_1, \dots, \overline{Z}_n$ at $p$. Then $p$ is a point of positive type; if fact, there exists a characteristic direction $\xi$ at $p$ such that $\Phi(p, \xi, \delta) > 0$ for all sufficiently small $\delta > 0$.*

Proof. The hypotheses imply that there cannot be the germ of any complex $n$-dimensional submanifold lying on $M$ and passing through $p$. Hence by Corollary 1, $p$ is a point of positive type. The existence of the $\xi$ is then a consequence of part (b) of Propostion 3. $\square$

## 10. CR extension and CR submanifolds

First we recall two theorems; one about holomorphic approximation of CR functions, and the other about holomorphic extension of germs of holomorphic functions:

1. Recall the approximation theorem of Baouendi and Trèves: In [BT1] it was shown that if $f$ is a continuous CR function in a neighborhood of $p$ on $M$, then on a smaller neighborhood of $p$ in $M$, $f$ is the limit, in the uniform norm, of a sequence of holomorphic polynomials $p_j$ in the holomorphic coordinates of $\mathbb{C}^{n+k}$. This result can be generalized to obtain convergence in higher norms: If $f$ is a CR function of class $C^t$, where $t$ is an integer with $0 \leq t \leq \infty$, then there is such a sequence $p_j$ that converges in $C^t$ (personal communication with M. S. Baouendi, or see [HT2]).

2. Recall Theorem 8.2 in [HT1] about the extension of germs of holomorphic functions on a fixed neighborhood of $p$ in $M$: They all have holomorphic extensions to the same locus $\mathcal{D}$, where $\mathcal{D}$ is the union of closures of images of a maximal family of local analytic discs having their boundaries within the fixed neighborhood of $p$ on $M$. (Using, say, $C^\alpha$ discs.)

However it is well known, by a maximum modulus principle argument for algebras of holomorphic functions, that the convergence of the holomorphic approximations also takes place uniformly on $\mathcal{D}$. In this way one obtains a continuous limit, which *by definition*, we regard as a CR function on $\mathcal{D}$. If the original CR function $f$ on $M$ is of class $C^t$, then the convergence of the $p_j$ to the extended continuous CR function $\tilde{f}$ takes place in $C^t$ on the closure of the image of each individual analytic disc involved in the construction. For this one has to use analytic discs of class $C^{t,\alpha}$, so that $\mathcal{D}$ is replaced by a perhaps somewhat smaller $\mathcal{D}_t$.

Thus one has CR extension of a CR function $f$ in a neighborhood $V$ of $p$ on $M$ to at least the locus $\mathcal{D}$ where one has the holomorphic extension of germs of holomorphic functions. Hence it is important to try to understand just what is the union of the closures of the images of these local analytic discs with boundaries contained in $V$.

We attempt to gain some insight into this question by tracking the centers of these discs. We introduce the open half spaces

$$H_\xi^+ = \{v \in N_pM \mid \langle {}^tJ\xi, v\rangle > 0\},$$

where $N_pM$ denotes the normal space to $M$ at $p$. Let $\mathcal{C} = \mathcal{P} \cap \{-\mathcal{Z}\}$ on $S^{k-1}$, where $\mathcal{P}$ and $\mathcal{Z}$ are as in §8.

Let $U$ be any sufficiently small neighborhood of $p$ in $\mathbb{C}^{n+k}$.

**Theorem 8.**

1. *There are centers of analytic discs in* $U \cap H_\xi^+$, *for each* $\xi \in \mathcal{P}$.
2. *There are no centers of analytic discs in* $U \cap \left[\bigcup_{\xi\in\mathcal{Z}} H_\xi^+\right]$.
3. *There are centers of analytic discs in* $U \cap \bigcap_{\xi\in\mathcal{C}} H_\xi^+$.

*Remark.* In 3. above we have centers of discs trapped in $\Gamma = \bigcap_{\xi\in\mathcal{C}} H_\xi^+$, which is a convex cone in the normal space $N_pM$.

We shall need the following two propositions. Set $\mathbb{C}^{m_2} = \mathbb{C}^{m_1} \times \mathbb{C}^{m_2-m_1}$.

The first was proved in [HT2, Proposition 1].

**Proposition 6.** *Let* $N^\flat$ *be a CR manifold* $N^\flat \subset \mathbb{C}^{m_1}$ *of type* $(n,l)$. *Let* $\Psi : N^\flat \to \mathbb{C}^{m_2-m_1}$, *and let* $N \subset \mathbb{C}^{m_2}$ *be the graph of* $\Psi$ *over* $N^\flat$. *Then* $N$ *is a CR manifold of the same type,* $(n,l)$, *if and only if* $\phi = (I,\Psi) : N^\flat \to N$ *is a CR map; i.e., if and only if each component of* $\Psi$ *is a CR function on* $N^\flat$.

Note that neither manifold in Proposition 6 is assumed to be generic. The proposition is often used locally by starting with $N$, choosing a point $p \in N$, and taking the holomorphic projection of $N$ onto $\mathbb{C}T_pN$, producing a generic $N^\flat$. In this circumstance we will refer to $N^\flat$ as the *generic image* of $N$ at $p$.

The second is an immediate consequence of the local version of a theorem of Sussmann [Su]. It was formulated in [BR1].

**Proposition 7.** *Let* $M$ *be a CR manifold and let* $p \in M$. *Then there is a unique germ* $N$ *of a CR submanifold of minimal dimension contained in* $M$, *which passes through* $p$, *and has the same CR dimension as* $M$.

The germ of the manifold $N$ consists of all points which can be reached from $p$ by a finite sequence of integral curves of $HM$, locally near $p$.

We now return to the consideration of a generic $M$ of type $(n,k)$ in $\mathbb{C}^{n+k}$. Fix a point $p$ on $M$, and consider the unique CR submanifold $N$ on $M$ through $p$ given in Proposition 7. Our Theorem 2 already gives a necessary and sufficient condition for when $N$ is of minimum possible dimension; i.e., when $N$ is of type $(n, 0)$.

Therefore in the remainder of this section we discuss the next case; that is where the $N$ is of type $(n, 1)$. Consider the generic image $N^\flat$ in $\mathbb{C}T_pN \cong \mathbb{C}^{n+1} \subset \mathbb{C}^{n+k}$, and denote its type function by $\Phi^\flat$. If $N^\flat$ were to contain the germ of a complex $n$-dimensional submanifold through $p$, then so would $M$, by Proposition 6. Hence $p$ is a point of positive type on $N^\flat$, which might or might not be a point of pseudoconcave type.

In this situation we have the following:

**Theorem 9.** *(a) If $p$ is not a point of pseudoconcave type on $N^\flat$ then:*

(i) *There is an $(n+1)$-dimensional complex submanifold $\tilde{N}$ of $\mathbb{C}^{n+k}$ whose boundary contains $N$. Moreover, each continuous CR function on $N$ has a holomorphic extension to $\tilde{N}$ with continuous boundary values on $N$.*

(ii) *There is a $(2n+k+1)$-dimensional real submanifold $N^\sharp$ in $\mathbb{C}^{n+k}$ (not necessarily CR), containing $\tilde{N}$, whose boundary contains a neighborhood of $N$ in $M$. Moreover, continuous CR functions on $\overline{N^\sharp} \cap M$ have continuous CR extensions to $N^\sharp$ with continuous boundary values on $\overline{N^\sharp} \cap M$.*

*(b) If $p$ is a point of pseudoconcave type on $N^\flat$, then $\tilde{N}$ contains $N$ in its interior, and each CR function on $N$ is the restriction to $N$ of a holomorphic function on $\tilde{N}$. Also $N^\sharp$ contains a neighborhood of $N$ on $M$ in its interior. The continuous CR functions on $N^\sharp \cap M$ have continuous CR extensions to $N^\sharp$.*

Proof. First we organize convenient holomorphic coordinates: In $\mathbb{C}^{n+k}$ we use our usual coordinates $(z, w) = (z_1, \ldots, z_k, w_1, \ldots, w_n)$, and take $p$ to be the origin. We decompose $\mathbb{C}^{n+k} = \mathbb{C}^1 \times \mathbb{C}^{k-1} \times \mathbb{C}^n$, with $\mathbb{C}^{n+1}$ being the space of the $(z_1, w_1, \ldots, w_n)$. The holomorphic projection $\pi : \mathbb{C}^{n+k} \to \mathbb{C}^{n+1}$ is given by $\pi(z_1, \ldots, z_k, w) = (z_1, w)$. Thus $N^\flat = \pi|_N(N)$, and without loss of generality we can assume that $N^\flat$ is given by the usual equation $y_1 = g(x_1, w)$, where $g$ vanishes to second order at the origin. As in §5, $y_1 > g(x_1, w)$ corresponds to the + side of the hypersurface $N^\flat$ in $\mathbb{C}^{n+1}$.

To prove (i) we may assume that $\Phi^\flat_+(0, \delta) > 0$ and $\Phi^\flat_-(0, \delta) \equiv 0$, for all sufficiently small $\delta > 0$. By (ii) in §6 we know that CR functions on

$N^\flat$ have local holomorphic extensions to the $+$ side of $N^\flat$ in $\mathbb{C}^{n+1}$. In particular the local graphing function $\psi : N^\flat \to \mathbb{C}^{k-1}$, which graphs $N$ over $N^\flat$ is CR, and hence has a holomorphic extension $\tilde{\psi}$. The $\tilde{N}$ is then obtained as the graph of $\tilde{\psi}$. If $f$ is a continuous CR function on $N$, then $f \circ \psi$ is a CR function on $N^\flat$, and hence has a holomorphic extension $\widetilde{f \circ \psi}$ to the $+$ side of $N^\flat$. The required holomorphic extension to $\tilde{N}$ is then given by $\tilde{f} = \widetilde{f \circ \psi} \circ \pi|_{\tilde{N}}$.

Note that if $p$ is a point of pseudoconcave type on $N^\flat$, then $\psi$ has a holomorphic extension $\tilde{\psi}$ to both sides of $N^\flat$ in $\mathbb{C}^{n+1}$, so $\tilde{N}$ contains $N$ in its interior. In that case each CR function on $N$ is the restriction to $N$ of a holomorphic function on $\tilde{N}$.

To prove (ii) we introduce the Banach spaces $\mathcal{B}_w = [C^\alpha(\overline{D}) \cap \mathcal{O}(D)]^n$, $\mathcal{B}^\flat = \mathbb{R} \times \mathcal{B}_w$, and $\mathcal{B} = \mathbb{R} \times \mathbb{R}^{k-1} \times \mathcal{B}_w$, and by a slight abuse of notation we sometimes write $\mathcal{B} = \mathcal{B}^\flat \times \mathbb{R}^{k-1}$. When $x \in \mathbb{R}^k$ we also write $x = (x_1, x')$, where $x_1 \in \mathbb{R}$ and $x' = (x_2, \ldots, x_k) \in \mathbb{R}^{k-1}$.

Let $U$ be a sufficiently small open ball about the origin in $\mathcal{B}^\flat$. For each parameter point $(c_1, w(\cdot))$ belonging to $U$ there is a unique analytic disc $A : \overline{D} \to \mathbb{C}^{n+1}$ which has its boundary on $N^\flat$, obtained by solving the Bishop equation for $N^\flat$ about the origin (see [HT1]). Note that here we do not require that $c_1 = 0$ and $w(0) = 0$. Let $\Omega$ be the union of the closures of the images of all such analytic discs $A$. Then $\Omega$ includes an open region in $\mathbb{C}^{n+1}$ which lies on the $+$ side of $N^\flat$. With the same abuse of notation as above, we consider $\tilde{A} : \overline{D} \to \mathbb{C}^{n+k}$, where $\tilde{A}(\zeta) = (A(\zeta), \tilde{\psi}(A(\zeta)))$. This gives an analytic disc in $\mathbb{C}^{n+k}$ with boundary on $N$.

We now shift our point of view, and consider the Bishop equations, about the origin in $\mathbb{C}^{n+k}$, associated to the manifold $M$. The parameter point associated to $\tilde{A}(\cdot)$ is $(c_1, x'(0), w(\cdot)) \in U \times \mathbb{R}^{k-1}$. We obtain a $(k-1)$-codimensional Banach submanifold $\Sigma$ of $\mathcal{B}$ by graphing $x'(0)$ over $U$. We remark that it can be arranged that $\Sigma$ is smooth provided that $M$ is smooth (see [HT1, Theorem 7.1, Theorem 8.1]). Let $V$ be an open tubular neighborhood of $\Sigma$ in $\mathcal{B}$ of radius $\varepsilon > 0$. If $\varepsilon$ and the radius of $U$ are sufficiently small, then each parameter point in $V$ corresponds to a unique analytic disc $A^\sharp$ in $\mathbb{C}^{n+k}$ whose boundary lies on $M$. Denote by $\mathcal{D}^\sharp$ the union of the closures of the images of all these discs $A^\sharp$. Then $\mathcal{D}^\sharp \cap M$ is a neighborhood of the origin in $M$. If $f$ is a continuous CR function on $M$, defined in this neighborhood of the origin, which is the union of the boundaries of all of these discs $A^\sharp$, then $f$ has a continuous CR extension

$\tilde{f}$ to $\mathcal{D}^\sharp$, and has continuous boundary values on $\mathcal{D}^\sharp \cap M$. Notice that if the origin is a point of pseudoconcave type on $N^\flat$, then $\mathcal{D}^\sharp$ includes a neighborhood, on $M$, of the origin in its interior. It is clear that $\mathcal{D}^\sharp$ has dimension at least $2n+k+1$. The details required to show that $\mathcal{D}^\sharp$ contains a real $(2n+k+1)$-dimensional manifold $N^\sharp$ are left to the reader. Also the paper [Tu2] can be consulted. $\square$

## Appendix
## Analytic discs, the Bishop equation and the Hilbert transform

Here we collect in a condensed form some essential background material. For more complete details see [HT1].

Fix an $\alpha$ with $0 < \alpha < 1$; we use the standard Hölder spaces: let $K$ be a compact subset of $\mathbb{R}^m$, then

$$C^\alpha(K) = \left\{ u : K \to \mathbb{R} \;\middle|\; |u|_\alpha \equiv \sup_{x\in K} |u(x)| + \sup_{x,y\in K} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty \right\}.$$

Likewise when $k$ is a nonnegative integer, we have

$$C^{k,\alpha}(K) = \left\{ u : K \to \mathbb{R} \;\middle|\; |u|_{k,\alpha} \equiv \sum_{|\beta|\le k} |D^\beta u|_\alpha < \infty \right\}.$$

Then $C^{k,\alpha}(K)$ is a Banach algebra with respect to the $|\ |_{k,\alpha}$ norm.

We use $D$ to denote the unit disc $\{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$, $S^1 = \partial D$, and $\mathcal{O}(D)$ to denote the space of holomorphic functions in $D$. When $K = S^1$ we indicate the norm by $|\cdot|_{k,\alpha}$, and when $K = \overline{D}$ we indicate it by $|\cdot|_{k,\alpha}^{\overline{D}}$. For $u \in \mathcal{O}(D) \cap C^{k,\alpha}(\overline{D})$ these two norms are equivalent:

$$|u|_{k,\alpha} \le |u|_{k,\alpha}^{\overline{D}} \le C_{k,\alpha} |u|_{k,\alpha}.$$

The operator

$$(Tx)(e^{it}) = \frac{1}{2\pi} p.v. \int_0^{2\pi} x(e^{i\theta}) \Im \left( \frac{e^{i\theta}+e^{it}}{e^{i\theta}-e^{it}} \right) d\theta, \tag{A.1}$$

is known as the Hilbert transform on $S^1$. It is a bounded linear operator $T : C^{k,\alpha}(S^1) \to C^{k,\alpha}(S^1)$, and $\|T\|_{k,\alpha} \le \|T\|_\alpha$ for $k = 0, 1, 2, \ldots,$ (see

[HT1, §3]). The significance of $T$ is as follows. Let $y = Tx$ with $x \in C^{k,\alpha}(S^1)$ and let $U$ be the unique harmonic function in $D$ having boundary values $x$. Then $U \in C^{k,\alpha}(\overline{D})$. Construct the unique conjugate harmonic function $V$ such that $V(0) = 0$. Then also $V \in C^{k,\alpha}(\overline{D})$. It follows that $f = U + iV \in \mathcal{O}(D) \cap C^{k,\alpha}(\overline{D})$, $\Im f(0) = 0$, and on $S^1$ has the boundary values $x + iy$.

For any function $x \in C(S^1)$, we denote its mean value by

$$E(x) = \frac{1}{2\pi} \int_0^{2\pi} x(e^{i\theta}) d\theta. \tag{A.2}$$

For $x, u \in C^\alpha(S^1)$, the following relations hold:

$$E(Tx) = 0 \tag{A.3}$$

$$T(xu - (Tx)(Tu)) = xTu + (Tx)u, \tag{A.4}$$

$$E(xu - (Tx)(Tu)) = 0 \quad \text{if} \quad E(x) = 0, \tag{A.5}$$

$$E(xTu + (Tx)u) = 0, \tag{A.6}$$

$$\text{If } u = Tx \text{ then } x = -Tu + E(x). \tag{A.7}$$

These relations also hold for complex valued functions $x$ and $u$.

A map $g : \overline{D} \to \mathbb{C}^{k+n}$, with each component belonging to $\mathcal{O}(D)$ and to some differentiability class on $\overline{D}$ (or sometimes the image $g(\overline{D})$) will be called an *analytic disc* in $\mathbb{C}^{k+n}$. The restriction of $g$ to $S^1$ (or sometimes the image $g(S^1)$) will be called the *boundary of the disc*, and the point $g(0)$ will be called the *center of the disc*.

We shall be interested in analytic discs whose boundaries lie on $M$, where $M$ is as in (2.1). To such a disc $g(\zeta) = (z(\zeta), w(\zeta))$ in $C^\alpha$ we associate its so called *parameters* $(c, w(\cdot))$ which are defined by $c = \Re z(0)$ and $w(\zeta)$. Note that $c$ is a column vector of constants in $\mathbb{R}^k$, and $w(\cdot)$ belongs to $[\mathcal{O}(D) \cap C^\alpha(\overline{D})]^n$. Let $x(e^{i\theta})$ now be the real part of the boundary values of $z(\zeta)$. Then the following relation, known as the *Bishop equation* must hold:

$$x = c - T[h(x, w)]. \tag{A.8}$$

Note that the Bishop equation is a $k \times k$ system of nonlinear singular integral equations on $S^1$.

Therefore if we want to find such an analytic disc $g$ (locally) with boundary on $M$ given by the equation $y = h(x, w)$, as in (2.1), then we have to solve the Bishop equation (A.8) for the column vector of $k$ unknown functions $x$, corresponding to the prescribed data $(c, w(\cdot))$. Namely, $x+iTx$ will then be the boundary values of a $k$-tuple of holomorphic functions $z(\zeta)$ in $D$, with $\Re z(0) = c$, and the required analytic disc is simply given by $g(\zeta) = (z(\zeta), w(\zeta))$. It is called the "lifted disc" corresponding to the "parameter disc" $(c, w(\zeta))$ in $T_0M = \mathbb{R}^k \times \mathbb{C}^n \subset \mathbb{C}^k \times \mathbb{C}^n$.

We measure the size of a parameter disc in a natural way by $|c| + |w(\cdot)|_\alpha^{\overline{D}}$, where $|c|$ denotes the Euclidean norm. If the assigned parameters are sufficiently small, in this sense, then there exists a unique solution $x$ to the Bishop equation such that $x \in C^\alpha(S^1)$, $E(x) = c$, and $|z|_\alpha^{\overline{D}} \le \text{const}\{|c| + |w|_\alpha^{\overline{D}}\}$, (see [HT1, p. 339]). One can also solve the Bishop equation with a greater amount of smoothness, such as $C^{k,\alpha}$, $C^\infty$ or real analyticity up to the boundary, provided one has enough smoothness on the defining function $h$. See [HT1] for further details.

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