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Some characterizations of π-solvable and supersolvable groups using θ-pairs

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Abstract. For a finite group G, $D_p(G)$ is a generalization of the Frattini subgroup of G. We obtain some results on π-solvable and supersolvable groups with the help of $D_p(G)$ using θ-pairs.

1. Introduction

In the process of developing various conditions characterizing solvable groups, some characteristic groups were defined as the generalization of Frattini subgroup $\phi(G)$ of G. Working in this context in [4] we have introduced a characteristic subgroup $D_p(G)$ and studied its influence on the solvable groups. In [5] N. P. Mukherjee and Prabir Bhattacharya obtained some results characterizing supersolvable groups using the class of maximal subgroups M with composite index and $[G:M]_p = 1$ where p is a given prime. In the present paper we obtained a condition characterizing supersolvable groups with the help of θ-pairs introduced by Mukherjee and Bhattacharya in 1990. Here the maximal subgroups considered are of composite index and the normal index is coprime to p where p is a prime. The family of such maximal subgroups has already been considered in [4]. The paper also contains characterization of π-solvable groups with the help of θ-pairs for a maximal subgroup M in $\delta_p(G)$.

All groups considered here are finite and we have used standard notations as in Gorenstein (1968). The notation $M \lessdot G$ is used to denote that M is a maximal subgroup of G.

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2. Preliminaries

Definition 2.1. Let M be a maximal subgroup of a group G, and H and K two normal subgroups of G with $K \subset H$. The factor group H/K is called a *chief factor* of G if there does not exist any normal subgroup A of G such that $K \subset A \subset H$ with proper inclusion. H is called a *normal supplement* of M in G if $MH = G$. The *normal index* of M in G is defined as the order of a chief factor H/K where H is minimal in the set of normal supplements of M in G, and is denoted by $\eta(G:M)$. It is proved that $\eta(G:M)$ is uniquely determined by M (Deskins 1959, 2.1) [3].

Definition 2.2. Let G be any group and p any prime. The characteristic subgroup$L(G)$ and $D_p(G)$ are defined as follows:

$$L(G) = \bigcap \{M : M \in \wedge(G)\}, \quad D_p(G) = \bigcap \{M : M \in \delta_p(G)\}$$

where $\wedge(G) = \{M : M \lessdot G$ and $[G:M]$ is composite$\}$ and $\delta_p(G) = \{M : M \lessdot G$ and $[G:M]$ is composite and $\eta(G:M)_p = 1\}$.

In case $\wedge(G)$ or $\delta_p(G)$ is empty we set $G = L(G)$ or $G = D_p(G)$ respectively.

Theorem 2.3 [2, Theorem 3]**.** *$L(G)$ is supersolvable.*

Lemma 2.4 [1, Lemma 2]**.** *If N is a normal subgroup of a group G and M is a maximal subgroup of G such that $N \subseteq M$ then $\eta(G/N : M/N) = \eta(G:M)$.*

Definition 2.5 [6]. For a maximal subgroup M of a group G, let $\theta(M) = \{(C, D) : C \leq G,\ D \lhd G,\ D \subsetneqq C,\ \langle M, C\rangle = G,\ \langle M, D\rangle = M$ and C/D contains properly no nontrivial normal subgroup of $G/D\}$.

Lemma 2.6 [6, Lemma 2.1]**.** *If (C, D) is a maximal θ-pair in $\theta(M)$ and $N \lhd G$, $N \subset D$ then $(C/N, D/N)$ is a maximal θ-pair in $\theta(M/N)$ and vice versa.*

Definition 2.7. Let L be a non-empty subset of a group G, the *core* of L or *normal interior* of L in G denoted by L_G, is defined to be the join of all the normal subgroups of G that are contained in L, with the convention that $L_G = 1$ if there are no such subgroups. Again $H_G = \bigcap_{g \in G} g^{-1}Hg$ where H is a subgroup.

Lemma 2.8 [3, 2.5]**.** *$[G:M]$ divides $\eta(G:M)$.*

Lemma 2.9 [4, Corollary 3.5]**.** *Let N be a normal subgroup of G. If $N \subseteq D_p(G)$ then $D_p(G/N) = D_p(G)/N$.*

Theorem 2.10 [4, Theorem 3.6]**.** *If $|D_p(G)|_p = 1$ then G is supersolvable if and only if $G/D_p(G)$ is supersolvable.*

Theorem 2.11 [4, Theorem 4.1]**.** *Let p be a prime taken in the definition of $D_p(G)$. Then $D_p(G)$ is solvable if G is a p-solvable group.*

Definition 2.12. A finite group G is called p-solvable if it has a subnormal series $1 = V_0 \subset V_1 \subset \cdots \subset V_n = G$ in which each factor group V_{i+1}/V_i, $i = 0, 1, \ldots, n-1$, is either a p-group or a p'-group.

Theorem 2.13 [7, Theorem 1]**.** *If M is a maximal subgroup of a group G and M is normal in G then $\eta(G:M) = [G:M] =$ a prime.*

3. Some conditions characterizing π-solvable groups

In the present article, we prove some results using θ-pairs in the case when M is a maximal subgroup of composite index such that $\eta(G:M)_p = 1$ where p is a given prime.

Theorem 3.1. *Let G be a p-solvable group. G is π-solvable if and only if for each M in $\delta_p(G)$, every maximal θ-pair (C, D) in $\theta(M)$ is such that C/D is π-solvable.*

Proof. Let G be a counter example of minimal order satisfying the hypothesis of the theorem. If $\delta_p(G)$ is empty or G is simple then we can show that G is π-solvable, a contradiction. So $\delta_p(G) \neq \phi$ and G is not simple. Let N be a minimal normal subgroup of G. Since G is p-solvable then G/N is p-solvable. We can assume that $\delta_p(G/N)$ is non-empty. Let M/N be any maximal subgroup of G/N in $\delta_p(G/N)$ and $(C/N, D/N)$ be a maximal θ-pair in $\theta(M/N)$. Then by Lemma 2.6, it follows that (C, D) is a maximal θ-pair in $\theta(M)$ where M is in $\delta_p(G)$. Then by the hypothesis C/D is π-solvable. Since $C/N\big/D/N$ is isomorphic to C/D, it follows that $C/N\big/D/N$ is π-solvable. Thus G/N satisfies the hypothesis of the theorem. Since $|G/N| < |G|$, G/N is π-solvable. If possible let N_1 be any other minimal normal subgroup of G. Then as above, it can be shown that G/N_1 is π-solvable. Again G, which is isomorphic to a subgroup of the π-solvable group $G/N \times G/N_1$, is π-solvable, a contradiction. So, we now assume that N is the unique minimal normal subgroup of G. Again as above it can be shown that G/N is π-solvable. Now if $N \subseteq D_p(G)$, then N is solvable and hence N is π-solvable. Consequently, G is π-solvable, a contradiction. We now assume that $N \not\subset D_p(G)$. Then there exists M in $\delta_p(G)$ such that $N \not\subset M$. So $G = MN$ and $\mathrm{Core}_G\, M = \langle 1\rangle$. We claim that $(N, \langle 1\rangle)$ is a maximal θ-pair in $\theta(M)$. Now $(N, \langle 1\rangle)$ is a θ-pair in $\theta(M)$ and if possible let (C, D) be a θ-pair such that $(N, \langle 1\rangle) \subset (C, D)$. Then we must have $D = \langle 1\rangle$. For, if not, let $D \neq \langle 1\rangle$. Since M is core free $D \not\subset M$. So $G = MD = M$, a contradiction. Then we have $(N, \langle 1\rangle) \subset (C, \langle 1\rangle)$ which implies that $N/\langle 1\rangle = N \subset C = C/\langle 1\rangle$, again a contradiction as $C/\langle 1\rangle$ cannot contain any non-trivial normal subgroup of $G/\langle 1\rangle$. Hence $(N, \langle 1\rangle)$ is a maximal θ-pair in $\theta(M)$. So by hypothesis $N = N/\langle 1\rangle$ is π-solvable. Also G/N is π-solvable. Hence G is π-solvable, a contradiction. All these contradictions prove the theorem.

The converse is obvious. □

Theorem 3.2. *Let G be a p-solvable group. G is π-solvable if and only if for each M in $\delta_p(G)$, there exists a normal maximal θ-pair (C, D) in $\theta(M)$ such that C/D is π-solvable.*

Proof. Let G satisfy the hypothesis of the theorem. If possible let G be a counter example of minimal order. It can be shown that $\delta_p(G)$ is non-empty, and G is not simple. Let N be a minimal normal subgroup of G. Then, we can assume that $\delta_p(G/N) \neq \phi$. Let $M/N \in \delta_p(G/N)$. Then $M \in \delta_p(G)$. By hypothesis there exists a normal maximal θ-pair (C, D) in $\theta(M)$ such that C/D is π-solvable. If $N \subseteq D$ then $(C/N, D/N)$ is a normal maximal θ-pair in $\theta(M/N)$ and $C/N\big/D/N$ is π-solvable. If $N \not\subset D$, we claim that $N \not\subset C$. If possible let $N \subseteq C$, then $D \subseteq DN \subseteq C$. Since C/D contains no proper non-trivial normal subgroup of G/D, either $D = DN$ or, $DN = C$. If $D = DN$, then $N \subseteq DN = D$, a contradiction. So $DN = C$. Then $G = \langle M, C\rangle = \langle M, DN\rangle = M$, a contradiction. Hence $N \not\subset C$. Now since C/D is π-solvable, CN/DN is also π-solvable. Let K be a maximal proper normal subgroup of G contained in $CN \cap M$ and containing DN. We now claim that CN/K is not a minimal normal subgroup of G/K. For, if not then $(CN, K) \in \theta(M)$. Also $(C, D) \leq (CN, K)$. Since (C, D) is a maximal θ-pair we have $C = CN$. So $N \subseteq C$, a contradiction. Let H/K be a minimal normal subgroup of G/K such that $H/K \subset CN/K$. We have $H \not\subset M$. So $G = MH$. Therefore (H, K) belongs to $\theta(M)$. Since $DN \subset K \subset H \subset CN$, so H/DN is a subgroup of the π-solvable group CN/DN. Therefore H/DN is π-solvable. Since H/K is an epimorphic image of the π-solvable group H/DN so H/K is π-solvable. Now if (H, K) is a maximal pair in $\theta(M)$ then $(H/N, K/N)$ is a normal maximal pair in $\theta(M/N)$ and $H/N\big/K/N$ is π-solvable. If (H, K) is not a maximal pair in $\theta(M)$ then let $(H, K) < (H_1, K_1)$ where (H_1, K_1) is a maximal pair in $\theta(M)$ and consequently $H \subset H_1$. Also $K_1 \subsetneqq HK_1$. For if $K_1 = HK_1$, then $H \subseteq K_1 \subset M$ and $G = \langle M, H\rangle = M$, a contradiction. Now $HK_1 = H_1$. For if $HK_1 \neq H_1$ then $K_1 \subset HK_1 \subset H_1$ and $HK_1/K_1 \lhd G/K_1$ and $HK_1/K_1 \subset H_1/K_1$, a contradiction. Now, $K \subseteq K_1$, so either $K = K_1$ or $K \subset K_1$. If $K = K_1$ then $H_1 = HK_1 = HK = H$, a contradiction. Hence $K \subset K_1$. Again it can be shown that $H/H \cap K_1$ is an epimorphic image of π-solvable group H/K and hence is π-solvable. Since $H_1/K_1 = HK_1/K_1 \cong H/H \cap K_1$ we have H_1/K_1 π-solvable. Thus $(H_1/N, K_1/N)$ is a normal maximal pair in $\theta(M/N)$ by Lemma 2.6, and $H_1/N\big/K_1/N$ is π-solvable. So by minimality G/N is π-solvable. Now as in Theorem 3.1 we can assume that N is the unique minimal normal subgroup of G. Let $N \subseteq D_p(G)$. Thus N is solvable, so N is π-solvable. So G is π-solvable as G/N is π-solvable. We now suppose that $N \not\subset D_p(G)$. Then there exists a core-free maximal subgroup M in $\delta_p(G)$. By hypothesis, there exists a normal maximal θ-pair (C, D) in $\theta(M)$ such that C/D is π-solvable. Since M is Core-free, $D = \langle 1\rangle$ and consequently C is a minimal normal subgroup of G. By uniqueness of N, we get $N = C$. This implies that N is π-solvable, and hence G is π-solvable, a contradiction. All these contradictions prove the theorem.

The converse is obvious. □

Theorem 3.3. *Let G be a p-solvable group. G is π-solvable if and only if for any two distinct maximal subgroups M_1 and M_2 in $\delta_p(G)$, whenever $\theta(M_1)$ and $\theta(M_2)$ have a common maximal θ-pair (C, D), it follows that C/D is π-solvable.*

Proof. Let G be a counter example of minimal order satisfying the hypothesis of the theorem. We can assume that $\delta_p(G) \neq \phi$. Let $\delta_p(G)$ consists of a single element M. Then $D_p(G) = M$. So M is a normal subgroup of G. By Theorem 2.13, we have $\eta(G : M) = [G : M] =$ a prime, a contradiction as $M \in \delta_p(G)$. So we assume that $\delta_p(G)$ consists of at least two elements, M_1 and M_2. We can assume that G is not simple. Let N be a minimal normal subgroup of G. As above we can show that $\delta_p(G/N)$ contains more than one element. Let $M_1/N, M_2/N \in \delta_p(G/N)$ and $(C/N, D/N)$ be a common maximal θ-pair in $\theta(M_1/N)$ and $\theta(M_2/N)$. Then M_1 and M_2 are maximal subgroups of G and by Lemma 2.6 we have (C, D) a maximal θ-pair in $\theta(M_1)$ and $\theta(M_2)$. Thus (C, D) is a common maximal θ-pair in $\theta(M_1)$ and $\theta(M_2)$ and by hypothesis we have C/D is π-solvable. Then we have $C/N\big/D/N$ is π-solvable. Since $|G/N| < |G|$, we have G/N is π-solvable. As in Theorem 3.1 it can be assumed that N is the unique minimal normal subgroup of G. Since G is p-solvable, by Definition 2.12, we have N is either a p-group or a p'-group. If N is a p-group, then it is solvable. This implies that N is π-solvable. Now let N be a p'-group i.e., $|N|_p = 1$. If $N \subseteq D_p(G)$ then N is solvable as $D_p(G)$ is solvable by 2.11. Consequently N is π-solvable. If $N \not\subset D_p(G)$, then there exists $M_1 \in \delta_p(G)$ such that $N \not\subset M_1$ and so $G = M_1N$. Then $\eta(G : M_1) = |N|$. Since N is the unique minimal normal subgroup of G, core of M_1 in G is $\langle 1\rangle$. If $N \subset L(G)$, then N is π-solvable. We now assume that $N \not\subset L(G)$. Then there exists M_2 in $\wedge(G)$ such that $N \not\subset M_2$. Then $G = M_2N$ and $\mathrm{Core}_G\, M_2 = \langle 1\rangle$. Also we have $\eta(G : M_2) = |N|$. So $\eta(G : M_2)_p = |N|_p = 1$. As $M_2 \in \wedge(G), [G : M_2]$ is composite. This shows that $M_2 \in \delta_p(G)$. Again as in Theorem 3.1 it can be verified that $(N, \langle 1\rangle)$ is a common maximal θ-pair in $\theta(M_1)$ and $\theta(M_2)$. By hypothesis $N = N/\langle 1\rangle$ is π-solvable. Consequently G is π-solvable, a contradiction. All these contradictions prove the theorem. Converse is obvious. □

Theorem 3.4. *Let G be a p-solvable group with a π-solvable maximal subgroup M. Then G is π-solvable if each maximal θ-pair (C, D) in $\theta(M)$ is such that $D_p(G/D) \neq \langle \bar{1}\rangle$, where $\bar{1}$ denotes the identity element of G/D.*

Proof. Let us consider that G satisfies the hypothesis of the theorem. We assume that G is not simple. Let $H = \mathrm{Core}_G\, M \neq \langle 1\rangle$. Since M is a π-solvable maximal subgroup of G, we have M/H is also a π-solvable maximal subgroup of G/H. Let $\big(C/H, D/H\big)$ be a maximal θ-pair in $\theta(M/H)$. Then by Lemma 2.6 it follows that (C, D) is a maximal θ-pair in $\theta(M)$. Then by hypothesis we have $D_p(G/D) \neq \langle \bar{1}\rangle$. Since $G/H\big/D/H \cong G/D$ we have $D_p(G/H\big/D/H) \neq \langle \bar{1}\rangle$. Thus G/H satisfies the hypothesis of the theorem, so by induction G/H is π-solvable. Again since $H \subseteq M$ and M is π-solvable, H is π-solvable. Hence G is

π-solvable. Let us now suppose that $H = \mathrm{Core}_G M = \langle 1 \rangle$. Let N be a minimal normal subgroup of G. Then $N \not\subset M$ and $G = MN$. Since $G/N = MN/N \cong M/M \cap N$ and M is π-solvable, we get G/N is π-solvable. We now assume that N is the unique minimal normal subgroup of G. As in Theorem 3.1, it can be verified that $(N, \langle 1 \rangle)$ is a maximal θ-pair in $\theta(M)$ and so by hypothesis we have $D_p(G) = D_p(G/\langle 1 \rangle) \neq \langle 1 \rangle$. Since $D_p(G)$ is a normal subgroup and N is the unique minimal normal subgroup of G, so, $N \subset D_p(G)$ and hence N is π-solvable. Thus G/N and N are π-solvable which implies that G is π-solvable. Hence the theorem. □

4. A supersolvability condition

In [5] Mukherjee and Bhattacharya proved some supersolvability conditions for a group where the hypothesis is satisfied by only the maximal subgroups of composite indices. Here we examine a supersolvability condition where the hypothesis is satisfied by maximal subgroups of composite indices of even smaller class.

Theorem 4.1. *Let G be a group with $|D_p(G)|_p = 1$. G is supersolvable if for every maximal subgroup M of $\delta_p(G)$, each θ-pair (C, D) in $\theta(M)$ is such that $D_p(G/D) \neq \langle \bar{1} \rangle$, where $\bar{1}$ denotes the identity element of G/D.*

Proof. Let $\delta_p(G)$ be empty. Then $G = D_p(G)$. We now claim that $\wedge(G)$ is also empty. If possible, let there exists M in $\wedge(G)$. Then $[G : M]$ is composite. Also by hypothesis $|G|_p = |D_p(G)|_p = 1$. Since $\eta(G : M)$ divides $|G|$, then $\eta(G : M)_p = 1$. Hence M is in $\delta_p(G)$ which contradicts the fact that $\delta_p(G)$ is empty. Hence $\wedge(G)$ is empty and then by definition $G = L(G)$. Since $L(G)$ is supersolvable (by 2.3), G is supersolvable. We now assume that $\delta_p(G)$ has at least one element M. Then G cannot be simple. For, if G is simple, then for each M in $\delta_p(G)$, $(G, \langle 1 \rangle)$ is a θ-pair in $\theta(M)$ and by hypothesis we have $D_p(G/\langle 1 \rangle) \neq \langle \bar{1} \rangle$ i.e., $D_p(G) \neq \langle 1 \rangle$. Since G is simple, we get $G = D_p(G) \subseteq M$, a contradiction. So G is not simple. If $D_p(G) = \langle 1 \rangle$, then for any minimal normal subgroup N of G we have $N \not\subset D_p(G)$. This implies that there exists M in $\delta_p(G)$ such that $N \not\subset M$, so $G = MN$. It can be shown that $(N, \langle 1 \rangle)$ is a θ-pair in $\theta(M)$ and by hypothesis we have $D_p(G/\langle 1 \rangle) \neq \langle \bar{1} \rangle$ i.e., $D_p(G) \neq \langle 1 \rangle$, a contradiction. So $D_p(G) \neq \langle 1 \rangle$. Let N be a minimal normal subgroup of G such that $N \subseteq D_p(G)$.

Then by Lemma 2.9, $|D_p(G/N)| = |D_p(G)/N| = \frac{|D_p(G)|}{|N|}$. Therefore, $|D_p(G)| = |D_p(G/N)|\,|N|$. Since $|D_p(G)|_p = 1$, therefore, $|D_p(G/N)|_p = 1$. Let $(C/N, D/N)$ be a θ-pair in $\theta(M/N)$ where M/N is in $\delta_p(G/N)$. Then $M \in \delta_p(G)$. Also by Lemma 2.6 we have (C, D) is a θ-pair in $\theta(M)$. Then by hypothesis we have $D_p(G/D) \neq \langle \bar{1} \rangle$. So $D_p(G/N/D/N) \neq \langle \bar{1} \rangle$. Hence by induction we get G/N is supersolvable. Now an epimorphism $\phi : G/N \to G/D_p(G)$ can be defined as $\phi(gN) = gD_p(G) \quad \forall g \in G$. So $G/D_p(G)$ is an epimorphic image of the supersolvable group G/N. So $G/D_p(G)$ is supersolvable. Also $|D_p(G)|_p = 1$. So by Theorem 2.10 we have G is supersolvable. Hence the theorem. □

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