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Functions (j,k)-symmetrical and functional equations with iterates of the unknown function

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Abstract. In the present paper we give a method of obtaining some solutions for functional equations in which the unknown function occurs in the form of its own iterates. We reduce the equation to one of the type we have solved in [2] using properties of (j,k)-symmetrical functions, which are collected in [1].

1. Introduction

Let $D \subset \mathbb{C}$ be a nonempty set. For a function $\varphi : D \to D$ and every integer n from the set $\mathbb{N}$ of all positive integers, by $\varphi^{(n)}$ we will denote the n-th iterate of the function φ on the set D; in addition we assume that $\varphi^{(0)} = \mathrm{id}_D$, (the identity on D).

Let $k \in \mathbb{N}$, $k > 2$, be arbitrarily fixed. We will consider the following functional equation

$$\sum_{m_1,\dots,m_{k-1}} a_{m_1,\dots,m_{k-1}}(z)\left(\varphi^{(1)}(z)\right)^{m_1} \cdot \ldots \cdot \left(\varphi^{(k-1)}(z)\right)^{m_{k-1}} = 0,\ z \in D,$$

where φ is an unknown function, $m_1,\dots,m_{k-1} \in \mathbb{N} \cup \{0\}$, the coefficients $a_{m_1,\dots,m_{k-1}}$ are polynomials and the multiple sum includes only finitely many components.

Since $z = \varphi^{(0)}(z)$, we can write the above equation in the form

$$P\left(\varphi^{(0)}(z), \varphi^{(1)}(z), \dots, \varphi^{(k-1)}(z)\right) = 0, \quad z \in D, \tag{1.1}$$

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where φ is an unknown function and $P(x_0, x_1, \dots, x_{k-1})$ is a given polynomial of complex variables $x_0, x_1, \dots, x_{k-1}$.

Before we reduce this equation to one of the type we have solved in [2] we give the following definition.

Definition 1 ([1]). Let $k \in \mathbb{N}$, $k > 2$, be arbitrarily fixed and let $\varepsilon := \exp(2\pi i/k)$. A nonempty subset U of the complex plane $\mathbb{C}$ will be called *k-symmetrical* if $\varepsilon U = U$. The family of all k-symmetrical sets will be denoted by $\mathcal{S}_k$.

The main result of the paper consists in reducing the problem of solving equation of the form (1.1) to finding the solutions of the functional equation

$$P\left(f(\varepsilon^0 w), f(\varepsilon^1 w), \dots, f(\varepsilon^{k-1} w)\right) = 0, \quad w \in U, \tag{1.2}$$

with an $U \in \mathcal{S}_k$ and the unknown function f, and proving that we can get some solutions φ for (1.1) from the solutions f of (1.2).

We will need the following definition.

Definition 2 ([1]). Let $U \in \mathcal{S}_k$ and j belongs to the set $\mathbb{Z}$ of all integers. A function $f : U \to \mathbb{C}$ will be called (j,k)-*symmetrical* if $f(\varepsilon w) = \varepsilon^j f(w)$ for each $w \in U$. The family of all (j,k)-symmetrical functions $f : U \to \mathbb{C}$ will be denoted by $\mathcal{F}_k^j(U)$.

Obviously, $\mathcal{F}_k^j(U) = \mathcal{F}_k^{j+mk}(U)$ for all $j, m \in \mathbb{Z}$. Because of this we can restrict our considerations to $\mathcal{F}_k^j(U)$, where $j = 0, 1, \dots, k-1$.

Further on we will use the following results.

Lemma 1 ([1]). *Let $U \in \mathcal{S}_k$. For every function $f : U \to \mathbb{C}$ there exists exactly one sequence of functions $f_0, f_1, \dots, f_{k-1}$ such that $f_j \in \mathcal{F}_k^j(U)$, for $j = 0, 1, \dots, k-1$ and*

$$f = \sum_{j=0}^{k-1} f_j. \tag{1.3}$$

Moreover

$$f_j(w) = \frac{1}{k} \sum_{l=0}^{k-1} \varepsilon^{-jl} f(\varepsilon^l w), \quad w \in U.$$

In view of the uniqueness of the above decomposition, the functions f_j will be called (j,k)-symmetrical parts of f.

Theorem 1 ([2]). *Let $U \in \mathcal{S}_k$. A function $f : U \to \mathbb{C}$ is a solution of equation* (1.2) *on the set U if and only if the functions f_j, $j = 0, 1, \ldots, k-1$, which occur in the partition* (1.3) *of f, fulfil on U the system of equations*

$$\sum_{m_0,m_1,\ldots,m_{k-1}} b_{l,m_0,m_1,\ldots,m_{k-1}} \prod_{j=0}^{k-1} f_j^{m_j}(w) = 0, \quad l = 0, 1, \ldots, k-1, \tag{1.4}$$

where $m_0, m_1, \ldots, m_{k-1} \in \mathbb{N} \cup \{0\}$, the coefficients $b_{l,m_0,m_1,\ldots,m_{k-1}}$ are determined by the coefficients of the polynomial P from (1.2)*, every sum includes only finitely many components and these components belong to $\mathcal{F}_k^l(U)$ for $l = 0, 1, \ldots, k-1$.*

The system of equations (1.4) has been obtained in [2] with the application of (1.3) to the unknown function f and with the use of the equalities $f_j(\varepsilon^l w) = \varepsilon^{jl} f_j(w)$, $j, l = 0, 1, \ldots, k-1$ to its (j,k)-symmetrical parts f_j.

The following notions and results are also useful.

Definition 3 ([2]). Let $U \in \mathcal{S}_k$ and $l \in \{0, 1, \ldots, k-1\}$. By the *sector* U_k^l of U we will mean the set

$$U_k^l := \left\{ w \in U \;\middle|\; \arg w \in \left\langle \frac{2\pi}{k} l, \frac{2\pi}{k}(l+1) \right) \right\}.$$

If 0 belongs to U, then we assume that $0 \in U_k^l$ for every $l = 0, 1, \ldots, k-1$.

Definition 4 ([2]). Let $h : U_k^0 \to \mathbb{C}$ and $j = 0, 1, \ldots, k-1$. By $[h]_k^j$ let us denote the function which is defined in U as follows:

$$[h]_k^j(w) := \varepsilon^{jl} h(\varepsilon^{-l} w), \quad w \in U_k^l \setminus \{0\},\ l = 0, 1, \ldots, k-1,$$

$$[h]_k^j(0) := \begin{cases} 0 & \text{for } j = 1, 2, \ldots, k-1, \\ h(0) & \text{for } j = 0. \end{cases}$$

The function $[h]_k^j$ will be called the (j,k)-*symmetrical extension* of h from U_k^0 onto U.

Lemma 2 ([2]). (i) *The (j,k)-symmetrical extension of every function $h : U_k^0 \to \mathbb{C}$ onto U is a (j,k)-symmetrical function.*

(ii) *Every function $f \in \mathcal{F}_k^j(U)$ is the (j,k)-symmetrical extension onto U of the function $h := f \mid U_k^0 : U_k^0 \to \mathbb{C}$.*

Theorem 2 ([2]). (i) *If some functions* $h_j : U_k^0 \to \mathbb{C}$, $j = 0, 1, \ldots, k-1$, *fulfil the system of equations* (1.4) *on the sector* U_k^0 *of* U, *then so do the functions* $f_j := [h_j]_k^j$, $j = 0, 1, \ldots, k-1$ *on* U.

(ii) *If the* (j,k)*-symmetrical functions* f_j, $j = 0, 1, \ldots, k-1$ *fulfil the system* (1.4) *on a set* $U \in \mathcal{S}_k$, *then there exist functions* $h_j : U_k^0 \to \mathbb{C}$, $j = 0, 1, \ldots, k-1$ *which fulfil this system on* U_k^0 *and* $f_j := [h_j]_k^j$.

2. Auxiliary results

Before we discuss equation (1.1) we introduce the basic notions and give some of their properties.

Let $U \in \mathcal{S}_k$. By $\mathcal{F}_k^*(U)$ we will denote the class of all functions $f : U \to \mathbb{C}$ which have the following property:

$$(*) \qquad \forall_{x,y \in U} \quad [f(x) = f(y) \Longrightarrow f(\varepsilon x) = f(\varepsilon y)] \,.$$

It is easily seen that all injective functions on U and all functions from the class $\mathcal{F}_k^j(U)$, $j = 0, 1, \ldots, k-1$ belong to the class $\mathcal{F}_k^*(U)$.

In the next theorem we will give a necessary and sufficient condition for a function $f : U \to \mathbb{C}$ to belong to the class $\mathcal{F}_k^*(U)$.

For every function $f : U \to \mathbb{C}$ and every $x \in U$ let us denote

$$U_x(f) := \{y \in U \mid f(x) = f(y)\} \,.$$

Theorem 3. *A function* $f : U \to \mathbb{C}$ *belongs to* $\mathcal{F}_k^*(U)$ *if and only if for every* $x \in U$

$$U_x(f) \subset \bigcap_{j=0}^{k-1} U_x(f_j),$$

where f_j *are the* (j,k)*-symmetrical components of* f *in partition* (1.3).

This theorem follows directly from the following lemma:

Lemma 3. *Let* $f = \sum_{j=0}^{k-1} f_j$ *be the unique partition of* f *onto* (j,k)*-symmetrical components. The function* f *belongs to* $\mathcal{F}_k^*(U)$ *if and only if*

$$(2.1) \qquad \forall_{x,y \in U} \quad [f(x) = f(y) \Longrightarrow f_j(x) = f_j(y)]$$

for $j = 0, 1, \dots, k-1$.

Proof. Let us assume that f satisfies relation (2.1). If for any $x, y \in U$ $f(x) = f(y)$, then $\varepsilon^j f_j(x) = \varepsilon^j f_j(y)$ for $j = 0, 1, \dots, k-1$, so $f_j(\varepsilon x) = f_j(\varepsilon y)$. Therefore, by (2.1) $f(\varepsilon x) = f(\varepsilon y)$, and f has property $(*)$. Thus $f \in \mathcal{F}_k^*(U)$.

Now let us assume that $f \in \mathcal{F}_k^*(U)$. If $f(x) = f(y)$ for any $x, y \in U$, then by property $(*)$ $f(\varepsilon^l x) = f(\varepsilon^l y)$ for $l = 0, 1, \dots, k-1$. Hence by Lemma 1,

$$f_j(x) - f_j(y) = \frac{1}{k}\sum_{l=0}^{k-1} \varepsilon^{-lj}(f(\varepsilon^l x) - f(\varepsilon^l y)) = 0, \quad j = 0, 1, \dots, k-1.$$

Therefore, f satisfies (2.1). $\square$

Now we will give another theorem which is very helpful in checking whether at a function $f : U \to \mathbb{C}$ belongs to the class $\mathcal{F}_k^*(U)$.

Theorem 4. *Let $U \in \mathcal{S}_k$, and $h_j : U_k^0 \to \mathbb{C}$, $j = 0, 1, \dots, k-1$ be arbitrarily chosen functions (we will assume $h_j(0) = 0$ for $j = 1, \dots, k-1$ when $0 \in U$). The function $f := \sum_{j=0}^{k-1} [h_j]_k^j$ belongs to $\mathcal{F}_k^*(U)$ if and only if for all points $v, w \in U_k^0$ and all integers $l, m, s \in \{0, 1, \dots, k-1\}$ there holds the relation*

$$\sum_{j=0}^{k-1} (\varepsilon^{jl} h_j(v) - \varepsilon^{jm} h_j(w)) = 0 \implies \varepsilon^{ls} h_s(v) = \varepsilon^{ms} h_s(w). \tag{2.2}$$

Proof. First, let us observe that for all $j, l \in \{0, 1, \dots, k-1\}$ and $x \in U_k^l$

$$[h_j]_k^j(x) = \varepsilon^{jl} h_j(\varepsilon^{-l} x), \tag{2.3}$$

(see Definition 4).

Now let us assume that condition (2.2) is fulfilled and $x \in U$. If $y \in U_x(f)$, then $f(x) = f(y)$, and consequently $\sum_{j=0}^{k-1}([h_j]_k^j(x) - [h_j]_k^j(y)) = 0$. Of course, $x \in U_k^l$ and $y \in U_k^m$ for any $l, m \in \{0, 1, \dots, k-1\}$, so by (2.3),

$$\sum_{j=0}^{k-1} (\varepsilon^{jl} h_j(\varepsilon^{-l} x) - \varepsilon^{jm} h_j(\varepsilon^{-m} y)) = 0.$$

From this, in view of (2.3) and (2.2), we obtain that $[h_s]_k^s(x) = [h_s]_k^s(y)$ for $s = 0, 1, \ldots, k-1$, because the points $v = \varepsilon^{-l}x$, $w = \varepsilon^{-m}y$ belong to U_k^0. Thus $y \in U_x([h_s]_k^s)$ for $s = 0, 1, \ldots, k-1$. Theorem 3 also gives that $f \in \mathcal{F}_k^*(U)$.

Now let us assume that $f \in \mathcal{F}_k^*(U)$ and for any $l, m \in \{0, 1, \ldots, k-1\}$

$$\sum_{j=0}^{k-1} (\varepsilon^{jl} h_j(v) - \varepsilon^{jm} h_j(w)) = 0, \quad v, w \in U_k^0.$$

Then there exist $x \in U_k^l$ and $y \in U_k^m$ such that $v = \varepsilon^{-l}x$, $w = \varepsilon^{-m}y$ and

$$\sum_{j=0}^{k-1} (\varepsilon^{jl} h_j(\varepsilon^{-l}x) - \varepsilon^{jm} h_j(\varepsilon^{-m}y)) = 0.$$

From this and from (2.3) we have $\sum_{j=0}^{k-1} [h_j]_k^j(x) = \sum_{j=0}^{k-1} [h_j]_k^j(y)$. This gives that $f(x) = f(y)$. Therefore, applying Lemma 3, we obtain $[h_s]_k^s(x) = [h_s]_k^s(y)$ for $s \in \{0, 1, \ldots, k-1\}$. Hence by (2.3), $\varepsilon^{ls} h_s(v) = \varepsilon^{ms} h_s(w)$ for $l, m, s \in \{0, 1, \ldots, k-1\}$. $\square$

3. Main results

Let $U \in \mathcal{S}_k$, $f \in \mathcal{F}_k^*(U)$ and let $D := f(U)$. We introduce the class of functions

$$[f]^* = \{g : D \longrightarrow U \ \mid f \circ g = \mathrm{id}_D\}.$$

The class $[f]^*$ is nonvoid, because for every $z \in D$ the equation $f(w) = z$ has at least one solution $w \in U$. Also it is clear that $[f]^* = \{f^{-1}\}$ when f is an injective function.

Now let us observe that, for every $g \in [f]^*$ the function $\varphi := f \circ \varepsilon g$ maps D into itself and, in view of the fact that $f \in \mathcal{F}_k^*(U)$, if $h \in [f]^*$ and $\psi := f \circ \varepsilon h$, then $\psi = \varphi$ on D. Hence the definition of the function φ does not depend on the choice of function $g \in [f]^*$. Thus we may write

$$\varphi = H(f), \ f \in \mathcal{F}_k^*(U),$$

as φ is uniquely assigned to f.

Theorem 5. *Let $U \in \mathcal{S}_k$ and $f \in \mathcal{F}_k^*(U)$. If $g \in [f]^*$ and $\varphi = f \circ \varepsilon g$, then for $n \in \mathbb{N} \cup \{0\}$ the following relations*

$$\varphi^{(n)} = f \circ \varepsilon^n g \tag{3.1}$$

hold on the set $f(U)$.

Proof. First let us observe that for every $w \in U$ we have $f(\varepsilon w) = f(\varepsilon g(f(w)))$. Indeed, if for every $w \in U$ we put $z = f(w)$ and $\widetilde{w} := g(z)$, then $z = f(\widetilde{w})$, so $f(w) = f(\widetilde{w})$. Using the fact that $f \in \mathcal{F}_k^*(U)$, we have

$$f(\varepsilon w) = f(\varepsilon \widetilde{w}) = f(\varepsilon g(z)) = f\big(\varepsilon g(f(w))\big).$$

Now we proceed by induction.

For $n = 0$ and $n = 1$ relation (3.1) is evident. Let us assume that it holds for an integer $n \in \mathbb{N}$. Then for every $z \in f(U)$ it follows that $g(z) \in U$ and $f(\varepsilon^{n+1} g(z)) = f\big(\varepsilon(\varepsilon^n g(z))\big) = f\big(\varepsilon g(f(\varepsilon^n g(z)))\big) = f\big(\varepsilon g(\varphi^{(n)}(z))\big) = \varphi^{(n+1)}(z)$. □

Now we will give the main theorem which permits to obtain some solutions φ for (1.1) from the the solutions f of (1.2).

Theorem 6. (i) *If $U \in \mathcal{S}_k$ and a function $f \in \mathcal{F}_k^*(U)$ satisfies equation* (1.2) *on U, then the function $\varphi = H(f)$ fulfils equation* (1.1) *on $D = f(U)$.*

(ii) *If for $U \in \mathcal{S}_k$ and $f \in \mathcal{F}_k^*(U)$ the function $\varphi = H(f)$ fulfils equation* (1.1) *on the set $f(U)$, then the function f fulfils equation* (1.2) *on U.*

Proof. (i) Let us assume that $U \in \mathcal{S}_k$ and that $f \in \mathcal{F}_k^*(U)$ satisfies equation (1.2). Consider $\varphi = H(f) = f \circ \varepsilon g$, where g is an arbitrary element of $[f]^*$. It is clear that f fulfils equation (1.2) on the set $U_g := g(D)$, because $U_g \subset U$.

As for every $z \in D = f(U)$ there exists such $w \in U_g$ that $w = g(z)$, so f fulfils equation (1.2) at the point w. Therefore

$$P(f(\varepsilon^0 g(z)), f(\varepsilon^1 g(z)), \dots, f(\varepsilon^{k-1} g(z))) = 0,\ z \in D,$$

i.e., φ fulfils equation (1.1) on D.

(ii) Now let us assume that $U \in \mathcal{S}_k$, $\varphi = H(f)$ with a function $f \in \mathcal{F}_k^*(U)$ and let φ satisfy equation (1.1) on the set $D = f(U)$.

As $\varphi = f \circ \varepsilon g$ for every arbitrarily chosen $g \in [f]^*$, so f satisfies equation (1.2) on every set $g(D)$, with $g \in [f]^*$. Consequently, f satisfies equation (1.2) on the set U, because $U = \bigcup_{g\in[f]^*} g(D)$. □

Theorem 6 shows that we should consider a solution of equation (1.1), or (1.2) as a function together with the set on which this function satisfies equation (1.1) or (1.2), respectively. To do it we introduce the following notions.

Definition 5. A pair $\langle f, U\rangle$, where f is a function satisfying equation (1.2) on a set $U \in \mathcal{S}_k$, (a pair $\langle \varphi, D\rangle$, where φ is a function satisfying equation (1.1) on a set $D \subset \mathbb{C}$) will be called a *solving element of equation* (1.1) and equation (1.2), respectively.

Now we reformulate Theorem 6.

Theorem 7. *Let $U \in \mathcal{S}_k$ and $f \in \mathcal{F}_k^*(U)$. The pair $\langle f, U\rangle$ is a solving element of equation* (1.2) *if and only if the pair $\langle H(f), f(U)\rangle$ is a solving element of equation* (1.1).

Further on, (1.2) be called *equation associated with* (1.1).

The method of solving of equation (1.1), which is included in Theorem 6 permits to find all solving elements $\langle \varphi, D\rangle = \langle H(f), f(U)\rangle$ of equation (1.1), where $f \in \mathcal{F}_k^*(U)$, and $\langle f, U\rangle$ is a solving element of equation (1.2) associated with (1.1).

4. Examples

In this section we shall present two examples. In the first one we shall illustrate the main idea of our method; in the second one we shall additionally point to the difficulties which may appear in practice.

Example 1. Let us consider the functional equation

$$\varphi^{(2)}\varphi^{(3)} - 1 = 0. \tag{4.1}$$

We will look for the solving elements $\langle \varphi, D\rangle$ of the above equation. We should take $k = 4$. For any $U \in \mathcal{S}_4$ the equation associated with the equation (4.1) is the following

$$f(-w)f(-iw) - 1 = 0, \quad w \in U. \tag{4.2}$$

First, we will use Theorem 1 and Theorem 2 to determine all solving elements $\langle f, U\rangle$ of equation (4.2). Let us fix arbitrarily a set $U \in \mathcal{S}_4$ and represent the unknown function f in form (1.3), that is $f = f_0 + f_1 + f_2 + f_3$. In view of Theorem 1 function f satisfies equation (4.2) on U if and only if the $(j,4)$-symmetrical components f_j, $j = 0,1,2,3$ of f fulfil on U the system of equations

$$\begin{aligned} &{f_0}^2 - {f_2}^2 = 1, \qquad f_0 f_1 - f_2 f_3 = 0 \\ &{f_1}^2 - {f_3}^2 = 0, \qquad f_0 f_3 - f_1 f_2 = 0 \end{aligned} \tag{4.3}$$

It is easy to check that this system of equations has infinitely many solutions on U. From Theorem 2 it follows that these solutions depend on one arbitrary function F defined on the sector U_4^0 of U and they have the form

$$f_0 = [F]_4^0, \quad f_1 = 0, \quad f_2 = \left[\sqrt{F^2 - 1}\right]_4^2, \quad f_3 = 0,$$

where, for every function $G : U_4^0 \to \mathbb{C}$, the symbol $\sqrt{G}$ means the function whose value $\sqrt{G}(w)$ at every w is arbitrarily chosen square root of the number $G(w)$. If $0 \in U$, then it is necessary to assume that $F^2(0) = 1$; compare the definition of the $(2,4)$-symmetrical extension of a function and the first equation of system (4.3).

In sequel we will understand the symbol $\sqrt{G}$ in the same sense.

From this by Theorem 1 we obtain that all solutions of equation (4.2) on the set $U \in \mathcal{S}_4$ are of the form

$$f = [F]_4^0 + \left[\sqrt{F^2 - 1}\right]_4^2, \tag{4.4}$$

where $F : U_4^0 \to \mathbb{C}$ is an arbitrary function (with $F^2(0) = 1$ if $0 \in U$). Using Theorem 7 we conclude: if the function f of the form (4.4) belongs to $\mathcal{F}_4^*(U)$, then $\langle f, U\rangle$ is a solving element of equation (4.2) if and only if $\langle H(f), f(U)\rangle$ is a solving element of equation (4.1).

Now we will show that all functions of the form (4.4) belong to $\mathcal{F}_4^*(U)$.

By Theorem 4 it suffices to show that for all points $v, w \in U_4^0$ and all integers $l, m \in \{0,1,2,3\}$ the equalities

$$i^{2l}\sqrt{F^2(v) - 1} = i^{2m}\sqrt{F^2(w) - 1}, \quad F(v) = F(w) \tag{4.5}$$

follow from the equality

$$(4.6)\qquad (F(v) - F(w)) + \left(i^{2l}\sqrt{F^2(v) - 1} - i^{2m}\sqrt{F^2(w) - 1}\right) = 0.$$

In order to check that the equality $F(v) = F(w)$ follows from (4.6) observe that after carrying the second component of sum (4.6) onto the right hand side and then raising both sides to the 2-nd power, we obtain $F(v)F(w) - 1 = (-1)^{l+m}\sqrt{F^2(v) - 1}\sqrt{F^2(w) - 1}$, hence $(F(v) - F(w))^2 = 0$.

The first part of (4.5) follows directly from (4.6) and from the second part of (4.5), proved above. Therefore $f \in \mathcal{F}_4^*(U)$.

Now we can use Theorem 7. It says that for every $U \in \mathcal{S}_4$ and every function F in the expression (4.4) we may find a corresponding solving element $\langle f, U\rangle$ of equation (4.2), so the solving elements $\langle \varphi, D\rangle$ of the equation (4.1) are in the form $\langle H(f), f(U)\rangle$, too.

Let us put $\varphi := H(f) = f \circ ih$ on $f(U)$, with an arbitrary $h \in [f]^*$. Then for $z \in f(U)$ we have $\varphi(z) = f(ih(z)) = f_0(ih(z)) + f_2(ih(z)) = f_0(h(z)) - f_2(h(z))$ and $z = f(h(z)) = f_0(h(z)) + f_2(h(z))$. From this and (4.3) we obtain that $z\varphi(z) = 1$. Hence, $z \neq 0$ and $\varphi(z) = z^{-1}$ for $z \in f(U)$.

Now we will consider two particular cases with $U = \mathbb{C}\setminus\{0\}$.

If we put $F = c$, where $c \in \mathbb{C}$ is a constant, then $f_0 = c$, $f_2 = \left[\sqrt{c^2 - 1}\,\right]_4^2$ and $D = f(U) = \left\{c + \sqrt{c^2 - 1},\ c - \sqrt{c^2 - 1}\,\right\}$. Therefore the solution of equation (4.1) is the function $\varphi(z) = z^{-1}$, $z \in \{c + \sqrt{c^2 - 1}$, $c - \sqrt{c^2 - 1}\,\}$, so a corresponding solving element of equation (4.1) has the form

$$\left\langle \frac{1}{\mathrm{id}_D}, \left\{c + \sqrt{c^2 - 1}, c - \sqrt{c^2 - 1}\,\right\}\right\rangle.$$

Now let us take $F(w) = w^4$ for $w \in \mathbb{C}_4^0$. Then we have $f_0(w) = w^4$ and $f_2(w) = \left[\sqrt{(w^8 - 1) \mid \mathbb{C}_4^0}\,\right]_4^2$ for $w \in \mathbb{C}\setminus\{0\}$; moreover $f(U) = D = \mathbb{C}\setminus\{0\}$. Consequently, we obtain that the solution of equation (4.1) is the function $\varphi(z) = z^{-1}$, $z \in \mathbb{C}\setminus\{0\}$, so the corresponding solving element of (4.1) has the form $\left\langle \frac{1}{\mathrm{id}_D}, \mathbb{C}\setminus\{0\}\right\rangle$.

Let us observe that in Example 1 function f appearing in solving element $\langle f, U\rangle$ of associated equation (4.2) belongs to class $F_4^*(U)$ for each U. In a general case it does not have to happen. In the next example we shall consider an equation (1.1) that function f from the solving element

$\langle f, U\rangle$ of the associated equation (1.2) belongs to class $F_k^*(U)$ only for the properly selected sets U. In order to obtain such sets U we shall apply Theorem 3; moreover, we shall demonstrate the possibility of a maximization of selected sets U.

Example 2. Let us consider the equation

$$\varphi^{(0)} - \varphi^{(1)} - \varphi^{(2)} + \varphi^{(3)} = 0. \tag{4.7}$$

We should take $k = 4$. For every $U \in \mathcal{S}_4$ the equation associated with the above has the form

$$f(\varepsilon^0 w) - f(\varepsilon^1 w) - f(\varepsilon^2 w) + f(\varepsilon^3 w) = 0, \quad w \in U$$

that is

$$f(w) - f(iw) - f(-w) + f(-iw) = 0, \quad w \in U. \tag{4.8}$$

We will determine the family of all solving elements $\langle f, U\rangle$ of equation (4.8).

Let us decompose the unknown function f onto the sum $f = f_0 + f_1 + f_2 + f_3$ of the form (1.3) on the set U. By Theorem 1 the function f fulfils equation (4.8) on the set U if and only if the $(j, 4)$-symmetrical components f_j of f fulfil on U the following equations : $f_1 = 0$, $f_3 = 0$. Thus $f = f_0 + f_2$, and the family of all solving elements $\langle f, U\rangle$ of equation (4.8) is identical with the set of pairs $\langle f_0 + f_2, U\rangle$, where f_0 and f_2 are arbitrary functions from $\mathcal{F}_4^0(U)$, $\mathcal{F}_4^2(U)$, respectively.

If the function $f = f_0 + f_2$ belongs to $\mathcal{F}_4^*(U)$ then, by Theorem 7, the pair $\langle f, U\rangle$ is a solving element of equation (4.8) if and only if the pair $\langle H(f), f(U)\rangle$ is a solving element of equation (4.7).

Let us put $U = \mathbb{C}$, $f_0 = c$, $c \in \mathbb{C}$ and let f_2 be an arbitrary function from $\mathcal{F}_4^2(\mathbb{C})$. Then it is easily seen that the function $f = c + f_2$ belongs to $\mathcal{F}_4^*(\mathbb{C})$ and in this case all pairs $\langle \varphi, f(\mathbb{C})\rangle$ are solving elements of equation (4.8), with $\varphi = H(f)$. If we fix the function $f_2 \in \mathcal{F}_4^2(\mathbb{C})$ arbitrarily, then for every $g \in [f]^*$ and $z \in D := f(\mathbb{C})$ we have $\varphi(z) = f(ig(z)) = c - f_2(g(z))$. From this we have $\varphi(z) = 2c - z$, because $z = f(g(z)) = c + f_2(g(z))$. Therefore in this case all solving elements of equation (4.8) have the form $\langle 2c - \mathrm{id}_D, D\rangle$, where $D = c + f_2(\mathbb{C})$ and f_2 is an arbitrary element of $\mathcal{F}_4^2(\mathbb{C})$. It is convenient to choose such a function $f_2 \in \mathcal{F}_4^2(\mathbb{C})$ for which the set $f_2(\mathbb{C})$ is possibly the biggest. If, for instance, we put $f_2(w) = w^2$, then $f_2(\mathbb{C}) = \mathbb{C}$, so for every $a \in \mathbb{C}$ the function $\varphi(z) = a - z$, $z \in \mathbb{C}$ fulfils equation (4.8) and the corresponding solving elements of the equation (4.8) have the form $\langle a - \mathrm{id}_{\mathbb{C}}, \mathbb{C}\rangle$, with every $a \in \mathbb{C}$.

Now we will consider the case in which f_0 is nonconstant. Let $f_0(w) = c + w^4$, $w \in \mathbb{C}$, $c \in \mathbb{C}$ and $f_2(w) = -w^2$, $w \in \mathbb{C}$. Then $f_0 \in \mathcal{F}_4^0(\mathbb{C})$, $f_2 \in \mathcal{F}_4^2(\mathbb{C})$ and the function f, $f(w) = c + w^4 - w^2$ is a solution of equation (4.8) on $\mathbb{C}$. We can use Theorem 7 if we find such a set $U \in \mathcal{S}_4$ that $f \in \mathcal{F}_4^*(U)$. Let us observe that for every $x \in U$

$$U_x(f_0) = \{x, -x, ix, -ix\}\,, \quad U_x(f_2) = \{x, -x\}\,,$$
$$U_x(f) = \left\{x, -x, \sqrt{1-x^2}, -\sqrt{1-x^2}\,\right\}.$$

Thus, in view of Theorem 3, $f \in \mathcal{F}_4^*(U)$ if and only if for every $x \in U$

$$\sqrt{1-x^2} \notin U \quad \text{or} \quad x = \pm\sqrt{1-x^2}. \tag{4.9}$$

Now we give an example of a set $U \in \mathcal{S}_4$, which satisfies condition (4.9). We restrict ourselves to sets $U = B(0,r)$, i.e., the open discs with the radius $r > 0$, centered at the origin. To obtain condition (4.9) it is sufficient to choose such an $r > 0$, that for every x the inequality $\left|\sqrt{1-x^2}\,\right| > r$ follows from the inequality $|x| < r$. The largest of such radii r is the number $r_0 = \frac{1}{\sqrt{2}}$. Hence $f \in \mathcal{F}_4^*(B(0,r_0))$. An easy computation shows that f satisfies condition (4.9) on the closed disc $U_0 = \overline{B(0,r_0)}$, too.

On the other hand for every $r > r_0$ there exists such x belonging to the circle $C(0,r) = \partial B(0,r)$ that $\sqrt{1-x^2} \in C(0,r)$ and $\sqrt{1-x^2} \neq x$ and $-\sqrt{1-x^2} \neq x$, so $f \notin \mathcal{F}_4^*(C(0,r))$ for $r > r_0$. Therefore $f \in \mathcal{F}_4^*(U_0)$ and U_0 is the largest disc with this property. (The maximality of U_0 guarantees that in every disc $B(0,r)$, $B(0,r) \not\supseteq U_0$, there exists a point x such that (4.9) is false, so we also have that $f \notin \mathcal{F}_4^*(\mathbb{C} - U_0)$). As $\langle f, U_0\rangle$ is a solving element of equation (4.8) so by Theorem 7 the pair $\langle H(f), f(U_0)\rangle$ is a solving element of equation (4.7) for every $c \in \mathbb{C}$. Let $\varphi := H(f)$. Then for every $g \in [f]^*$ and every $z \in f(U_0)$ we have $\varphi(z) = f(ig(z)) = c + g(z)^4 + g(z)^2$, and $g(z) := w$ is an arbitrary solution of the equation $f(w) = z$, that is of the equation $c + w^4 - w^2 - z = 0$. From this we obtain $(g(z))^2 = \frac{1}{2}(1 + \sqrt{1-4c+4z}\,)$. Therefore

$$\varphi(z) = z + 1 + \sqrt{1-4c+4z},$$

where $c \in \mathbb{C}$ and $z \in f(U_0)$.

Consequently, we obtain the next class of solving elements $\langle \varphi, f(U_0)\rangle$ of equation (4.7), where $U_0 = \overline{B(0,r_0)}$ and $f(w) = c + w^4 - w^2$ for $w \in U_0$.

In a similar way we can look for other solving elements of equation (4.7) by choosing other functions $f_0 \in \mathcal{F}_4^0(U)$, $f_2 \in \mathcal{F}_4^2(U)$.

In this example we have determined the solutions φ of equation (4.7) which are defined only on the set $f(B(0, \frac{1}{\sqrt{2}}))$. However, the disc $B(0, \frac{1}{\sqrt{2}})$ can be replaced by sets $U \in \mathcal{S}_4$ of other types which satisfy condition (4.9).

Now we will show that the above method does not give the solving element $\langle 1 + \mathrm{id}_{\mathbb{C}}, \mathbb{C}\rangle$ if $k = 4$. To this purpose observe that for every solution f of equation (4.8) and for the corresponding solving element $\langle \varphi, f(U_0)\rangle$ of equation (4.7), the function $\varphi = H(f)$ has the following property: $\varphi^{(2)} = \mathrm{id}_{f(U_0)}$. Indeed, for $z \in f(U_0)$ and $g \in [f]^*$ we have $\varphi^{(2)}(z) = f(-g(z)) = f_0(g(z)) + f_2(g(z)) = f(g(z)) = z$. Simultaneously, the pair $\langle \psi, \mathbb{C}\rangle$, with $\psi = 1 + \mathrm{id}_{\mathbb{C}}$, is also a solving element of equation (4.7), but $\psi^{(2)} = 2 + \mathrm{id}_{\mathbb{C}} \neq \mathrm{id}_{\mathbb{C}}$.

5. Final Observations

The results presented so far can be generalized when we consider the equation

$$P\left(\varphi^{(0)}(z), \varphi^{(1)}(z), \ldots, \varphi^{(n)}(z)\right) = 0, \quad z \in D \tag{5.1}$$

and the associated equation

$$P\left(f(\varepsilon_k^0 w), f(\varepsilon_k^1 w), \ldots, f(\varepsilon_k^n w)\right) = 0, \quad w \in U, \tag{5.2}$$

where $k - 1 \geq n$, $\varepsilon_k := \exp(2\pi i/k)$, $U \in \mathcal{S}_k$, and φ, f are unknown functions and $P(x_0, x_1, \ldots x_{k-1})$ is a given polynomial of complex variables $x_0, x_1, \ldots, x_n$. It is clear that to equations (5.1), (5.2) there can be applied the main Theorem 6 (see its proof).

Using this observation we can take $k > n + 1$. Consequently, for equation (5.1) we can sometimes obtain more solving elements than in the case $k = n + 1$. Let us illustrate it with a short example. Consider the following simple equation

$$\varphi^{(1)} + \varphi^{(2)} = 1. \tag{5.3}$$

It can be treated as equation (5.1), with $n = 2$.

Taking $k = 3$ we obtain the associated equation (5.2) of the form

$$f(\varepsilon_3 w) + f(\varepsilon_3^2 w) = 1, \quad w \in U, \tag{5.4}$$

where $\varepsilon_3 := \exp(2\pi i/3)$ and U is an arbitrary set from $\mathcal{S}_3$.

Equation (5.4) has only the constant solution $f = \frac{1}{2}$ on U. Of course, $f \in \mathcal{F}_3^*(U)$. Since $D = f(U) = \left\{\frac{1}{2}\right\}$, we have only one solving element $\langle \varphi, D\rangle = \left\langle \frac{1}{2}, \left\{\frac{1}{2}\right\}\right\rangle$ of equation (5.3).

Now let $k = 4$. Then the equation associated with (5.3) has the form

$$f(\varepsilon_4 w) + f(\varepsilon_4^2 w) = 1, \quad w \in U, \tag{5.5}$$

where $\varepsilon_4 := \exp(2\pi i/4)$ and U is an arbitrary set from $\mathcal{S}_4$. Equation (5.5) has infinitely many solutions f on U and they have the form $f = \frac{1}{2} + f_2$, where f_2 is an arbitrary function from $\mathcal{F}_4^2(U)$. Of course, $f \in \mathcal{F}_4^*(U)$. Thus, putting $f_2 = 0$, we obtain the same solving element $\langle \varphi, D\rangle = \left\langle \frac{1}{2}, \left\{\frac{1}{2}\right\}\right\rangle$ for (5.3) as above. However if we put $f_2 \neq 0$, we obtain other solving elements for (5.3). They have the form $\langle 1 - \mathrm{id}_{D_4}, D_4\rangle$, where $D_4 = \frac{1}{2} + f_2(U)$, for instance $\langle 1 - \mathrm{id}_{\mathbb{C}}, \mathbb{C}\rangle$ when $U = \mathbb{C}$, and $f_2(w) = w^2$ for $w \in U$.

Finally, let us observe that sometimes by enlarging k we can obtain less solutions of (5.3). Indeed, for every odd $k \geq 3$ we obtain only the solving element $\left\langle \frac{1}{2}, \left\{\frac{1}{2}\right\}\right\rangle$ for (5.3). It is also easy to check that for every even $k \geq 4$ we obtain the solving elements of (5.3) only of the form $\langle 1 - \mathrm{id}_{D_k}, D_k\rangle$, where $D_k = \frac{1}{2} + f_{\frac{k}{2}}(U)$ and $f_{\frac{k}{2}}$ is an arbitrary function from $\mathcal{F}_k^{\frac{k}{2}}(U)$.

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References

[1] P. Liczberski and J. Połubiński, On (j,k)-symmetrical functions, *Mathematica Bohemica* **120** (1995), 13–28.

[2] P. Liczberski and J. Połubiński, On the application of (j,k)-symmetrical functions to solving some functional equations, Proceedings of the Fifth Environmental Mathematical Conference Rzeszów–Lublin–Lesko 1998, *Lublin Catholic University Press*, 1999, 89–97.

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