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Divisors of shifted primes

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Abstract. Let p range over the set of primes and let a be a non-zero integer. Here we prove that many properties of the divisors of the natural numbers which can be expressed by inequalities are true for the set $\{p+a\}$ of shifted primes, too.

We obtain estimates for the quantities

$$|\{p : p+a \le x,\ d_m(p+a) \le (\log x)^{(1-\alpha)\log m} \text{ or } d_m(p+a) \ge (\log x)^{(1+\alpha)\log m}\}|,$$
$$0 < \alpha < 1,$$

(see Theorem 1) and

$$|\{p : p+a \le x,\ p+a \text{ has at least one divisor } d \text{ such that } y < d \le z\}|,$$

(see Corollary 2) where $d_m(n)$ denotes the number of representations of n as the product of m positive integers.

Erdős conjectured that almost all integers have two divisors d, d' such that $d < d' \le 2d$ and this has recently been confirmed by H. Maier and G. Tenenbaum. We prove that this conjecture is true for the shifted primes, too (see Theorem 7).

Let f be a multiplicative function and define

$$M(n,f) = \max_y \Big| \sum_{\substack{d|n \\ d \le y}} f(d) \Big|, \quad \Delta(n,f) = \max_y \Big| \sum_{\substack{d|n \\ y<d\le ey}} f(d) \Big|.$$

Under some conditions on f we prove estimates for

$$|\{p : p+a \le x,\ \rho(p+a,f) \in [K,S)\}|,$$

where $\rho(n,f) = \Delta(n,f)$ or $M(n,f)$. In particular we show

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Corollary 12. *Let* $\alpha < \alpha_0 = -\frac{\log 2}{\log\left(1-\frac{1}{\log 3}\right)}$ *and let* $\psi(x) \to \infty$ *as* $x \to \infty$*. Then*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \left|\{p : p+a \le x,\ (\log_2 p)^\alpha < M(p+a,\mu) < \psi(p)\log_2 p\}\right| = 1,$$

where $\log_2 x = \log\log x$ *and* μ *denotes the Möbius function.*

Corollary 10. *Let* $x \ge x_0$*. Then*

$$c_1 \log_2 x \le \frac{1}{\pi(x)} \sum_{p+a\le x} \Delta(p+a,1) \le c_2 \exp\left(\left(1+\frac{30}{\log_3 x}\right)\sqrt{\log_2 x \log_3 x}\right),$$

where $c_1 > 0$ *and* $\log_3 x = \log(\log_2 x)$*.*

1. Introduction

Suppose p ranges over the set of primes and a is a non-zero integer. There are at present many conjectures proceeding from the assumption that the set $\{p+a\}$ of shifted primes has similar properties as the set of natural numbers. Here we prove that many results for divisors of natural numbers which can be expressed by inequalities are true for the shifted primes, too.

For example, it is easy to show that for any fixed m

$$\sum_{p\le x} d_m(p+a) \asymp \pi(x)\log^{m-1} x$$

where $\pi(x)$ is the number of primes $p \le x$ and $d_m(n)$ denotes the number of ordered m-tuples $(t_1,\dots,t_m)$ of positive integers such that $t_1t_2\dots t_m = n$. Thus $d_2(n) = d(n)$ is the number of positive divisors of n. The upper bound can be obtained from the result of Barban, Levin [1] and Wolke [2] who proved that for any multiplicative function f with $0 \le f(p^r) \le Ar^c$ for all prime powers p^r

$$\sum_{p\le x} f(p+a) \ll \pi(x)\exp\left(\sum_{p\le x}\frac{f(p)-1}{p}\right).$$

The lower bound follows easily from the obvious formula

$$d_{m+1}(n) = \sum_{t\mid n} d_m(t)$$

and A. I . Vinogradov–Bombieri's Theorem.

K. Norton [3] proved the following result about the distribution of values of the divisor function on the set of natural numbers.

Proposition 1 (see K. Norton [3]). *Let $m \geq 2$. For any x, α with $x \geq 3$, let*

$$D(x,\alpha,m) = \Big|\{n : n \leq x,\ \text{either}\ d_m(n) \leq (\log x)^{(1-\alpha)\log m} \\ \text{or}\ d_m(n) \geq (\log x)^{(1+\alpha)\log m}\}\Big|.$$

If $x \geq 3$ and $0 < \alpha \leq \beta < 1$, then

$$D(x,\alpha,m) \leq c_1(\beta)\alpha^{-1}x(\log_2 x)^{-1/2}(\log x)^{Q(\alpha)}.$$

Furthermore, if $0 < \beta < 1$, then there is a number $c_3(\beta)$ such that whenever $x \geq c_2(\beta)$ and $(\log_2 x)^{-\frac{1}{2}} \leq \alpha \leq \beta$, we have

$$D(x,\alpha,m) \geq c_3(\beta)\alpha^{-1}x(\log_2 x)^{-1/2}(\log x)^{Q(\alpha)}.$$

where $c_3(\beta) > 0$, $Q(\alpha) = \alpha - (1+\alpha)\log(1+\alpha)$ and $\log_2 x = \log(\log x)$.

In the sequel $|A|$ denotes the number of elements of the set A.

We prove that Norton's theorem is true for the set of shifted primes.

Theorem 1. *Let*

$$D_1(x,\alpha,m,P) = |\{p : p+a \leq x,\ d_m(p+a) \leq (\log x)^{(1-\alpha)\log m}\}|,$$

$$D_2(x,\alpha,m,P) = |\{p : p+a \leq x,\ d_m(p+a) \geq (\log x)^{(1+\alpha)\log m}\}|.$$

If $x \geq 3$ and $0 < \alpha \leq \beta < 1$, then for $i = 1, 2$

$$D_i(x,\alpha,m,P) \leq c_4(\beta)\alpha^{-1}\pi(x)(\log_2 x)^{-1/2}(\log x)^{Q((-1)^i\alpha)}.$$

If $(\log_2 x)^{-1/2} \leq \alpha \leq \beta < 1$ and $a = 2$, then there are constants $c_5(\beta) > 0$ and $c_6(\beta) > 0$ such that for $x \geq c_5(\beta)$ and $i = 1, 2$, we have

$$D_i(x,\alpha,m,P) \geq c_6(\beta)\alpha^{-1}\pi(x)(\log_2 x)^{-1/2}(\log x)^{Q((-1)^i\alpha)}.$$

For example, if $m = 2$, $\alpha = -1 + \frac{1}{\log 2}$ we obtain

$$|\{p : p+a \leq x,\ d(p+2) \geq \log x\}| \asymp \pi(x)(\log_2 x)^{-1/2}(\log x)^{-\delta}$$

where $-\delta = Q(-1+\frac{1}{\log 2}) = -\left(1-\frac{1+\log\log 2}{\log 2}\right) \approx -0{,}086$.

Let $N(y,z)$ denote the set of natural numbers n which have at least one divisor d such that $y < d \le z$ and

$$H(x,y,z) := \{n : n \le x,\ n \in N(y,z)\}.$$

Estimates for $|H(x,y,z)|$ are needed in many arithmetical problems. Concerning the history of this question see R. R. Hall and G. Tenenbaum ([4], Chapter 2). In particular we have

Proposition 2 (see Theorem 21, (iii), [4] p. 29)**.** *Let $y \ge 2$ and $2y \le z \le \min(y^{\frac{3}{2}}, \sqrt{x})$. Then we have*

$$xu^{\delta}(L(1/u))^{-c_2} \ll |H(x,y,z)| \ll xu^{\delta}\frac{\log_2 3/u}{\sqrt{\log 2/u}}$$

where

$$c_2 > 0, \quad \delta = -Q\left(-1+\frac{1}{\log 2}\right) \approx 0.086, \quad u = \frac{\log z/y}{\log y},$$

$$L(x) = \exp\left(\sqrt{\log x}\cdot\sqrt{\log_2 3x}\right).$$

Moreover, for each fixed $B \ge 2$, if $z \le By$ we may strike out the factor $\log_2(3/u)$ provided we replace $\ll$ by $\ll_B$ on the right.

For the set of shifted primes we prove

Theorem 2. *Let $T := (1+(\log x)^{-c})$, where $c \ge 2$ and $3 \le y < z \le \sqrt{x}$. Then we have*

$$|\{p : p+a \le x,\ p+a \in N(y,z)\}| \ll \frac{1}{\log x}\sum_{n\in H(x,yT^{-1},zT)}\frac{n}{\varphi(n)} + \pi(x)\frac{(\log_2 10x)^2}{\log x}.$$

The constant implied in $\ll$ depends only on c.

A consequence of this result is

Corollary 1. *If $3 \le y < z \le \sqrt{x}$ then we have*

$$|\{p : p + a \in H(x,y,z)\}| \ll \frac{1}{\log x}|H(x,y,z)| \log\left(\frac{1}{x}\left(|H(x,y,z)|+1\right)\right)^{-1} + \pi(x)\left(\frac{(\log_2 10x)^2}{\log x} + \frac{1}{y}\right).$$

For the PROOF of Corollary 1 we need the following simple result.

Lemma 1. *Let $A(x)$ be a subset of the natural numbers $\le x$. Then for any $3 \le t \le x$ we have*

$$\sum_{n \in A(x)} \frac{n}{\varphi(n)} \ll |A(x)| \log t + \frac{\log t}{t} x.$$

PROOF of Lemma 1. Let $P(t) = \prod_{p \le t} p$ and $p(n)$ denote the largest prime divisor of n. Put $n = n_1 n_2$ where $p(n_1) \le t$ and $(n_2, P(t)) = 1$. Then

$$\frac{n}{\varphi(n)} \le \frac{n_1}{\varphi(n_1)} + \frac{n_1}{\varphi(n_1)} \sum_{\delta | n_2, \delta > 1} \frac{1}{\varphi(\delta)} \le \prod_{p \le t}\left(1 - \frac{1}{p}\right)^{-1}\left(1 + \sum_{\delta > t} \frac{1}{\varphi(\delta)}\right)$$

and therefore

$$\sum_{n \in A(x)} \frac{n}{\varphi(n)} \ll |A(x)| \log t + x \log t \sum_{\substack{\delta > t \\ (\delta, P(t)) = 1)}} \frac{1}{\delta \varphi(\delta)}.$$

Because of

$$\sum_{\substack{\delta > t \\ (\delta, P(t)) = 1}} \frac{1}{\delta\varphi(\delta)} \le \frac{1}{\log t} \sum_{(\delta, P(t)) = 1} \frac{\log \delta}{\delta \varphi(\delta)} \ll \frac{1}{\log t} \sum_{\substack{p > t \\ r}} \frac{\log p^r}{p^r \varphi(p^r)}$$

$$\ll \frac{1}{\log t} \sum_{p > t} \frac{\log p}{p^2} \ll \frac{1}{t \log t}$$

the assertion of Lemma 1 holds true. □

Using Theorem 2 and Lemma 1 we obtain

$$|\{p : p+a \in H(x,y,z)\}| \ll \frac{1}{\log x}|\{n : n \in H(x, yT^{-1}, zT)\}| \cdot \log t$$
$$+ \pi(x)\frac{\log t}{t} + \frac{\pi(x)}{\log x}(\log_2 10x)^2.$$

Since

$$|\{n : n \in H(x, uT^{-1}, u)\}| \le \sum_{uT^{-1} \le d \le u} \left(\frac{x}{d}+1\right) \ll \left(\frac{x}{u}+1\right)\cdot(u - uT^{-1}+1)$$

and $c \ge 2$ the right-hand side can be estimated by

$$\ll |\{n : n \in H(x,y,z)\}|\frac{\log t}{\log x} + \pi(x)\left(\frac{\log t}{t} + \frac{(\log_2 10x)^2}{\log x} + \frac{1}{y}\right).$$

Hence, putting

$$t = \left(\frac{1}{x}|\{n : n \in H(x,y,z)\}| + \frac{1}{x}\right)^{-1}$$

we prove Corollary 1. □

Concerning Theorem 21 of [4] and the further results from Chapter 2 of [4] Corollary 1 yields the following corresponding estimates for the shifted primes.

Corollary 2. *Let $y \ge 2$, $2y \le z \le \min(y^{3/2}, \sqrt{x}\,)$. Then we have*

$$|\{p : p+a \in H(x,y,z)\}| \ll \pi(x)u^{\delta}\log_2\frac{3}{u}\cdot\sqrt{\log\frac{2}{u}}$$

where $\delta = -Q\left(-1+\frac{1}{\log 2}\right) \approx 0.086$, $u = \frac{\log\frac{z}{y}}{\log y}$. If $z \le By$ we may strike out the factor $\log_2\frac{3}{u}$ provided we replace $\ll$ by $\ll_B$. In particular for any $y \le \sqrt{x}$

$$|\{p : p+a \in H(x,y,2y)\}| \ll \frac{\pi(x)}{(\log y)^{\delta}}(\log_2 y)^{\frac{1}{2}}. \tag{1}$$

We are now in the position to prove applications which correspond to results ([4]) given for the set of all natural numbers, but in our case we obtain only upper bounds.

Corollary 3. *Let $A(x,P)$ be the set of shifted primes $p+a\le x$ such that $p+a=mn$ where $m\le\sqrt{x}$ and $n\le\sqrt{x}$. Then, for $x\ge 3$ we have*

$$|A(x,P)| \ll \frac{\pi(x)}{\log^{\delta} x}\cdot(\log_2 x)^{\frac{1}{2}},$$

where $\delta\approx 0.086\ldots$.

Proof. The assertion follows immediately from (1) and the estimate

$$|A(x,P)| \le \sum_{0\le k\le \log(\log x)^2}\left|\left\{p: p+a\in H\left(\frac{x}{4^k},\frac{\sqrt{x}}{2^{k+1}},\frac{\sqrt{x}}{2^k}\right)\right\}\right| + \frac{\pi(x)}{\log x}. \quad \square$$

Corollary 4. *Let*

$$M(n,y) := \sum_{d|n,\, d\le y}\mu(d)$$

and

$$\varepsilon(x,y) := \frac{1}{\pi(x)}|\{p: p+a\le x,\ M(p+a,y)\ne 0\}|.$$

Then, for $x\ge y\ge 3$

$$\varepsilon(x,y) \ll (\log y)^{-\frac{\delta}{1+\delta}}\log_2 y.$$

Proof. We follow the proof given in [4] on page 34.

Put $m(p)=\prod q$, where q runs over the prime factors of $p+a$, and where q^- denotes the least prime factor of $p+a$. Then

$$M(p+a,y) = \sum_{\substack{d|\frac{m(p)}{q^-}\\ d\le y}}\mu(d) + \sum_{\substack{d|\frac{m(p)}{q^-}\\ dq^-\le y}}\mu(q^- d) = \sum_{\substack{d|\frac{m(p)}{q^-}\\ \frac{y}{q^-}<d\le y}}\mu(d)$$

and thus $M(p+a,y)\ne 0$ implies that $p+a$ has a divisor $d\in(y/q^-,y]$. We obtain

$$\begin{aligned}\varepsilon(x,y) &\le \frac{1}{\pi(x)}|\{p: p+a\le x, (p+a,P(t))=1\}|\\ &\quad+\frac{1}{\pi(x)}|\{p: p+a\in H(x,yt^{-1},y)\}|\\ &=: A_1+A_2.\end{aligned}$$

Let $t = \exp(\log^{\alpha} y)$ where $\alpha = \frac{\delta}{1+\delta}$. Using Selberg's sieve we get $A_1 = O(\log^{-\alpha} y)$. Applying Corollary 2 to A_2 gives

$$\varepsilon(x,y) \ll (\log y)^{-\alpha} + (\log y)^{-\delta(1-\alpha)} \sqrt{\log_2 y} \log_3 y$$

where $\log_3 y = \log(\log_2 y)$. This proves Corollary 4. $\square$

In [13] the estimate of

$$|\{p : p \le x,\ p-1 \in N(y,z)\}| \tag{2}$$

is used for analysing the following problems.

Let r be a natural number. We call the r non-zero integers $a_1, \dots, a_r$ *multiplicatively independent* in case the representation

$$a_1^{m_1} \dots a_r^{m_r} = 1 \qquad (m_1, \dots, m_r \in \mathbb{Z})$$

implies $m_1 = \dots = m_r = 0$. We assume that none of $a_1, \dots, a_r$ is a perfect square or ± 1. Let Γ denote the subgroup of Q^* generated by $a_1, \dots, a_r$ and let $|\Gamma_p|$ denote the order of such a group Γ (mod p). We say that a sequence $\{a_1, \dots, a_r, \dots\}$ is a multiplicatively independent sequence, if for any r the integers $\{a_1, \dots, a_r\}$ are multiplicatively independent. If $r = r(p)$ is a given function of p we, as in [13], will still denote by $|\Gamma_p|$ the order of the group generated by a_i, $i \le r(p) \mod p$. Note that this is well defined for all primes that do not divide any of the a_i's and the number of such primes $p \le x$ is $O(\sum_{i \le r(x)} \log a_i)$. In the case $r = 1$, $\Gamma = \langle a \rangle$ and $|\Gamma_p| = \mathrm{ord}_p\, a$ is the order of $a \pmod p$.

Using an estimate for (2) (see Theorem 3.1 of [13]) Pappalardi, (see Theorem 2.1 of [13]) proved that

$$|\Gamma_p| \ge p^{\frac{r}{r+1}} \exp\left(\frac{\log^{\tau} p}{\exp\left(\psi(p)\sqrt{\log\log p}\,\right)} \right)$$

for almost all p, where, $\tau = (1 - \log 2)/2$ and $\psi(x)$ is any function that steadily tends to infinity with x.

The result of [13] is a consequence of the estimate of (2) and of the following result of Matthews.

Proposition 3 (see [14] and Lemma 1.2 [13]). *Suppose that* $rt^{-1/r} \leq C$. *Then*

$$|\{p : |\Gamma_p| \leq t\}| \ll \frac{t^{1+1/r}}{\log t} 2^r r \sum_i \log a_i$$

holds uniformly for all t, r *and* $\{a_1, \ldots, a_r\}$.

Using Lemma 1.2 [13] and Corollary 2 we prove

Theorem 3. *Let* $\{a_1, \ldots, a_r, \ldots\}$ *be a multiplicatively independent sequence such that*

$$\sum_{i=1}^{r} \log a_i \ll 2^r r^{-1}.$$

Then

$$|\Gamma_p| > 4^{r\varepsilon} \cdot \left(\frac{p}{4^r}\right)^{\frac{r+\varepsilon}{r+1}}$$

holds for all but $O\left(\pi(x)\varepsilon^{\delta}(\log\log \frac{3}{\varepsilon})\sqrt{\log \frac{2}{\varepsilon}}\right)$ *primes* $p \leq x$ *and uniformly in* r *and* $0 < \varepsilon < 1$ *such that* $r \log \frac{1}{\varepsilon} \leq \varepsilon \log x$, $r \leq \frac{\log x}{\log\log x}$, *where* $\delta = 1 - \frac{1+\log\log 2}{\log 2} \approx 0.086$.

Proof. We take $t = \left(\frac{x\varepsilon}{4^r}\right)^{\frac{r}{r+1}}$.

For $x \geq x_0$ we have

$$\begin{aligned} rt^{-1/r} &= r\frac{4^{\frac{r}{r+1}}}{(x\varepsilon)^{\frac{1}{r+1}}} \leq 4\exp\left(\frac{1}{r+1}\left((r+1)\log r + \log\frac{1}{\varepsilon} - \log x\right)\right) \\ &\leq 4\exp\left(\frac{1}{r+1}\left((r+1)\log r + \varepsilon\frac{\log x}{r} - \log x\right)\right) \leq 4. \end{aligned}$$

Thus the condition of Lemma 1.2 [13] is fulfilled and we obtain

$$|\{p : |\Gamma_p| \leq t\}| \ll \frac{t^{1+1/r}}{\log t} 2^r r \sum_i \log a_i \ll 4^r \frac{t^{1+\frac{1}{r}}}{\log t} \ll \pi(x). \tag{3}$$

Let $r \geq 2$. Then $t > \sqrt{x}$ and, if $t \leq |\Gamma_p| \leq t_1$ for $p \in [\varepsilon x, x]$ we have $m_p = \frac{p-1}{|\Gamma_p|} \in \left[\frac{\varepsilon x}{t_1}, \frac{x}{t}\right]$. Hence we get

$$|\{p : p \leq x,\ |\Gamma_p| \in [t, t_1]\}| \leq \left|\left\{p : p \leq x,\ p-1 \in N\left(\frac{\varepsilon x}{t_1}, \frac{x}{t}\right)\right\}\right| + \varepsilon\pi(x).$$

Let $t_1 = x^{\frac{\varepsilon}{1+\varepsilon}} t^{\frac{1}{1+\varepsilon}} \varepsilon^{-\frac{r}{r+1}\frac{1}{1+\varepsilon}}$. We choose in Corollary 2 $y = \frac{\varepsilon x}{t_1}$, $z = \frac{x}{t}$ and

$$u = \frac{\log z/y}{\log y} = \frac{\log \frac{t_1}{\varepsilon t}}{\log \frac{\varepsilon x}{t_1}} = \frac{\frac{\varepsilon}{1+\varepsilon}\log\frac{x}{t} + \left(\log\frac{1}{\varepsilon}\right)\left(1 + \frac{r}{(r+1)(1+\varepsilon)}\right)}{\frac{1}{1+\varepsilon}\log\frac{x}{t} - \left(\log\frac{1}{\varepsilon}\right)\left(1 + \frac{r}{(r+1)(1+\varepsilon)}\right)}.$$

Then, if $r\log\frac{1}{\varepsilon} \le \varepsilon\log x$, we have

$$\log\frac{x}{t} \ge \frac{1}{r+1}\log x$$

and $u \le 10\varepsilon$.

Using Corollary 2 and (3) gives

$$|\{p : p \le x,\ |\Gamma_p| \le t_1\}| \ll \pi(x)\varepsilon^{\delta}\left(\log\log\frac{3}{\varepsilon}\right)\sqrt{\log\frac{2}{\varepsilon}}$$

where $\delta \approx 0.086$ and

$$\begin{aligned} t_1 &= \varepsilon^{-\frac{r}{r+1}\frac{1}{1+\varepsilon}}\ x^{\frac{\varepsilon}{1+\varepsilon}}\ t^{\frac{1}{1+\varepsilon}} = x^{\frac{\varepsilon}{1+\varepsilon}}\left(\frac{x}{4^r}\right)^{\frac{r}{r+1}\frac{1}{1+\varepsilon}} \\ &= 4^{\frac{r\varepsilon}{1+\varepsilon}}\left(\frac{x}{4^r}\right)^{\frac{r}{r+1}+\frac{\varepsilon}{(r+1)(1+\varepsilon)}}. \end{aligned}$$

This proves the theorem for $r \ge 2$.

Now, let $r = 1$. We take $t = \sqrt{\varepsilon x}$, $t_1 = x^{\frac{1}{2}+\varepsilon}$ and obtain

$$\begin{aligned} &|\{p : p \le x, \operatorname{ord}_p(a) \le x^{1/2+\varepsilon}\}| \\ \le\ &|\{p : p \le x, \operatorname{ord}(a) \le \sqrt{\varepsilon x}\,\}| + |\{p : p \le x,\ p-1 \in N(\sqrt{\varepsilon x}, \sqrt{x}\,)\}| \\ &+|\{p : p \le x,\ p-1 \in N(\varepsilon x^{1/2-\varepsilon}, \sqrt{x}\,)\}| + \varepsilon\pi(x). \end{aligned}$$

By (3) and Corollary 2 we get

$$|\{p : p \le x, \operatorname{ord}_p(a) \le x^{1/2+\varepsilon}\}| \ll \pi(x)\varepsilon^{\delta}\left(\log\log\frac{3}{\varepsilon}\right)\sqrt{\log\frac{2}{\varepsilon}}.$$

This completes the proof of the theorem. □

Arguing as in [13] we obtain

Corollary 5. *For any fixed r with $\delta = 1 - \frac{1+\log\log 2}{\log 2} \approx 0.086$ we have*

$$R_\Gamma(x) = \sum_{p \le x} \frac{1}{|\Gamma_p|} \ll \frac{x^{\frac{1}{r+1}}}{\log^{1+\frac{\delta}{r+1}} x} (\log\log x)^{1+\delta} \ll \frac{x^{\frac{1}{r+1}}}{\log^{1+\frac{\gamma}{r+1}} x}$$

if $\gamma < \delta$.

Remark. This improves the result in [13], Corollary 2.4, where $\gamma = 0.0306$.

Proof. We have

$$R_\Gamma(x) \le \sum_{|\Gamma_p| \le y} \frac{1}{|\Gamma_p|} + \sum_{\substack{y < \Gamma_p \le z \\ p \le x}} \frac{1}{|\Gamma_p|} + \sum_{\substack{z \le \Gamma_p \le x \\ p \le x}} \frac{1}{|\Gamma_p|}.$$

Using Lemma 1.2 of [13] we get that the number of primes p for which $|\Gamma_p| \le u$ is $O(u^{1+\frac{1}{r}}/\log u)$. Hence, by partial summation, the first sum is $O(y^{1/r}/\log y)$.

The third sum is $O(\frac{1}{z}\pi(x))$. Thus using Corollary 2 with $y = x^{\frac{r}{r+1}}(\log x)^{-\frac{\delta r}{r+1}}$, $z = x^{\frac{r}{r+1}} \log^{\frac{\delta}{r+1}} x$ leads to

$$R_\Gamma(x) \ll \frac{x^{\frac{1}{r+1}}}{(\log x)^{1+\frac{\delta}{r+1}}} + \frac{x}{\log x}\left(\frac{\log\log x}{\log x}\right)^{\delta} (\log\log x)^{\frac{1}{2}+\varepsilon} (\log x)^{\frac{\delta r}{r+1}} x^{-\frac{r}{r+1}},$$

and this completes the proof. □

Using Theorem 3 we obtain

Corollary 6. *Let r be a fixed number. Suppose that $\varepsilon(x)$ tends to zero as $x \to \infty$, $\varepsilon(x) \gg \frac{\log\log x}{\log x}$. Then we have*

(1) *For all but $O\left(\pi(x)(\eta(x))^{\delta}\left(\log\log\frac{3}{\eta(x)}\right)\sqrt{\log\frac{2}{\eta(x)}}\right)$ primes $p \le x$*

$$\Gamma(p) \ge p^{\frac{r}{r+1}+\varepsilon(p)},$$

where $\eta(x) = \sup_{[\sqrt{x},x]} \varepsilon(x)$ and $\delta \approx 0,086$.

(2) *For all but $O(\pi(x)(\log x)^{-\delta(1-\alpha)}\log\log x)$ primes $p \le x$*

$$\Gamma(p) \ge p^{\frac{r}{r+1}} \exp((\log p)^{\alpha}),$$

where $0 < \alpha < 1$ and $\delta \approx 0,086$.

Remark. Similar results have been proven by P. Erdős and M. R. Murty. In particular, they obtained (see [15], Theorem 5): *Let* $\varepsilon(x)$ *tend to zero as* $x \to \infty$.

(1) For all but $o(\frac{x}{\log x})$ *primes* $p \le x$,

$$\Gamma_p \ge p^{\frac{r}{r+1}+\varepsilon(p)}.$$

(2) There exist $\alpha > 0$ *and* $\delta > 0$ *such that*

$$\Gamma_p \ge p^{\frac{r}{r+1}} \exp\left((\log p)^{\delta}\right)$$

for all but $O\left(\frac{x}{(\log x)^{1+\alpha}}\right)$ *primes* $p \le x$.

Now we turn to the average order of Hooley's function (see [5], [4])

$$\Delta(n) := \max_u |\{d : d \mid n,\ u < d \le eu\}|$$

on the set of shifted primes. In many applications it is necessary to obtain an upper bound for the sum

$$\frac{1}{\pi(x)} \sum_{p \le x} \Delta(p+a).$$

Using the results of [5], [4] we prove an upper bound, which is better than that obtained from the trivial inequality $\Delta(n) \le d(n)$.

First we recall the main results about $\Delta(n)$ on the set of natural numbers.

Proposition 4 (see Theorem 55 of [4]). *Let* $\sigma < \sigma_0 = \log 2/\log\{\log 3/(\log 3 - 1)\} = 0.28754\dots$. *Then we have*

$$\lim_{x\to\infty} \frac{1}{x} |\{n : n \le x,\ \Delta(n) > (\log_2 n)^{\sigma}\}| = 1.$$

Proposition 5 (see Theorem 56 of [4]). *Let* $\psi(n) \to \infty$ *as* $n \to \infty$. *Then*

$$\lim_{x\to\infty} \frac{1}{x} |\{n : n \le x,\ \Delta(n) < \psi(n) \log_2 n\}| = 1.$$

Observe that Proposition 4 implies

$$\lim_{x\to\infty} \frac{1}{x(\log_2 x)^{\sigma}} \sum_{n \le x} \Delta(n) \ge 1 \qquad (\sigma < \sigma_0).$$

In [4], Theorem 60, it was proven that the average order of $\Delta(n)$ is at least $c\log_2 n$ and (see Theorem 70 of [4])

$$\sum_{n\le x}\Delta(n)\ll x\exp\left(\left(1+\frac{30}{\log_3 x}\right)\sqrt{2\log_2 x\cdot\log_3 x}\right). \tag{4}$$

where $\log_3 x=\log\log_2 x$.

Here we prove

Theorem 4. *Let F be a non-negative arithmetical function such that*

$$c_1F(n)\le F(np)\le c_2F(n)$$

for some fixed $c_1>0$, $c_2>0$ and $(p,n)=1$. Then, for any K, S, we have

$$|\{p:p+a\le x,\ F(p+a)\in[K,S)\}|$$
$$\ll\frac{1}{\log x}\sum_{\substack{n\le x\\ F(n)\in\left[\frac{c_1}{c_2}K,\frac{c_2}{c_1}S\right)}}\frac{n}{\varphi(n)}+x\frac{(\log_2 x)^2}{\log^2 x}.$$

Using Lemma 1 we obtain

Corollary 7. *Let F satisfy the conditions of Theorem 4. Then*

$$\frac{1}{\pi(x)}|\{p:p+a\le x,\ F(p+a)\in[K,S)\}|$$
$$\ll\frac{1}{x}\left|\left\{n:n\le x,\ F(n)\in\left[\frac{c_1}{c_2}K,\frac{c_2}{c_1}S\right)\right\}\right|$$
$$\cdot\log\left(\frac{1}{x}\left|\left\{n:n\le x,\ F(n)\in\left[\frac{c_1}{c_2}K,\frac{c_2}{c_1}S\right)\right\}\right|+\frac{1}{x}\right)^{-1}+\frac{(\log_2 x)^2}{\log x}.$$

For $\Delta(n)$ we have

$$\Delta(n)\le\Delta(np)\le 2\Delta(n).$$

Hence we get

$$\frac{1}{\pi(x)}|\{p : p+a \le x,\ \Delta(p+a) \le (\log_2 x)^\sigma\}|$$
$$\ll \frac{1}{x}|\{n : n \le x,\ \Delta(n) \le 2(\log_2 x)^\sigma\}|$$
$$\times \log\left(\frac{1}{x}|\{n : n \le x,\ \Delta(n) \le 2(\log_2 x)^\sigma\}| + \frac{1}{x}\right)^{-1} + \frac{(\log_2 x)^2}{\log x}.$$

Evidently for $\sqrt{x} \le p \le x$ we have $2(\log_2 x)^\sigma \le (\log_2 p)^{\sigma_1}$ where $\sigma < \sigma_1 < \sigma_0$. Hence by Theorem 55 of [4]

$$\lim_{x\to\infty} \frac{1}{\pi(x)}|\{p : p+a \le x,\ \Delta(p+a) > (\log_2 p)^\sigma\}| = 1$$

for every $\sigma < \sigma_0 = 0.28574\ldots$.

Using Theorem 56 of [4] and Corollary 7 shows

$$\lim_{x\to\infty} \frac{1}{\pi(x)}|\{p : p+a \le x,\ \Delta(p+a) \ge \psi(p)\log_2 p\}| = 0$$

for every $\psi(x) \to \infty$ as $x \to \infty$.

Thus we have

Corollary 8. *Let $\sigma < \sigma_0 = 0.28574\ldots$ and $\psi(x) \to \infty$ as $x \to \infty$. Then*

$$\lim_{x\to\infty} \frac{1}{\pi(x)}|\{p : p+a \le x,\ (\log_2 p)^\sigma < \Delta(p+a) < \psi(p)\log_2 p\}| = 1.$$

Using Theorem 4 we prove

Theorem 5. *Let F satisfy the conditions of Theorem 4 and*

$$\sum_{n\le x} F^2(n) \ll x\log^r x, \quad (r > 0).$$

Then

$$\frac{1}{\pi(x)}\sum_{p\le x} F(p+a) \ll \frac{1}{x}\sum_{n\le x} F(n)\frac{n}{\varphi(n)} + \frac{(\log_2 x)^3}{\log x}.$$

Proof. Let $0 < \Theta < \min\left(\frac{c_1}{c_2}, 1\right)$. Then

$$\frac{1}{\pi(x)} \sum_{p \le x} F(p+a) \le \frac{1}{\pi(x)} \sum_{-K \le l \le K} \sum_{p+a \in \sum(x,l)} F(p+a)$$
$$+\Theta^K \cdot \log x + \frac{\Theta^{K+1}}{\log x} \cdot \frac{1}{\pi(x)} \sum_{p+a \le x} F^2(p+a),$$

where $\sum(x,l) = \{n : n \le x,\ \Theta^l \log x \le F(n) < \Theta^{l-1} \log x\}$. Set $K = [A \cdot (\log 1/\Theta)^{-1} \cdot \log_2 x]$, where $A > r+2$. Using Theorem 3 we obtain

$$\frac{1}{\pi(x)} \sum_{p \le x} F(p+a) \ll \sum_{|l| \le K} \frac{1}{x} \sum_{n \in \sum(x,l-1) \cup \sum(x,l) \cup \sum(x,l+1)} \frac{n}{\varphi(n)} F(n)$$
$$+ \frac{(\log_2 x)^3}{\log x} \ll \frac{1}{x} \sum_{n \le x} \frac{n}{\varphi(n)} F(n) + \frac{(\log_2 x)^3}{\log x}.$$

This completes the proof of Theorem 5. □

Corollary 9. *Suppose the conditions of Theorem 4 for F are fulfilled and $F(dn) \le d_r(d) F(n)$. Then*

$$\frac{1}{\pi(x)} \sum_{p \le x} F(p+a) \ll \max_{x \log^{-s} x \le y \le x} \frac{1}{y} \sum_{n \le y} F(n) + \frac{(\log_2 x)^3}{\log x}$$

where $s \ge 2r + 2$.

Proof. We have

$$\sum_{n \le x} \frac{n}{\varphi(n)} F(n) \le \sum_{d < x} \frac{1}{\varphi(d)} \sum_{nd \le x} F(nd) \le \sum_{d \le x} \frac{d_r(d)}{\varphi(d)} \sum_{n \le \frac{x}{d}} F(n)$$
$$\ll x \sum_{d} \frac{d_r(d)}{d\varphi(d)} \cdot \max_{x \log^{-s} x \le y \le x} \frac{1}{y} \sum_{n \le y} F(n) + x \log^r x \sum_{d > \log^s x} \frac{d_r(d)}{d\varphi(d)}.$$

Since

$$\sum_{d > \log^s x} \frac{d_r(d)}{d\varphi(d)} \le (\log x)^{-s/2} \sum_{d} \frac{d_r(d)}{\sqrt{d}\varphi(d)}$$

the proof of Corollary 9 is finished. □

Using Theorem 70 (see (4)) and Corollary 9 we get the proof of the first part of

Corollary 10. *Let* $x \geq x_0$*. Then*

$$\sum_{p+a\leq x} \Delta(p+a) \ll \pi(x) \exp\left(\left(1+\frac{30}{\log_3 x}\right)\sqrt{2\log_2 x \cdot \log_3 x}\right)$$

and

$$\sum_{p+a\leq x} \Delta(p+a) \geq c\pi(x)\log_2 x$$

with some positive constant c.

Proof. Let

$$T(n) := |\{d_1, d_2 : d_1|n,\ d_2|n,\ |\log(d_1/d_2)| \leq 1\}|.$$

We show (cf. Lemma 60.1 of [4]) that $2\Delta(n)d(n) \geq T(n)$.

Now,

$$T(n) = \sum_{d_1|n} \big(|\{d_2 : d_2|n,\ \log d_1 - 1 \leq \log d_2 < \log d_1\}| \\ + |\{d_2 : d_2|n,\ \log d_1 \leq \log d_2 < \log d_1 + 1\}|\big) \leq 2\Delta(n)d(n)$$

and therefore

$$\sum_{p+a\leq x} \Delta(p+a) \geq \frac{1}{2} \sum_{p+a\leq x} \frac{1}{d(p+a)} \sum_{\substack{\delta_1,\delta_2|p+a \\ (\delta_1,\delta_2)=1,\ |\log\delta_1/\delta_2|\leq 1}} d\left(\frac{p+a}{\delta_1\delta_2}\right).$$

Notice that $d(nm) \leq d(n)d(m)$. Therefore

$$\sum_{p+a\leq x} \Delta(p+a) \geq \frac{1}{2} \sum_{\delta_1\leq x^{1/4-\varepsilon}} \sum_{\substack{\delta_1\leq\delta_2<e\delta_1 \\ (\delta_2,\delta_1)=1}} \frac{1}{d(\delta_1)d(\delta_2)} \pi(x, -a, \delta_1\delta_2).$$

By A. I. Vinogradov–Bombieri's Theorem we obtain

$$\sum_{p+a\leq x} \Delta(p+a) \geq \frac{1}{2}\pi(x)S(x) + O(\pi(x)),$$

where

$$S(x) := \sum_{\substack{\delta_1 \le x^{1/4-\varepsilon} \\ (\delta_1,2)=1}} \frac{1}{d(\delta_1)\varphi(\delta_1)} \sum_{\substack{\delta_1 \le \delta_2 < e\delta_1 \\ (\delta_2, 2\delta_1)=1}} \frac{1}{d(\delta_2)\varphi(\delta_2)}.$$

Put

$$S'(x) := \sum_{\substack{\delta_1 \le x^{1/4-\varepsilon} \\ (\delta_1,2)=1}} \frac{1}{d(\delta_1)\varphi(\delta_1)} \sum_{\substack{\delta_1 \le \delta_2 < e\delta_1 \\ (\delta_2, 2)=1}} \frac{1}{d(\delta_2)\varphi(\delta_2)}.$$

We have

$$\begin{aligned} S(x) &= S'(x) - \sum_{\substack{\delta_1 \le x^{1/4-\varepsilon} \\ (\delta_1,2)=1}} \frac{1}{d(\delta_1)\varphi(\delta_1)} \sum_{\substack{\delta_1 \le \delta_2 < e\delta_1 \\ (\delta_2, 2)=1,\ (\delta_2,\delta_1)>1}} \frac{1}{d(\delta_2)\varphi(\delta_2)} \\ &\ge S'(x) - \sum_{p \ge 3} \frac{1}{(p-1)^2} S'(x) \ge \frac{1}{3} S'(x). \end{aligned}$$

Hence, using

$$\sum_{\substack{\delta_2 \le y \\ (\delta_2,2)=1}} \frac{1}{d(\delta_2)} = y a_0 (\log y)^{-\frac{1}{2}} + O(y(\log y)^{-\frac{3}{2}})$$

where $a_0 > 0$ we obtain

$$\begin{aligned} S(x) &\ge \frac{1}{2e} \sum_{\substack{\delta_1 \le x^{1/4-\varepsilon} \\ (\delta_1,2)=1}} \frac{1}{\delta_1 d(\delta_1)\varphi(\delta_1)} \sum_{\substack{\delta_1 \le \delta_2 < e\delta_1 \\ (\delta_2, 2)=1}} \frac{1}{d(\delta_2)} \\ &\ge c \sum_{\substack{2 < \delta_1 \le x^{1/4-\varepsilon} \\ (\delta_1,2)=1}} \frac{(\log \delta_1)^{-\frac{1}{2}}}{d(\delta_1)\delta_1}. \end{aligned}$$

By partial summation we have

$$S(x) \ge c \log_2 x$$

and hence the assertion of Corollary 10 holds true. □

Following P. Erdős and I. Kátai (see [6]) we define

$$M(n) := \sup_y \Big| \sum_{\substack{d|n \\ d \le y}} \mu(d) \Big|$$

where μ denotes the Möbius function. H. MAIER [7] has shown that

$$\lim_{x\to\infty}\frac{1}{x}|\{n : n\le x,\ M(n) > (\log_2 n)^{\alpha}\}| = 1 \tag{5}$$

where $\alpha < \alpha_0 := -\log 2/\log\left(1-\frac{1}{\log 3}\right)$. As a corollary of Theorem 56 from [4] (cf. [7], [4]) the limit law

$$\lim_{x\to\infty}\frac{1}{x}|\{n : n\le x,\ M(n) < \psi(n)\log_2 n\}| = 1 \tag{4}$$

holds if $\psi(n)\to\infty$ as $n\to\infty$.

Now, let f be a multiplicative function and define

$$M(n,f) = \max_y\Big|\sum_{\substack{d|n\\ d\le y}} f(d)\Big| \quad \text{and} \quad \Delta(n,f) = \max_y\Big|\sum_{\substack{d|n\\ y<d\le ey}} f(d)\Big|.$$

Then we prove

Theorem 6. *Let f be a multiplicative function such that $|f(n)|\le 1$ and f takes on the set of primes $l+s$ values $a_1,\dots,a_s,b_1,\dots,b_l$. Suppose that the set $\{p : f(p)=b_j,\ j=1,\dots,l\}$ is finite and that for every a_i, $i=1,\dots,s$ there exists $\delta_i>0$ such that*

$$\begin{aligned}&\left|\left\{p : p\in\left[M, M(1+\log^{-c}x)\right)\right\}\right|\\ &\le \delta_i\left|\left\{p : p\in\left[M, M(1+\log^{-c}x)\right), f(p)=a_i\right\}\right|\end{aligned}$$

where $M\ge\exp((\log_2 x)^2)$, $x\ge x_0$, $c>0$. Then

$$\begin{aligned}&\left|\left\{p : p+a\le x,\ \rho(p+a,f)\in\left[K,S\right)\right\}\right|\\ &\ll \frac{1}{\log x}\sum_{\substack{n\le x\\ \rho(n,f)\in[K-4,S+4)}}\frac{n}{\varphi(n)} + \pi(x)\frac{(\log_2 x)^2}{\log x},\end{aligned}$$

where $\rho(n,f)=\Delta(n,f)$ or $\rho(n,f)=M(n,f)$, $0\le K<S\le\infty$.

An immediate consequence of this result and Lemma 1 is

Corollary 11. *Let f satisfy the conditions of Theorem 6 and put $\rho(n,f)=\Delta(n,f)$ or $M(n,f)$, respectively. Then*

$$\frac{1}{\pi(x)}|\{p : p+a\le x,\ \rho(p+a,f)\in\big[K,S\big)\}|$$
$$\ll \frac{1}{x}|\{n : n\le x,\ \rho(n,f)\in[K-4,S+4)\}|$$
$$\cdot\log\left(\frac{1}{x}|\{n : n\le x,\ \rho(n,f)\in[K-2,S+2)\}|+\frac{1}{x}\right)^{-1}+\frac{(\log_2 x)^2}{\log x},$$

where $0\le K<S\le\infty$.

Using Corollary 11 and Maier's results (see (3), (4)) we prove

Corollary 12. *Let $\alpha<\alpha_0=-\frac{\log 2}{\log\left(1-\frac{1}{\log 3}\right)}$ and $\psi(x)\to\infty$ as $x\to\infty$. Then*

$$\lim_{x\to\infty}\frac{1}{\pi(x)}|\{p : p+a\le x,\ (\log_2 p)^{\alpha}<M(p+a)<\psi(p)\log_2 p\}|=1.$$

Proof. In our case $f(n)=\mu(n)$ and $\mu(p)=-1$. We see that the conditions of Corollary 9 are fulfilled. Using Corollary 9 we have

$$\frac{1}{\pi(x)}|\{p : p+a\le x,\ M(p+a)\in\big[0,(\log_2 x)^{\alpha}\big)\cup\big[\psi(x)\log_2 x,+\infty\big)\}|$$
$$\le\frac{1}{x}\left|\left\{n : n\le x,\ M(n)\in\Big[0,2(\log_2 x)^{\alpha}\Big)\cup\left[\frac{1}{2}\psi(x)\log_2 x,+\infty\right)\right\}\right|$$
$$\times\log\left(\frac{1}{x}\left|\left\{n : n\le x,\ M(n)\in\Big[0,2\log_2^{\alpha}x\Big)\cup\left[\frac{1}{2}\psi(x)\log_2 x,+\infty\right)\right\}\right|+\frac{1}{x}\right)^{-1}$$
$$+\,o(1).$$

Let $\alpha<\alpha_1<\alpha_0$ and $\psi_1(x)=\min_{[\sqrt{x},x]}\frac{1}{4}\psi(y)$. Then for $x\ge x_0$ we obtain

$$\left|\left\{n : n\le x,\ M(n)\in\Big[0,2(\log_2 x)^{\alpha}\Big)\cup\left[\frac{1}{2}\psi(x)\log_2 x,+\infty\right)\right\}\right|$$
$$\le\left|\left\{n : n\le x,\ M(n)\in\Big[0,2\log_2^{\alpha_1}n\Big)\cup\left[\frac{1}{2}\psi_1(n)\log_2 n,+\infty\right)\right\}\right|+\sqrt{x}.$$

Using these inequalities together with (3), (4) completes the proof. □

Erdős conjectured that almost all integers have two divisors d, d' such that $d < d' \le 2d$ which recently has been confirmed by Maier and Tenenbaum (see Theorem 52 of [4]). We prove that this conjecture is true for the shifted primes, too.

Theorem 7. *The number of shifted primes $p+a \le x$ without divisors d, d' such that $d < d' \le 2d$ is $O(\pi(x)(\log_2 x)^{-\beta}(\log_3 x)^{4\beta})$ where $\beta = 1 - \frac{(1+\log_2 3)}{\log 3} = 0.00415\ldots$.*

2. Preliminaries

We shall use sieve estimates and need the following result.

Lemma 2 (see Lemma 2.1 of [8]). *Let ρ be a real-valued nonnegative arithmetical function. Let λ_n, $n = 1, 2, \ldots, N$, be integers. Let r be a positive real number and $p_1 < p_2 < \cdots < p_s < r$ be primes. Set $Q = p_1 \ldots p_s$. If $d \mid Q$ then let*

$$\sum_{\substack{n=1 \\ \lambda_n \equiv 0 \pmod d}}^{N} \rho(n) = \eta(d)X + R(N,d)$$

where X, R are real numbers, $X \ge 0$ and $\eta(d_1 d_2) = \eta(d_1)\eta(d_2)$ whenever d_1 and d_2 are coprime divisors of Q. Assume that for each p, $0 \le \eta(p) < 1$. Let $I(N,Q)$ denote the sum

$$\sum_{\substack{n=1 \\ (\lambda_n, Q)=1}}^{N} \rho(n).$$

Then the estimate

$$I(N,Q) = (1 + 2\Theta_1 H)X \prod_{p \mid Q} (1 - \eta(p)) + 2\Theta_2 \sum_{\substack{d \mid Q \\ d \le z^3}} 3^{\omega(d)} |R(N,d)|$$

holds uniformly for $r \geq 2, \max(\log r, S) \leq \frac{1}{8}\log z$, *where* $|\theta_1| \leq 1$, $|\theta_2| \leq 1$, *and*

$$S := \sum_{p|Q} \frac{\eta(p)\log p}{1-\eta(p)},$$

$$H := \exp\left(-\frac{\log z}{\log r}\left(\log\frac{\log z}{S} - \log\log\frac{\log z}{S} - \frac{2S}{\log z}\right)\right).$$

When these conditions are satisfied there is a positive absolute constant c *so that* $2H \leq c < 1$.

For the estimate of the remainder terms in sieve applications we can use the following lemma, which is a corollary of Lemma 3 in [9].

Lemma 3. *Let* $\{a_n\}$, $\{b_m\}$ *be bounded sequences of complex numbers, such that* $a_n = 0$ *for* $n \leq z_1$ *and* $b_m = 0$ *for* $m \leq z_2$. *Let* $r(x,v,u) := \{n : n \leq x, (n, P(v,u)) = 1\}$, *where* $P(u) = \prod_{p\leq u} p$, $P(v,u) = \prod_{v<p\leq u} p$. *Let* $T \leq \sqrt{x}$, $y_1 \leq z_3$ *be real numbers. Then*

$$\Delta := \sum_{\substack{d\leq T\\(d,P(z_3))=1}} 3^{\omega(d)} \max_{t\leq x}\max_{(l,d)=1}\Bigg|\sum_{\substack{nm\leq t\\nm\equiv l \pmod d)\\nm-l\in r(x,y,y_1)}} a_n b_m - \frac{1}{\varphi(d)}\sum_{\substack{nm\leq t\\nm-l\in r(x,y,y_1)}} a_n b_m\Bigg|$$

$$\ll \log^7 x\left(\sqrt{Ty_2}\cdot x^{\frac{3}{4}} + \frac{x}{\sqrt[4]{z_1}} + \frac{x}{\sqrt[4]{z_2}} + \frac{x}{\sqrt{z_3}}\right) + x\exp\left(-\frac{1}{2}\frac{\log y_2}{\log y_1}\right)\log^7 x.$$

Proof. The sum of the left-hand side (see Lemma 3 of [9]) without the factor $3^{\omega(d)}$ is

$$\ll R := \log^4 x\left(Ty_2\sqrt{x} + \frac{x}{\sqrt{z_1}} + \frac{x}{\sqrt{z_2}} + \frac{x}{z_3}\right) + x\exp\left(-\frac{\log y_2}{\log y_1}\right)\log^8 x.$$

Hence by Cauchy's inequality

$$\Delta \ll \sqrt{R}\cdot\left(\sum_{d\leq\sqrt{x}} 9^{\omega(d)}\left(\frac{x\log x}{\varphi(d)} + \sqrt{x}\right)\right)^{\frac{1}{2}} \ll \sqrt{R\cdot x\log^9 x}$$

which proves Lemma 3. □

Lemma 4. *Let A be a subset of $\mathbb{N}$. Let $3 \le u,\, v \le x^{\varepsilon}$. Then*

$$|\{p : p+a \le x,\ p+a \in A\}| \le \frac{2}{\log x} \sum_{\substack{qn=p+a\le x\\ qn\in A, q>u, n>v}} \log q + O\left(\frac{\log uv}{\log x}\pi(x)\right).$$

Proof. We have

$$|\{p : p+a \le x,\ p+a \in A\}| \le \sqrt{x} + \frac{2}{\log x} \sum_{\substack{p+a\le x\\ p+a\in A}} \log(p+a)$$

$$\le \frac{2}{\log x} \sum_{\substack{qn=p+a\le x\\ qn\in A,\, q>u,\, n>v}} \log q + \frac{2}{\log x}\sum_{q\le u} \log q\ \pi(x,-a,q)$$

$$+ \frac{2}{\log x} \sum_{\substack{q^r\le x\\ r\ge 2}} \log q^r \pi(x,-a,q^r) + \frac{2}{\log x}\sum_{n\le v}\sum_{\substack{q\le x/n\\ qn-a=p}} \log q,$$

where $\pi(x,-a,d)$ is the number of primes $p \le x$, $p \equiv -a \pmod d$. For the estimate of the second and the third sum we apply Brun–Titchmarsh's inequality $\pi(x,-a,d) \ll \frac{x}{\varphi(d)\log 2x/d}$. Using a sieve result (see [10]) we conclude that the fourth sum is

$$\ll \frac{1}{\log x}\sum_{n\le v} \sum_{\substack{m\le x/n\\ (m\cdot(nm-a), P(\sqrt{x}))=1}} \log m \ll \frac{\pi(x)}{\log x}\sum_{n\le v}\frac{1}{\varphi(n)}.$$

Therefore

$$|\{p : p+a \le x,\ p+a \in A\}| \le \frac{2}{\log x} \sum_{\substack{qn=p+a\le x\\ qn\in A,\, q>u,\, n>v}} \log q$$

$$+ O\left(\pi(x)\left(\sum_{q\le u}\frac{\log q}{q\log\frac{2x}{q}} + \sum_{r\ge 2,\, q^r\le x}\frac{\log q^r}{q^r\log\frac{2x}{q^r}} + \frac{\log v}{\log x}\right)\right).$$

This proves the desired result. □

A first application of Lemmata 2, 3 and 4 gives

Lemma 5. *Let g be an additive function such that*

$$|\{g(p) : p \le x\}| \le c_1 (\log x)^{c_2}.$$

Then, for any $A \subset \mathbb{C}$,

$$|\{p : p \le x,\ g(p+a) \in A\}|$$
$$\ll \frac{\log_2 10x}{\log x} |\{n : n \le x,\ g(n) \in A, (n-a, P(t)) = 1\}| + \pi(x) \frac{\log_2 10x}{\log x}$$

where $t = \log^B x$, $B \ge 4(c_1 + D + 10)$, $D > 2$. The constant implied in $\ll$ depends only on c_1, c_2 and a.

Proof. Using Lemma 4 with $u = v = \log^B x$ we obtain

$$|\{p : g(p+a) \in A,\ p \le x\}| \tag{7}$$
$$\le \frac{2}{\log x} \sum_{\substack{u \le q \le x \\ g(q)+g(n) \in A}} \log q \sum_{qn = p+a \le x} 1 + O\left(\frac{\log_2 x}{\log x} \cdot \pi(x)\right).$$

Here we use the fact that the part of the sum for which $q \mid n$ is estimated by

$$\ll \frac{1}{\log x} \sum_{u < q \le \sqrt{x}} \frac{x}{q^2} \ll \frac{\pi(x)}{\log x}$$

if $B \ge 1$. Let

$$A_i = \{b : b = s - a_i,\ s \in A\}.$$

We apply Lemma 2, with $z^3 = \sqrt[3]{x}$, $r = x^{\frac{1}{100}}$, $Q = \prod_{t<p\le r} p$, $t = \log^B x$, $\eta(d) = \frac{1}{\varphi(d)}$, $X = X_i(1,t)$, $\lambda_m = m - a$, $\rho(m) = |\{qn : qn = m,\ q \in I_k,\ g(q) = a_i,\ g(n) \in A_i,\ n \ge v\}|$,

$$X_i(d,t) = \sum_{\substack{q \in I_k \\ g(q) = a_i}} \sum_{\substack{v \le n \le \frac{x}{q} \\ g(n) \in A_i,\, (qn-a, P(t)) = 1,\, qn \equiv a \pmod d}} 1,$$

$$I_k = \big[u(1 + (\log x)^{-D})^k,\ u(1 + \log^{-D} x)^{k+1}\big),$$

$$D > 2,\ 0 \le k \le K = O((\log x)^{D+1}).$$

Then

$$|\{qn : g(q) = a_i,\ g(n) \in A_i,\ q \in I_k,\ v \le n \le x/q,\ (qn - a, Q) = 1\}|$$
$$\ll \frac{\log t}{\log x} |\{qn : g(q) = a_i,\ g(n) \in A_i,\ q \in I_k,\ v \le n \le x/q,\ (qn - a, P(t)) = 1\}|$$
$$+ \sum_{\substack{d \le \sqrt[3]{x} \\ (d, P(t)) = 1}} 3^{\omega(d)} \left| X_i(d,t) - \frac{1}{\varphi(d)} X_i(1,t) \right|.$$

For the estimate the second sum we apply Lemma 3 with $T = \sqrt[3]{x}$, $y_1 = t = z_3 = \log^B x$, $z_1 = z_2 = \log^B x$, $y_2 = \exp((\log_2 x)^3)$. Thus the second sum is $\ll x(\log x)^{-\frac{B}{4}+7}$. Hence

$$\sum_{u \le q \le x} \log q \sum_{\substack{qn = p + a \le x \\ g(n) + g(q) \in A}} 1 \ll \frac{\log t}{\log x} \sum_{\substack{qn \le x \\ (qn - a, P(t)) = 1, g(qn) \in A}} \log q$$
$$+ x(\log x)^{c_1 + D - \frac{B}{4} + 8} + \frac{\pi(x)}{\log x}.$$

Let $B \ge 4(c_1 + D + 10)$. Using (5), we obtain the assertion of Lemma 5. $\square$

Lemma 6. *For any $t \ge e^3$, y we have*

$$\sum_{\substack{n > y \\ p(n) \le t}} \frac{1}{n} \ll \log^4 t \exp\left(-\frac{\log y}{\log t}\right)$$

where $p(n)$ denotes the largest prime divisor of n.

Proof. Applying Rankin's method we obtain that the sum on the left-hand side is

$$\le y^{-\frac{1}{\log t}} \sum_{p(n) \le t} n^{-1 + \frac{1}{\log t}} \ll \exp\left(-\frac{\log y}{\log t} + \sum_{p \le t} \frac{1}{p} \exp\left(\frac{\log p}{\log t}\right)\right)$$
$$\ll \exp\left(-\frac{\log y}{\log t} + e \log_2 t\right). \qquad \square$$

Lemma 7 (see [11]). *Let Λ be the von Mangoldt function and put*

$$\delta(Q,x,y)=\sum_{d\le Q}\max_{(a,d)=1}\max_{x/2\le N\le x}\max_{h\le y}\Big|\sum_{\substack{N<n\le N+h\\ n\equiv a \pmod d}}\Lambda(n)-\frac{h}{\varphi(d)}\Big|.$$

Suppose that $y=x^{\Theta}$, $\frac{3}{4}\le\Theta\le 1$. Then

$$\delta\left(yx^{-\frac{1}{2}}(\log x)^{-B},x,y\right)\ll y(\log x)^{-A}$$

where A is an arbitrary positive constant. The constant implied in $\ll$ and the constant B depend only on A.

3. Proofs of theorems

For proving Theorems 2, 4, 5 and 6 we use a method analogue to that involved in the proof of Lemma 5.

Proof of Theorem 2. Notice that

$$\sum_{\substack{qn\in N(y,z)\\ qn=m}}\log q\le\sum_{\substack{n\in N(y,z)\\ qn=m}}\log q+\sum_{\substack{n\in N(\frac{y}{q},\frac{z}{q})\\ qn=m}}\log q.$$

Therefore, by Lemma 4,

$$|\{p:p+a\in H(x,y,z)\}|\le\frac{2}{\log x}\sum_{\substack{u<q\\ qn=p+a\le x}}\log q\sum_{\substack{v<n\\ n\in H(x,y,z)}}1$$
$$+\frac{2}{\log x}\sum_{\substack{u<q\\ qn=p+a\le x}}\log q\sum_{\substack{v<n\\ n\in H(\frac{x}{q},\frac{y}{q},\frac{z}{q})}}1+O\left(\pi(x)\frac{\log(uv)}{\log x}\right)$$

where $u\ge v=\log^B x$. Now we divide the interval $(u,x/v]$ into the intervals $I_k=(M_k,M_{k+1}]$, $M_k=u(1+(\log x)^{-c})^k$, where $c\ge 2$ and $0\le k\le K=O(\log^{c+1}x)$. If $q\in I_k$ we replace the condition $n\in H\left(\frac{x}{q},\frac{y}{q},\frac{z}{q}\right)$ by the condition $n \in H\left(\frac{x}{M_k}, \frac{y}{M_{k+1}}, \frac{z}{M_k}\right)$. Then

$$|\{p : p+a \in H(x,y,z)\}| \leq \frac{2}{\log x} \sum_k \log M_{k+1}$$
$$\cdot \left\{ \left| \left\{ qn : q \in I_k,\ n > v,\ n \in H\left(\frac{x}{M_k}, y, z\right),\ qn - a = p \right\} \right| \right.$$
$$\left. + \left| \left\{ qn : q \in I_k,\ n \in H\left(\frac{x}{M_k}, \frac{y}{M_{k+1}}, \frac{z}{M_k}\right),\ n > v,\ qn - a = p \right\} \right| \right\}$$
$$+ O\left(\pi(x) \frac{\log_2^2 x}{\log x}\right).$$

Concerning the second sum we apply Lemma 2 with $z^3 = \sqrt[3]{x}$, $r = x^{\frac{1}{100}}$, $Q = \prod_{t<p\leq r} p$, $\lambda_m = m - a$, $\eta(d) = \frac{1}{\varphi(d)}$,

$$\rho(m) = \left| \left\{ qn : qn = m,\ q \in I_k,\ n > v,\ n \in N\left(\frac{y}{M_{k+1}}, \frac{z}{M_k}\right) \right\} \right|$$

$X = X_k(1,t)$, where $t = \log^B x$ and

$$X_k(d,t) = \sum_{\substack{m \leq x \\ (m-a, P(t))=1,\ m \equiv a \pmod d}} \rho(m).$$

This yields

$$|\{qn : q \in I_k,\ n > v,\ n \in H\left(\frac{x}{M_k}, \frac{y}{M_{k+1}}, \frac{z}{M_k}\right),\ qn - a = p\}|$$
$$\ll \frac{\log t}{\log x} \left| \left\{ qn : q \in I_k,\ n > v,\ n \in H\left(\frac{x}{M_k}, \frac{y}{M_{k+1}}, \frac{z}{M_k}\right), (qn-a, P(t)) = 1 \right\} \right|$$
$$+ \sum_{\substack{d \leq z^3 \\ d|Q}} 3^{\omega(d)} \left| X_k(d,t) - \frac{1}{\varphi(d)} X_k(1,t) \right|.$$

Now, Lemma 3 with $z_3 = t = \log^B x$, $y_1 = \log^B x$, $T = \sqrt[3]{x}$, $z_1 = z_2 = \log^B x$, $y_2 = \exp((\log_2 x)^3)$ shows that the second sum is $O(x \log^{-B/4+7} x)$.

Further, using Lemmata 2, 3 we obtain similar estimates for

$$\left|\left\{qn : q \in I_k,\ n > v,\ n \in H\left(\frac{x}{M_k}, y, z\right),\ qn - a = p\right\}\right|.$$

Let $B \geq 4(c+10)$. Then we have

$$|\{p : p + a \in H(x,y,z)\}| \tag{8}$$

$$\ll \frac{\log t}{\log^2 x} \sum_k \log M_k \{X_k'(1,t) + X_k(1,t)\} + \pi(x)\frac{\log_2^2 x}{\log x},$$

where $X_k'(1,t)$ is the same as $X_k(1,t)$ with the condition $n \in N\Big(\frac{y}{M_{k+1}}, \frac{z}{M_k}\Big)$ replaced by $n \in N(y,z)$.

The sieve of Eratosthenes gives

$$X_k'(1,t) + X_k(1,t)$$

$$\leq 2 \sum_{\substack{n \leq x/M_k \\ n \in N\left(\frac{x}{M_k}, \frac{y}{M_{k+1}}, \frac{z}{M_k}\right) \cup N(y,z)}} \sum_{\substack{d \leq \sqrt[3]{u} \\ d | P(t),\ (d, 2an)=1}} \mu(d) \sum_{\substack{q \in I_k \\ qn \equiv a \pmod d}} 1 + R_1$$

$$\ll \frac{1}{\log t} r_k + R_1 + R_2,$$

where

$$r_k = \sum_{\substack{n \leq \frac{x}{M_k} \\ n \in N(y,z) \cup N\left(\frac{y}{M_{k+1}}, \frac{z}{M_k}\right)}} \frac{n}{\varphi(n)} \sum_{q \in I_k} 1,$$

$$R_1 \ll (\log x)^2 x \sum_{\substack{d > \sqrt[3]{u} \\ d | P(t)}} \frac{1}{d}$$

and

$$R_2 \ll \sum_{\substack{v < n \leq x/M_k \\ n \in N(y,z) \cup N\left(\frac{y}{M_{k+1}}, \frac{z}{M_k}\right)}} \sum_{\substack{d \leq \sqrt[3]{u} \\ (d, 2na)=1}} \left| \sum_{\substack{q \in I_k \\ qn \equiv a \pmod d}} 1 - \frac{1}{\varphi(d)} \sum_{q \in I_k} 1 \right|.$$

Let $u=\exp(B^2(\log_2 x)^2)$, $t=\log^B x$. By Lemma 6

$$R_1 \ll x\log^7 x \exp\left(-\frac{1}{3}\frac{\log u}{\log t}\right) \ll x(\log x)^{-B/3+7}.$$

Using Lemma 7 we see that

$$R_2 \ll \frac{1}{\log^D u} r_k$$

where $D>0$ is arbitrary constant. Thus we arrive at

$$X_k'(1,t)+X_k(1,t) \ll \frac{1}{\log t} r_k + x(\log x)^{-B/3+7}.$$

Let $B=4(c+10)$. Then (8) gives

$$|\{p : p+a \in N(y,z)\}| \ll \frac{1}{\log^2 x}\Bigg\{\sum_q \log q \sum_{\substack{n\le \frac{x}{q}\\ n\in N(y,z)}} \frac{n}{\varphi(n)}$$

$$+\sum_q \log q \sum_{\substack{n\le \frac{x}{q}\\ qn\in N(y(1+\log^{-c} x)^{-1}, z(1+\log^{-c} x)))}} \frac{n}{\varphi(n)}\Bigg\} + \frac{(\log_2 x)^2}{\log x}\pi(x).$$

This completes the proof of Theorem 2. □

Proof of Theorem 4. Let $u=\exp(B^2\log_2^2 x)$, $v=\log^B x$, $B>0$. Setting

$$Y_k(d,t)=\Bigg|\Bigg\{qn : q\in I_k,\ v<n\le x/M_k,\ (qn-a,P(t))=1,$$
$$F(n)\in\left[\frac{1}{c_2}K, \frac{1}{c_1}S\right),\ qn\equiv a \pmod d\Bigg\}\Bigg|$$

where $I_k=(M_k,M_{k+1}]$, $M_k=(1+\log^{-c} x)^k$, $c\ge 2$, $0\le k\le K_1=O(\log^{c+1} x)$, we obtain by Lemma 4

$$|\{p : p+a\le x,\ F(p+a)\in[K,S]\}|$$

$$\le \frac{1}{\log x}\sum_k \log M_{k+1}\cdot Y_k(1, x^{\frac{1}{100}}) + O\left(\frac{\log_2^3 x}{\log x}\right).$$

Here we made use of the assumption

$$c_1 F(n) \le F(np) \le c_2 F(n), \quad (n,p)=1.$$

Applying Lemma 2 with $z^3 = \sqrt[3]{x}$, $r = x^{\frac{1}{100}}$,

$$Q = \prod_{t<p\le r} p, \quad t = \log^B x, \quad \eta(d) = \frac{1}{\varphi(d)},$$

$$X = X_k(1,t), \quad \lambda_m = m - a,$$

$$\rho(m) = \left|\left\{qn : qn = m,\ q \in I_k,\ F(n) \in \left[\frac{1}{c_2}K, \frac{1}{c_1}S\right],\ n > v\right\}\right|,$$

we obtain

$$\text{(9)}\quad Y_k(1, x^{\frac{1}{100}}) \ll \frac{\log t}{\log x} Y_k(1,t) + \sum_{\substack{d \le z^3 \\ (d,P(t))=1}} 3^{\omega(d)} \left| Y_k(d,t) - \frac{1}{\varphi(d)} Y_k(1,t) \right|.$$

By Lemma 3 with $T = \sqrt[3]{x}$, $y_1 = z_3 = t = \log^B x$, $z_1 = u$, $z_2 = v$, $y_2 = \exp(\log_2^3 x)$ it follows that the second sum is $O\big(x \log^{-B/4+7} x\big)$. Using the sieve of Eratosthenes we conclude

$$Y_k(1,t) \ll \sum_{\substack{d|P(t) \\ (d,2a)=1}} \frac{\mu(d)}{\varphi(d)} Y_k + x \log^2 x \sum_{\substack{d > \sqrt[3]{u} \\ d|P(t)}} \frac{1}{d}$$
$$+ \sum_{\substack{v<n\le x/m_k \\ F(n)\in\left[\frac{1}{c_2}K, \frac{1}{c_1}S\right)}} \sum_{\substack{d \le \sqrt[3]{u} \\ (d,2an)=1}} \left| \sum_{\substack{q\in I_k \\ qn\equiv a \pmod d}} 1 - \frac{1}{\varphi(d)} \sum_{q \in I_k} 1 \right|,$$

where

$$Y_k = \left|\left\{qn : q \in I_k,\ v < n \le \frac{x}{M_k},\ F(n) \in \left[\frac{K}{c_2}, \frac{S}{c_1}\right]\right\}\right|.$$

Recall that $u = \exp(B^2(\log_2 x)^2)$, $t = \log^B x$. By Lemmata 6, 7 we get

$$Y_k(1,t) \ll \frac{1}{\log t} Y_k + x(\log x)^{-\frac{B}{3}+7}.$$

Let $B \geq 4(c+10)$. From the last estimate and (9) we have

$$|\{p : p+a \leq x, F(p+a) \in [K,S]\}| \ll \frac{1}{\log^2 x} \sum_{u<q} \log q \sum_{\substack{v<n\leq \frac{x}{q} \\ F(n)\in\left[\frac{K}{c_2},\frac{S}{c_1}\right)}} + \pi(x)\frac{\log_2^2 x}{\log x}.$$

If $F(n) \in [K,S]$ and $(q,n) = 1$ we see that $F(qn) \in [c_1K, c_2S]$, and this completes the proof of Theorem 4. □

Proof of Theorem 6. Let $\rho(n,f)$ denote $F(n,f)$ or $\Delta(n,f)$, respectively. By Lemma 4 we obtain

$$\begin{aligned} &|\{p : p+a \leq x,\ \rho(p+a,f) \in [K,S]\}| \\ &\ll \pi(x)\frac{\log(uv)}{\log x} + \frac{1}{\log x}\sum_{q>u} \log q \sum_{\substack{v<n\leq \frac{x}{q} \\ qn=p+a,\ \rho(q\cdot n,f)\in[K,S)}} 1. \end{aligned}$$

Let $u = \exp(B^2 \log_2^2 x)$, $v = \log^B x$. The part of the last sum for which $q \mid n$ is $O\left(\pi(x)\frac{1}{u}\right)$. Hence we may assume that $(n,q) = 1$. Next, we note that, for $(n,q) = 1$,

$$F(nq,f) = \max_y \Big| \sum_{\substack{d|n \\ d\leq y}} f(d) + f(q) \sum_{\substack{d|n \\ d\leq y/q}} f(d) \Big|,$$

$$\Delta(nq,f) = \max_y \Big| \sum_{\substack{d|n \\ y<d\leq ey}} f(d) + f(q) \sum_{\substack{d|n \\ \frac{y}{q}<d\leq e\frac{y}{q}}} f(d) \Big|.$$

Let us make the summation over q, n independent from each other. For this aim we split the interval $(u, x/v)$ into the intervals $I_k = (M_k, M_{k+1}]$, $M_k = u(1+\log^{-c} x)^k$, $c \geq 2$. The number of such intervals equals $O(\log^{c+1} x)$. For $q \in I_k$ we replace $\rho(nq,f)$ by $\rho(n, f(q), M_k, f)$ where, for example,

$$\Delta(n, f(q), M_k, f) = \max_y \Big| \sum_{\substack{d|n \\ y<d\leq ey}} f(d) + f(q) \sum_{\substack{d|n \\ \frac{y}{M_k}<d\leq e\frac{y}{M_k}}} f(d) \Big|.$$

For every n such that

$$\Gamma(n,c) := \max_y \sum_{\substack{d|n \\ y<d\le y(1+\log^{-c} x)}} 1 \le 1$$

we have

$$|F(nq,f) - F(n,f(q),M_k,f)| \le 1, \tag{10}$$
$$|\Delta(nq,f) - \Delta(n,f(q),M_k,f)| \le 2.$$

If $f(q) = a_i$, $i = 1,\dots,s$, we put $\rho_{i,k}(n,f) = \rho(n,a_i,M_k,f)$. Thus we have

$$\begin{aligned} &|\{p : p+a \le x,\ \rho(p+a,f) \in [K,S\}| \\ &\ll \pi(x)\frac{(\log_2 x)^2}{\log x} + |\{n : n \le x,\ \Gamma(n,c) \ge 2\}| \\ &\quad + \frac{1}{\log x}\sum_k \log M_k \sum_{i=1}^{s} A_{k,i}(1,r) \end{aligned} \tag{11}$$

where $r = x^{\frac{1}{100}}$ and

$$A_{k,i}(d,r) = \Big|\Big\{\ qn;\ q \in I_k,\ v < n \le \frac{x}{M_k},\ \rho_{k,i}(n,f) \in (K-2,S+2],\ (qn-a,P(r)) = 1,\ qn-a \equiv 0 \pmod d\Big\}\Big|.$$

We apply Lemma 2 with $z^3 = \sqrt[3]{x}$, $r = x^{\frac{1}{100}}$, $\lambda_m = m-a$, $Q = \prod_{t<p\le r} p$, $t = \log^B x$, $\eta(d) = \frac{1}{\varphi(d)}$,

$$\rho(m) = |\{qn;\ q \in I_k,\ n > v,\ \rho_{k,i}(n,f) \in (K-2,S+2]\}|.$$

By Lemma 2 we obtain

$$A_{k,i}(1,r) \ll \frac{\log t}{\log x} A_{k,i}(1,t) + \sum_{\substack{d\le z^3 \\ d|Q}} 3^{\omega(d)} \Big| A_{k,i}(d,t) - \frac{1}{\varphi(d)} A_{k,i}(1,t) \Big|.$$

Put in Lemma 3 $z_3 = t = \log^B x$, $y_1 = \log^B x$, $T = \sqrt[3]{x}$, $z_1 = z_2 = \log^B x$, $y_2 = \exp((\log_2 x)^3)$. Using Lemma 3 we obtain that the second sum is

$O(x(\log x)^{-\frac{B}{4}+7})$. Hence

$$A_{k,i}(1,r) \ll \frac{\log t}{\log x} A_{k,i}(1,t) + x(\log x)^{-\frac{B}{4}+7}.$$

Observing (11) we obtain

$$\begin{aligned} &|\{p : p+a \le x,\ \rho(p+a,f) \in [K,S\}| \ll \pi(x)\frac{(\log_2 x)^2}{\log x} \\ &+|\{n : n \le x,\ \Gamma(n,c) \ge 2\}| + \frac{\log t}{(\log x)^2}\sum_k \log M_k \sum_{i=1}^{s} A_{k,i}(1,t), \end{aligned} \tag{12}$$

if $B \ge 4(c+10)$. By the sieve of Eratosthenes we conclude

$$\begin{aligned} A_{k,i}(1,t) &\le \sum_{\substack{v<n\le \frac{x}{M_k} \\ \rho_{k,i}(n,f)\in(K-2,S+2]}} \sum_{\substack{q\in I_k \\ (qn-a,P(t))=1}} 1 \\ &\ll \sum_{\substack{v<n\le \frac{x}{M_k} \\ \rho_{k,i}(n,f)\in(K-2,S+2]}} \sum_{\substack{d\le \sqrt[3]{u} \\ d|P(t),\ (d,nP(t_1))=1}} \mu(d) \sum_{\substack{q\in I_k \\ qn\equiv a \pmod d}} 1 + R_1. \end{aligned}$$

where $t_1 > \max(2a, b_1, \dots, b_l)$ and

$$R_1 \ll \log^2 x \quad x \sum_{\substack{d>\sqrt[3]{u} \\ d|P(t)}} \frac{1}{d}.$$

Recall that $u = \exp(B^2 \log_2^2 x)$, $t = \log^B x$, $B > 4(c+10)$. Hence by Lemma 6 $R_1 \ll x(\log x)^{-c-2}$. Thus

$$\begin{aligned} A_{k,i}(1,t) \ll \frac{1}{\log t} &\sum_{\substack{v<n\le \frac{x}{M_k} \\ \rho_{k,i}(n,f)\in(K-2,S+2]}} \frac{n}{\varphi(n)} \sum_{q\in I_k} 1 + x(\log x)^{-c-2} \\ &+ \sum_{\substack{v<n\le \frac{x}{M_k} \\ \rho_{k,i}(n,f)\in(K-2,S+2]}} \sum_{\substack{d\le \sqrt[3]{u} \\ (d,2na)=1}} \Bigg| \sum_{\substack{q\in I_k \\ qn\equiv a \pmod d}} 1 - \frac{1}{\varphi(d)} \sum_{q\in I_k} 1 \Bigg|. \end{aligned}$$

By Lemma 7

$$A_{k,i}(1,t) \ll \frac{1}{\log t} \sum_{\substack{v<n\le \frac{x}{M_k} \\ \rho_{k,i}(n,f)\in(K-2,S+2]}} \frac{n}{\varphi(n)} \sum_{q\in I_k} 1 + x\log^{-c-2} x.$$

The estimate

$$\sum_{q\in I_k} 1 \le \delta_i \sum_{\substack{q\in I_k \\ f(q)=a_i}} 1$$

and (12), (10) yield

$$|\{p : p+a\le x,\ \rho(p+a,f)\in[K,S\}|$$
$$\ll \pi(x)\frac{(\log_2 x)^2}{\log x} + \frac{1}{\log x} \sum_{\substack{n\le x \\ \rho(n,f)\in(K-4,S+4]}} \frac{n}{\varphi(n)} + |\{n : n\le x,\ \Gamma(n,c)\ge 2\}|.$$

Further, if $c\ge 3$

$$|\{n : n\le x,\ \Gamma(n,c)\ge 2\}|$$
$$\le |\{n : n\le x,\ n=d_1d_2m,\ d_1<d_2\le d_1(1+\log^{-c} x)\}|$$
$$\le x\sum_{d_1\le x}\frac{1}{d_1}\sum_{d_1<d_2\le d_1(1+\log^{-c} x)}\frac{1}{d_2}$$
$$\ll x\sum_{\log^c x\le d_1\le x}\frac{1}{d_1^2}(d_1\log^{-c} x+1) \ll x\log^{-2} x$$

and thus we obtain the proof of Theorem 6. □

Proof of Theorem 7. Actually we prove a more general result.

Theorem 8. *Let E be a divisor closed set, i.e. if $n\in E$ then every divisors of n belongs to E, too. Then*

$$|\{p : p+a\le x,\ p+a\in E\}| \ll \frac{\pi(x)}{\log x}\sum_{\substack{n\le x \\ n\in E}}\frac{1}{\varphi(n)} + \pi(x)\frac{(\log_2 x)^2}{\log x}.$$

Proof. We use Lemma 4 with $u = \exp(B^2 \log_2^2 x)$, $v = \log^B x$ and get

$$(13)\quad |\{p : p + a \le x,\ p + a \in E\}| \ll \pi(x)\frac{(\log_2 x)}{\log x} + \frac{1}{\log x}\sum_{0\le k\le K_1} |E_k| \log M_k$$

where

$$E_k = \Big\{nq : 2^k u < q \le 2^{k+1}u,\ v < n \le \frac{x}{2^k u},\ n \in E,\ qn - a = p\Big\}$$

and $K_1 \le \log x$. Applying Lemma 2 and Lemma 3 in the same way as before gives

$$|E_k| \ll \frac{\log t}{\log x}|E_k(t)| + x(\log x)^{-\frac{B}{4}+7}$$

where

$$E_k(t) = \Big|\Big\{nq : 2^k u < q \le 2^{k+1}u, v < n \le \frac{x}{2^k u},\ n \in E,\ (qn-a, P(t)) = 1\Big\}\Big|$$

Hence, by (13),

$$|\{p : p+a \le x,\ p+a \in E\}| \ll \pi(x)\frac{(\log_2 x)^2}{\log x} + \frac{\log t}{(\log x)^2}\sum_{\substack{n\le \frac{2x}{u}\\ n\in E}}\ \sum_{\substack{q\le \frac{2x}{n}\\ (nq-a,P(t))=1}} \log q.$$

Using Selberg's sieve (see [10]) yields

$$\sum_{\substack{q\le \frac{x}{n}\\ (qn-a,P(t))=1}} \log q \ll \frac{x}{\varphi(n)\log t},$$

and this completes the proof of Theorem 8. □

Let E_0 be a set of natural numbers without divisors d, d' such that $d < d' \le 2d$. E_0 is a divisor closed set. Hence

$$(14)\quad |\{p : p+a \le x,\ p+a \in E_0\}| \ll \frac{\pi(x)}{\log x}\sum_{d\le x}\frac{1}{d\varphi(d)}\sum_{\substack{n\le x\\ n\in E_0}}\frac{1}{n} + \pi(x)\frac{\log_2^2 x}{\log x}.$$

We shall apply the estimate (see Theorem 52 of [4])

$$|\{n : n \le x,\ n \in E_0\}| \ll xR(x),$$

where $R(x) = (\log_2 x)^{-\beta}(\log_3 x)^{4\beta}$, $\beta > 0$. Let $y = \exp((\log x)\cdot R(x))$. For $y \le t \le x$ we see that

$$R(t) \le R(y) = (\log(\log x + R(x)))^{-\beta}(\log_2(\log x \cdot R(x)))^{4\beta} \ll R(x).$$

Therefore

$$\frac{1}{\log x}\sum_{\substack{n\le x\\ n\in E_0}} \frac{1}{n} \ll \frac{1}{\log x}\sum_{n\le y}\frac{1}{n} + \max_{t\in[y,x]}\frac{1}{t}\sum_{\substack{n\le 2t\\ n\in E_0}} 1 \ll R(x).$$

Combining this estimate with (14) completes the proof of Theorem 7. □

Proof of Theorem 1. Put

$$P(m,x,g) := |\{p : p + a \le x,\ g(p+a) = m\}|,$$

where $g(n)$ denotes $\omega(n)$ or $\Omega(n)$, respectively. We use the following lemma which is a corollary of Theorems 1, 2 from [12].

Lemma 8. *For $x \ge |a| + 2$*

$$P(m,x,g) \le c_1 \frac{x(\log_2 x)^{m-1}}{(m-1)!\log^2 x}. \tag{15}$$

If $g(n) = \omega(n)$, then (15) holds for $m \le b\log_2 x$, and the constant c_1 depends on a and b. If $g(n) = \Omega(n)$, (15) holds for $m \le (2-\delta)\log_2 x$ and the constant c_1 depends on a and $\delta > 0$. In the case $a = 2$ the estimate

$$P(m,x,g) + P(m+1,x,g) \le c_2 \frac{x(\log_2 x)^{m-1}}{(m-1)!\log^2 x} \tag{16}$$

($c_2 > 0$) holds for $g(n) = \omega(n)$ if $m \le b\log_2 x$ and for $g(n) = \Omega(n)$ if $1 \le m \le (2-\delta)\log_2 x$.

The inequalities

$$m^{\omega(n)} \le d_m(n) \le m^{\Omega(n)} \quad (m \ge 2, n \ge 1) \tag{17}$$

follow easily from the formula $d_{m+1}(n) = \sum_{t|n} d_m(t)$ by induction on m. Hence

$$D_1(x,\alpha,m,P) \le \sum_{k\le(1-\alpha)\log x} P(k,x,\omega), \tag{18}$$

and

$$\begin{aligned} &D_2(x,\alpha,m,P) \\ &\le \sum_{(1+\alpha)\log_2 x\le k\le(2-\delta)\log_2 x} P(k,x,\Omega) + \sum_{k>(2-\delta)\log_2 x} P(k,x,\Omega). \end{aligned} \tag{19}$$

By (15) the right-hand side of (18) is

$$\begin{aligned} &\ll \sum_{k\le(1-\alpha)\log_2 x} \frac{x(\log_2 x)^{k-1}}{(k-1)!(\log x)^2} \\ &\ll \frac{\pi(x)}{\log^2 x}\frac{(\log_2 x)^{[(1-\alpha)\log_2 x]-1}}{([(1-\alpha)\log_2 x]-1)!}\sum_{k\ge 0}(1-\alpha)^k. \end{aligned}$$

Using Stirling's formula

$$n! = \sqrt{2\pi n}\; n^n e^{-n+\frac{\Theta}{12n}} \qquad (0<\Theta<1)$$

shows that for $0<\alpha\le\beta<1$

$$\begin{aligned} \frac{(\log_2 x)^{[(1-\alpha)\log_2 x]-1}}{([(1-\alpha)\log_2 x]-1)!} &\ll_\beta e^{(1-\alpha)\log_2 x}(\log_2 x)^{-\frac12}\cdot(1-\alpha)^{-(1-\alpha)\log_2 x} \\ &= (\log x)^{Q(-\alpha)}(\log_2 x)^{-\frac12}\cdot\log x, \end{aligned}$$

and therefore

$$D_1(x,\alpha,m,P) \ll_\beta \pi(x)(\log x)^{Q(-\alpha)}\alpha^{-1}(\log_2 x)^{-\frac12}.$$

Arguing in the same way as before we obtain by (15) that the first sum on the right-hand side of (19) is

$$\begin{aligned} &\ll \sum_{T_1(\alpha)\le k\le T_2(\alpha)} \frac{x(\log_2 x)^{k-1}}{(k-1)!(\log x)^2} \\ &\ll \frac{\pi(x)}{\log x}\frac{(\log_2 x)^{T_1(\alpha)-1}}{(T_1(\alpha)-1)!}\sum_k \frac{1}{(1+\alpha)^k}, \end{aligned}$$

where $T_1(\alpha) = [(1+\alpha)\log_2 x]$, $T_2(\alpha) = [(2-\delta)\log_2 x]$. Applying Stirling's formula we get

$$D_2(x,\alpha,m,P) \ll_\beta \pi(x)\alpha^{-1}(\log_2 x)^{-\frac{1}{2}}(\log x)^{Q(\alpha)} + \sum_{k>(2-\delta)\log_2 x} P(k,x,\Omega).$$

By Lemma 5 the second sum is

$$\begin{aligned} &\ll \frac{\log_2 x}{\log x}|\{n : n \le x, (2-\delta)\log_2 x \le \Omega(n)\}| + \frac{\pi(x)\log_2(x)}{\log x} \\ &\ll \frac{\log_2 x}{\log x} q^{-(2-\delta)\log_2 x} \sum_{n\le x} q^{\Omega(n)} + \pi(x)\frac{\log_2(x)}{\log x} \end{aligned} \tag{20}$$

where $1 < q < 2$. Put $\delta = 2-q$ and let $0 < \alpha \le \beta < q-1 < 1$. The first expression in (20) is

$$\ll \pi(x)\log_2 x(\log x)^{Q(q-1)} \ll_\beta \pi(x)(\log x)^{Q(\alpha)-\varepsilon}$$

where $\varepsilon > 0$. Thus

$$D_2(x,\alpha,m,P) \ll_\beta \pi(x)\alpha^{-1}(\log_2 x)^{-\frac{1}{2}}(\log x)^{Q(\alpha)}$$

and we complete the proof of the first part of Theorem 1.

By (17) we have

$$2D_1(x,\alpha,m,P) \ge \sum_{k\le(1-\alpha)\log_2 x-1} (P(k,x,\Omega) + P(k+1,x,\Omega)).$$

Using Lemma 8 we see that the left side is larger than

$$c_2 \sum_{K\le K(\alpha)} \frac{\pi(x)}{(\log x)^2}\frac{(\log_2 x)^{K-1}}{(K-1)!} \ge c_2\frac{\pi(x)}{\log x}\cdot\frac{(\log_2(x))^{K(\alpha)-1}}{(K(\alpha)-1)!}$$

$$\cdot\left(1 + \frac{K(\alpha)-1}{\log_2 x} + \frac{(K(\alpha)-1)(K(\alpha)-2)}{(\log_2 x)^2} + \dots + \frac{(K(\alpha)-1)\dots 1}{(\log_2 x)^{K(\alpha)-1}} + \right)$$

where $K(\alpha) = [(1-\alpha)\log_2 x] - 1$. Thus

$$\begin{aligned}
&\frac{(K(\alpha)-1)(K(\alpha)-2)\dots(K(\alpha)-r)}{(\log_2 x)^r} \\
&\quad \ge \left(1-\alpha-\frac{2}{\log_2 x}\right)\dots\left((1-\alpha)-\frac{r+1}{\log_2 x}\right) \\
&\quad \ge (1-\alpha)^r\left(\left(1-\frac{r+1}{(1-\alpha)\log_2 x}\right)^{\frac{(1-\alpha)\log_2 x}{r}}\right)^{\frac{r^2}{(1-\alpha)\log_2 x}}.
\end{aligned}$$

Let $r \le \left[\sqrt{\log_2 x}\,\right] = r_0$. For $x \ge x_0$

$$\frac{(K(\alpha)-1)\dots(K(\alpha)-r)}{(\log_2 x)^r} \ge c_3(1-\alpha)^r, \quad c_3 > 0.$$

Using Stirling's formula we get

$$D_1(x,\alpha,m,P) \ge c_4\pi(x)(\log x)^{Q(-\alpha)}(\log x)^{-\frac{1}{2}}\frac{1-(1-\alpha)^{r_0}}{\alpha}$$

and if $\alpha \ge \left(\sqrt{\log_2 x}\,\right)^{-1}$ we obtain the lower bound for $D_1(x,\alpha,m,P)$.

In a similar way we can prove the lower estimate for $D_2(x,\alpha,m,P)$, too. □

References

[1] M. B. Barban and B. V. Levin, Multiplicative functions on the set of shifted prime numbers, *Dokl. Acad. Nauk SSSR* **181** (1968), 778–780.

[2] D. Wolke, Multiplikative Funktionen auf schnell wachsenden Folgen, *Journal für die reine und angew. Math.* **251** (1971), 54–67.

[3] K. K. Norton, On the number of restricted prime factors of an integer, I, *Illinois J. Math.* **20** (1976), 681–705.

[4] R. R. Hall and G. Tenenbaum, Divisors, *Cambridge University Press*, 1988.

[5] C. Hooley, On a new technique and its applications to the theory of numbers, *Proc. London Math. Soc.* **38** (3) (1979), 115–151.

[6] P. Erdős and I. Katai, Non-complete sums of multiplicative functions, *Periodica Math. Hungarica* **1** (1977), 209–212.

[7] H. Maier, On the Möbius function, *Trans. Amer. Math. Soc.* **301** (1987), 649–664.

[8] P. D. T. A. Elliot, Probabilistic Number Theory I, Grundlehren der Math. Wissenschaften 239, 1979.

[9] N. M. Timofeev, The Erdős–Kubilius conjecture on the distribution of the values of additive functions on the sequence of shifted primes, *Acta Arithm.* **52** (2) (1991), 113–131.

[10] K. Prachar, Primzahlverteilung, *Springer-Verlag*, *Berlin*, 1979.

[11] M. N. Huxley and H. Iwaniec, Bombieri's theorem in short intervals, *Mathematika* **22** (1975), 188–194.

[12] N. M. Timofeev, The Hardy–Ramanujan and Halász inequalities for shifted primes, *Mathematical Notes* **57**(N5) (1995), 522–535.

[13] F. Pappalardi, On the Order of Finitely Generated Subgroups of Q^* (mod p) and Divisors of $p-1$, *Journal of Number Theory* **57** (1996), 207–222.

[14] C. R. Matthews, Counting points modulo p for some finitely generated subgroups of algebraic group, *Bull. London Math. Soc.* **14** (1982), 149–154.

[15] P. Erdős and M. R. Murty, On the order of a (mod p), *Centre de Recherches Mathematiques, CRM Proceedings and Lecture Notes* **19** (1999), 87–97.

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