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Paley–Wiener type theorems for Chébli–Trimèche transforms

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Abstract. In this note we establish new Paley–Wiener type theorems for the Chébli–Trimèche transform. We use exclusively real methods to prove our Paley–Wiener theorems, in contrast with the complex procedures that appear in the literature.

1. Introduction

In this note we investigate new properties for the generalized Fourier transformation (also called Chébli–Trimèche transformation) $\mathcal{F}$ defined, when f is a suitable function defined on $(0,\infty)$, by

$$\mathcal{F}(f)(y)=\int_0^\infty \psi_y(x)f(x)A(x)dx,\quad y\geq 0,$$

where, for every $y\geq 0$, ψ_y represents the solution of the equation

$$\Delta\psi_y(x)=(y^2+\rho^2)\psi_y(x),\quad x>0, \tag{1.1}$$

satisfying that

$$\psi_y(0)=1\quad\text{and}\quad \frac{d}{dx}\psi_y(0)=0.$$

Here $\rho\geq 0$, Δ denotes the differential operator

$$\Delta=-\frac{1}{A(x)}\frac{d}{dx}\left(A(x)\frac{d}{dx}\right), \tag{1.2}$$

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and we suppose that the function A is continuous on $[0,\infty)$, twice continuously differentiable on $(0,\infty)$ and fulfils the following conditions

(i) $A(0)=0$ and $A(x)>0$, $x>0$,

(ii) A is increasing and unbounded on $(0,\infty)$,

(iii) There exist $\alpha>-\frac{1}{2}$, $\delta>0$ and an odd C^∞-function B on $\mathbb{R}$ such that

$$\frac{A'(x)}{A(x)}=\frac{2\alpha+1}{x}+B(x),\quad x\in(0,\delta).$$

Moreover there exist η and $M>0$ and a smooth function $\mathcal{C}$, whose derivatives of any order are bounded on $(0,\infty)$, in such a way that

$$\frac{A'(x)}{A(x)}=\begin{cases}2\rho+e^{-\eta x}\mathcal{C}(x), & \text{if } \rho>0\\ \dfrac{2\alpha+1}{x}+e^{-\eta x}\mathcal{C}(x), & \text{if } \rho=0,\end{cases}$$

for every $x\in(M,\infty)$.

(iv) $\frac{A'}{A}$ is a decreasing C^∞-function on $(0,\infty)$. Hence there exists $\lim_{x\to\infty}\frac{A'(x)}{A(x)}\geq 0$.

In the sequel the positive real number ρ appearing in (1.1) is defined by

$$\rho=\frac{1}{2}\lim_{x\to\infty}\frac{A'(x)}{A(x)},$$

when Δ is given by (1.2).

In particular, the generalized Fourier transform $\mathcal{F}$ reduces to the Hankel transform ([8]) when $A(x)=x^{2\alpha+1}$, $x\in[0,\infty)$, and $\alpha>-\frac{1}{2}$. Also, the Jacobi transform ([6] and [9]), that can be interpreted in certain cases as the spherical transform on noncompact symmetric spaces of rank one, appears when $A(x)=\sinh^{2\alpha+1}x\cosh^{2\beta+1}x$, $x\in[0,\infty)$, with $\alpha\geq\beta\geq-\frac{1}{2}$ and $\alpha\neq-\frac{1}{2}$.

The inversion formula of the transformation $\mathcal{F}$ is given by [5]

$$f(x)=\int_0^\infty\psi_y(x)\mathcal{F}(f)(y)\frac{dy}{|c(y)|^2}$$

where $c(y)$ is a continuous function without zeros on $[0,\infty)$. The function $c(y)$ can be seen as a Harish–Chandra type function and we refer to [3] and [4] for details.

For the Chébli–Trimèche transform the following Plancherel formula ([12] and [2, Theorem 2.2.13]) holds

$$\int_0^\infty |f(x)|^2 A(x)dx = \int_0^\infty |\mathcal{F}(f)(y)|^2 \frac{dy}{|c(y)|^2}, \tag{1.3}$$

for every $f \in L^2((0,\infty), A(x)dx)$. As usual, for every $1 \le p \le \infty$, by $L^p((0,\infty), d\mu(x))$ we represent the Lebesgue p-space on $(0,\infty)$ with respect to the positive measure μ.

It is well-known that $\mathcal{F}$ maps $L^1((0,\infty), A(x)dx)$ into the space $\mathcal{C}_0$ of the continuous functions on $\mathbb{R}$ vanishing in ∞. Hence, by (1.3), Riesz–Thorin interpolation theorem implies that $\mathcal{F}$ can be extended as a bounded operator from $L^p((0,\infty), A(x)dx)$ into $L^{p'}((0,\infty), \frac{dy}{|c(y)|^2})$, provided that $1 \le p \le 2$, where p' denotes the exponent conjugated to p.

As in [11, Section 6], for every $a > 0$, the space $\mathcal{D}_a(\mathbb{R})$ is constituted by all those even and C^∞-functions ϕ on $\mathbb{R}$ such that $\phi(x) = 0$, $|x| \ge a$. In [11, Théorème 7.2, (i)] it was proved that a function ϕ is in $\mathcal{D}_a(\mathbb{R})$ if, and only if, $\mathcal{F}(\phi)$ is an even entire function and, for every $m \in \mathbb{N}$,

$$\sup_{y\in\mathbb{C}} (1 + |y|^2)^m |\mathcal{F}(\phi)(y)| e^{-a|\operatorname{Im} y|} < \infty.$$

Also the image by $\mathcal{F}$ of the (even) distributions of compact support was characterized ([11, Théorème 7.2, (ii)]).

In Section 2 of this paper we obtain new versions of the Paley–Wiener theorem for the Chébli–Trimèche transform. Our results, that extend other ones proved by V. K. Tuam [14] for the Hankel transform, are established by using real methods in contrast with the complex procedure followed in [5] and [11].

Throughout this work by C we always represent a positive constant not necessarily the same in each occurrence.

2. Paley–Wiener theorems for Chébli–Trimèche transform

H. Chébli [5] and K. Trimèche [11] established a Paley–Wiener theorem for the Chébli–Trimèche transform that extends the classical Paley–Wiener theorem for the Fourier transform ([10]) and the Griffith's theorem for the Hankel transform ([7]). In this Section, inspired in the results obtained by V. K. Tuam for the Hankel transform ([14]) and the Airy transform ([13]), we obtain new Paley–Wiener theorems for the Chébli–Trimèche transform. We use a real procedure in contrast with the complex method employed in [5] and [11].

W. Bloom and Z. Xu [3] studied the generalized Fourier transform on Schwartz spaces. They introduced the space $S_p((0,\infty),A)$, for each $0<p\leq 2$, as follows. A complex valued function ϕ defined on $(0,\infty)$ is in $S_p((0,\infty),A)$ if, and only if, there exists an even function $\Phi\in C^\infty(\mathbb{R})$ such that $\phi=\Phi$, on $(0,\infty)$, and

$$\mu^p_{n,m}(\phi)=\sup_{x\in(0,\infty)}(1+x^2)^m\psi_0(x)^{-2/p}\left|\frac{d^n}{dx^n}\phi(x)\right|<\infty,$$

for every $n,m\in\mathbb{N}$. The image by the Chébli–Trimèche transform of $S_p((0,\infty),A)$ is characterized in [3, Proposition 4.26]. Note that if $\rho=0$ then the space $S_p((0,\infty),A)$ is actually independent of $p\in(0,2]$. Moreover, when $\rho=0$ the space $S_p((0,\infty),A)$ coincides with the space $S_{\text{even}}(\mathbb{R})$ that is constituted by all those even functions in the Schwartz space $S(\mathbb{R})$ ([12]). The topology of $S_{\text{even}}(\mathbb{R})$ is defined by the family $\{\mu_{n,m}\}_{n,m\in\mathbb{N}}$ of seminorms, where, for every $n,m\in\mathbb{N}$,

$$\mu_{n,m}(\phi)=\sup_{x\in(0,\infty)}(1+x^2)^m\left|\frac{d^n}{dx^n}\phi(x)\right|,\quad \phi\in S_{\text{even}}(\mathbb{R}).$$

We now establish our first Paley–Wiener type result. Previously we need to show a useful property concerning the topology of $S_{\text{even}}(\mathbb{R})$.

Lemma 2.1. *Let $1\leq q\leq\infty$ and $\rho=0$. For every $n,m\in\mathbb{N}$ we define the seminorm $\gamma^q_{n,m}$ on $S_{\text{even}}(\mathbb{R})$ by*

$$\gamma^q_{n,m}(\phi)=\|(1+x^2)^m\Delta^n\phi(x)\|_{L^q((0,\infty),A(x)dx)},\quad \phi\in S_{\text{even}}(\mathbb{R}).$$

Then the family $\{\gamma^q_{n,m}\}_{n,m\in\mathbb{N}}$ of seminorms generates the topology of $S_{\text{even}}(\mathbb{R})$.

Proof. The family of seminorms $\{\gamma_{n,m}\}_{n,m\in\mathbb{N}}$ where, for every $n,m\in\mathbb{N}$,

$$\gamma_{n,m}(\phi)=\sup_{x\in(0,\infty)}\left|\frac{d^n}{dx^n}((1+x^2)^m\phi(x))\right|,\quad \phi\in S_{\text{even}}(\mathbb{R}),$$

generates the topology of $S_{\text{even}}(\mathbb{R})$ as well. Indeed, it is clear that the topology defined by $\{\mu_{n,m}\}_{n,m\in\mathbb{N}}$ is finer than the one generated by $\{\gamma_{n,m}\}_{n,m\in\mathbb{N}}$. Moreover, $\{\mu_{n,m}\}_{n,m\in\mathbb{N}}$ and $\{\gamma_{n,m}\}_{n,m\in\mathbb{N}}$ define Fréchet topologies on $S_{\text{even}}(\mathbb{R})$. Hence, the open mapping theorem implies that $\{\gamma_{n,m}\}_{n,m\in\mathbb{N}}$ and $\{\mu_{n,m}\}_{n,m\in\mathbb{N}}$ generate the same topology of $S_{\text{even}}(\mathbb{R})$.

By proceeding as in the proof of [3, Proposition 4.24] we can obtain, for every $n,m\in\mathbb{N}$, and $\phi\in S_{\text{even}}(\mathbb{R})$,

$$\frac{d^n}{dy^n}(y^{2m}(\mathcal{F}\phi)(y))=\int_0^\infty \Delta^m\phi(x)\frac{d^n}{dy^n}(\psi_y(x))A(x)dx,\quad y\in(0,\infty).$$

Let $1\le q\le\infty$. According to [3, Lemma 3.4, (iv), and (3.5)], and by using Hölder's inequality, it follows

$$\sup_{y\in(0,\infty)}\left|\frac{d^n}{dy^n}\left(y^{2m}(\mathcal{F}\phi)(y)\right)\right|\le C\|(1+x^2)^l\Delta^m\phi(x)\|_{L^q((0,\infty),A(x)dx)},\quad \phi\in S_{\text{even}}(\mathbb{R}),$$

for certain $C>0$ and $l\in\mathbb{N}$, that are not depending on $\phi\in S_{\text{even}}(\mathbb{R})$.

Then, from [3, Theorem 4.27] one infers that the topology defined by $\{\gamma^q_{n,m}\}_{n,m\in\mathbb{N}}$ is finer than the topology associated to $\{\mu_{n,m}\}_{n,m\in\mathbb{N}}$.

On the other hand, by [3, Lemma 4.18 and (3.5)], for every $n,m\in\mathbb{N}$, we get

$$\begin{aligned}\|(1+x^2)^n\Delta^m\phi(x)\|_{L^q((0,\infty),A(x)dx)}&\le C\sup_{x\in(0,\infty)}(1+x^2)^l|\Delta^m\phi(x)|\\&\le C\sum_{j=0}^{2m}\sup_{x\in(0,\infty)}(1+x^2)^l\left|\frac{d^j}{dx^j}\phi(x)\right|,\quad \phi\in S_{\text{even}}(\mathbb{R}),\end{aligned}$$

for certain $C>0$ and $l\in\mathbb{N}$ independent of $\phi\in S_{\text{even}}(\mathbb{R})$. Hence $\{\mu_{n,m}\}_{n,m\in\mathbb{N}}$ defines a topology finer than the one induced by $\{\gamma^q_{n,m}\}_{n,m\in\mathbb{N}}$. □

Proposition 2.2. *Let $0<p<2$, $1\le q\le\infty$ and $\phi\in S_p((0,\infty),A)$. Then there exists the following limit*

$$\sigma_\phi=:\lim_{k\to\infty}\|(\Delta-\rho^2)^k\phi\|^{1/2k}_{L^q((0,\infty),A(x)dx)}.$$

Moreover, we have that

$$\sigma_\phi = \sup\{|y| : y \in \operatorname{supp}(\mathcal{F}\phi)\}, \quad \textit{when } \phi \neq 0,$$

and

$$\sigma_\phi = 0, \quad \textit{when } \phi = 0.$$

In particular, the support of $\mathcal{F}\phi$ *is contained in* $[-\sigma, \sigma]$ *if, and only if,* $\sigma_\phi \leq \sigma$.

Proof. Denote by

$$w_\phi = \sup\{|y| : y \in \operatorname{supp}(\mathcal{F}\phi)\}, \quad \text{when } \phi \neq 0,$$

and

$$w_\phi = 0, \quad \text{when } \phi = 0.$$

Suppose firstly that $w_\phi = 0$. Then $\phi = 0$ and $\sigma_\phi = 0$. Note that, by virtue of [3, Theorem 4.27], if $\rho > 0$, then $\mathcal{F}\phi$ is a holomorphic function in the strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < \rho(\frac{2}{p} - 1)\}$. Hence, if $\rho > 0$ and $w_\phi < \infty$ then $\phi = 0$ and $w_\phi = \sigma_\phi = 0$.

Assume now that $w_\phi > 0$. For every $k \in \mathbb{N}$, $\Delta^k \phi \in S_p((0,\infty), A)$ and by partial integration we obtain

$$\mathcal{F}((\Delta - \rho^2)^k \phi)(y) = y^{2k} \mathcal{F}(\phi)(y), \quad y \in (0,\infty). \tag{2.1}$$

We divide our proof in different cases.

(i) Let $q = 2$. From Plancherel's formula for the Chébli–Trimèche transformation ([12, Theorem II.4]) and (2.1), it follows that

$$\begin{aligned}\|(\Delta - \rho^2)^k \phi\|^2_{L^2((0,\infty), A(x)dx)} &= \|\mathcal{F}((\Delta - \rho^2)^k \phi)\|^2_{L^2\left((0,\infty), \frac{dy}{|c(y)|^2}\right)} \\ &= \int_0^\infty y^{4k} |\mathcal{F}(\phi)(y)|^2 \frac{dy}{|c(y)|^2}, \quad k \in \mathbb{N}.\end{aligned}$$

Assume that $w_\phi = +\infty$. Then, for every $k, N \in \mathbb{N}$, we find that

$$\begin{aligned}\int_0^\infty y^{4k} |\mathcal{F}(\phi)(y)|^2 \frac{dy}{|c(y)|^2} &\geq \int_N^\infty y^{4k} |\mathcal{F}(\phi)(y)|^2 \frac{dy}{|c(y)|^2} \\ &\geq N^{4k} \int_N^\infty |\mathcal{F}(\phi)(y)|^2 \frac{dy}{|c(y)|^2}.\end{aligned}$$

Hence,

$$\|(\Delta-\rho^2)^k\phi\|^{1/k}_{L^2((0,\infty),A(x)dx)} \geq N^2\left\{\int_N^\infty |\mathcal{F}(\phi)(y)|^2\frac{dy}{|c(y)|^2}\right\}^{1/2k}, \quad k, N\in\mathbb{N}.$$

Since $\int_N^\infty |\mathcal{F}(\phi)(y)|^2\frac{dy}{|c(y)|^2}>0$, for every $N\in\mathbb{N}$, we conclude that

$$\lim_{k\to\infty}\|(\Delta-\rho^2)^k\phi\|^{1/2k}_{L^2((0,\infty),A(x)dx)}=+\infty.$$

Suppose now that $w_\phi\in(0,\infty)$. Then $\rho=0$. Moreover, if $0<\varepsilon<w_\phi$, we have

$$\int_{w_\phi-\varepsilon}^{w_\phi}|\mathcal{F}(\phi)(y)|^2\frac{dy}{|c(y)|^2}>0.$$

Therefore, when $0<\varepsilon<w_\phi$, we are led to

$$\begin{aligned}\|\Delta^k\phi\|^2_{L^2((0,\infty),A(x)dx)} &= \int_0^\infty y^{4k}|\mathcal{F}(\phi)|^2\frac{dy}{|c(y)|^2}\\ &\geq \int_{w_\phi-\varepsilon}^{w_\phi} y^{4k}|\mathcal{F}(\phi)(y)|^2\frac{dy}{|c(y)|^2}\\ &\geq (w_\phi-\varepsilon)^{4k}\int_{w_\phi-\varepsilon}^{w_\phi}|\mathcal{F}(\phi)(y)|^2\frac{dy}{|c(y)|^2}.\end{aligned}$$

Then

$$\liminf_{k\to\infty}\|\Delta^k\phi\|^{1/2k}_{L^2((0,\infty),A(x)dx)}\geq w_\phi-\varepsilon,\quad 0<\varepsilon<w_\phi.$$

Thus we conclude that

$$\liminf_{k\to\infty}\|\Delta^k\phi\|^{1/2k}_{L^2((0,\infty),A(x)dx)}\geq w_\phi.$$

(ii) Let $2\leq q\leq\infty$. As it was mentioned in Section 1, the inverse of the generalized Fourier transform is given by ([3, p. 91])

$$f(x)=\int_0^\infty \psi_y(x)\mathcal{F}(f)(y)\frac{dy}{|c(y)|^2},$$

when f is, for instance, in $S_p((0,\infty),A)$. We define the transformation $\mathcal{F}^{-1}$ by

$$\mathcal{F}^{-1}(f)(x)=\int_0^\infty\psi_y(x)f(y)\frac{dy}{|c(y)|^2},\quad f\in L^1\Big((0,\infty),\frac{dy}{|c(y)|^2}\Big).$$

Since $|\psi_y(x)| \leq 1$, $x, y \geq 0$ ([3, Lemma 3.4, (i)]), $\mathcal{F}^{-1}$ is a continuous mapping from $L^1((0,\infty), \frac{dy}{|c(y)|^2})$ into $L^\infty((0,\infty), A(x)dx)$. Moreover Plancherel's identity for the transformation $\mathcal{F}$ ([12]) says that $\mathcal{F}^{-1}$ is an isometry from $L^2((0,\infty), \frac{dy}{|c(y)|^2})$ onto $L^2((0,\infty), A(x)dx)$. Hence, Riesz–Thorin interpolation theorem implies that $\mathcal{F}^{-1}$ can be extended to $L^r((0,\infty), \frac{dy}{|c(y)|^2})$ as a continuous mapping from $L^r((0,\infty), \frac{dy}{|c(y)|^2})$ into $L^{r'}((0,\infty), A(x)dx)$, for each $1 \leq r \leq 2$. Here, r' represents the conjugated of r, that is, the equality $\frac{1}{r} + \frac{1}{r'} = 1$ holds.

Our next objective will be to prove

$$\limsup_{k\to\infty} \|(\Delta - \rho^2)^k \phi\|^{1/2k}_{L^q((0,\infty),A(x)dx)} \leq w_\phi.$$

Note that this inequality is clear when $w_\phi = +\infty$. So we can assume that $w_\phi \in (0,\infty)$. Hence $\rho = 0$.

Let $k \in \mathbb{N}$. Since $\phi \in S_p((0,\infty), A)$, $\Delta^k \phi$ is also in $S_p((0,\infty), A)$, and consequently $y^{2k}\mathcal{F}(\phi)(y) = \mathcal{F}(\Delta^k\phi)(y)$, $y \in (0,\infty)$, is in $L^1((0,\infty), \frac{dy}{|c(y)|^2})$. Then, it is inferred that

$$\begin{aligned} \|\Delta^k \phi\|_{L^q((0,\infty),A(x)dx)} &= \|\mathcal{F}^{-1}(y^{2k}\mathcal{F}(\phi)(y))\|_{L^q((0,\infty),A(x)dx)} \\ &\leq C\|y^{2k}\mathcal{F}(\phi)(y)\|_{L^{q'}((0,\infty),\frac{dy}{|c(y)|^2})} \\ &\leq C w_\phi^{2k} \|\mathcal{F}(\phi)(y)\|_{L^{q'}((0,\infty),\frac{dy}{|c(y)|^2})}. \end{aligned}$$

Therefore, since $\|\mathcal{F}(\phi)\|_{L^{q'}((0,\infty),\frac{dy}{|c(y)|^2})} \in (0,\infty)$,

$$\limsup_{k\to\infty} \|\Delta^k \phi\|^{1/2k}_{L^q((0,\infty),A(x)dx)} \leq w_\phi.$$

(iii) Let $1 \leq q < 2$. Suppose firstly that $\rho = 0$ and $w_\phi \in (0,\infty)$. Note that, according to [3, (3.5)], there exist $C > 0$ and $m \in \mathbb{N}$ for which

$$0 \leq A(x) \leq C(1 + x^2)^m, \quad x \geq 0.$$

In this case the space $S_p((0,\infty), A) = S_{\text{even}}(\mathbb{R})$. Moreover, $S_{\text{even}}(\mathbb{R}) = \mathcal{F}(S_{\text{even}}(\mathbb{R}))$. Hence, according to Lemma 2.1, for every $k \in \mathbb{N}$, there exist $C > 0$ and $\beta \in \mathbb{N}$ for which

$$\|\Delta^k \phi\|_{L^q((0,\infty),A(x)dx)} \leq C \max_{0\leq s\leq\beta} \|(1 + x^2)^\beta \Delta^s(x^{2k}\mathcal{F}(\phi))\|_{L^2((0,\infty),A(x)dx)}.$$

According to the hypotheses imposed to the function A we have that

$$\frac{A'(x)}{A(x)} = \frac{2\alpha+1}{x} + B(x), \quad x \in (0,\infty),$$

where $\alpha > -\frac{1}{2}$ and B is in $C^\infty(0,\infty)$ and $\frac{d^s}{dx^s}B$ is bounded on $(0,\infty)$, for every $s \in \mathbb{N}$. Hence the operator Δ can be written

$$\Delta = -\frac{d^2}{dx^2} - \frac{2\alpha+1}{x}\frac{d}{dx} - B(x)\frac{d}{dx}.$$

A straighforward manipulation allows us to obtain

$$\max_{0\le s\le\beta} \|(1+x^2)^\beta \Delta^s(x^{2k}\mathcal{F}(\phi))\|_{L^2((0,\infty),A(x)dx)} \le C\boldsymbol{p}(k)(1+w_\phi^{2k}),$$

where C is a positive constant that is not depending on k and $\boldsymbol{p}$ is a polynomial.

Therefore we conclude that

$$\limsup_{k\to\infty} \|\Delta^k\phi\|^{1/2k}_{L^q((0,\infty),A(x)dx)} \le w_\phi. \tag{2.2}$$

On the other hand, if $\rho \ge 0$ and $w_\phi = \infty$, then (2.2) is clear.

(iv) Let $1 \le q \le \infty$. For every $k \in \mathbb{N}$, partial integration and Hölder's inequality lead to

$$\begin{aligned}\int_0^\infty |(\Delta-\rho^2)^k\phi(x)|^2 A(x)dx &= \int_0^\infty (\Delta-\rho^2)^k\phi(x)\overline{(\Delta-\rho^2)^k\phi(x)}A(x)dx\\ &= \int_0^\infty \overline{\phi(x)}(\Delta-\rho^2)^{2k}(\phi(x))A(x)dx\\ &\le \|\phi\|_{L^{q'}((0,\infty),A(x)dx)}\|(\Delta-\rho^2)^{2k}\phi(x)\|_{L^q((0,\infty),A(x)dx)}.\end{aligned}$$

Hence, by (i) and (ii) (case $q=2$),

$$\begin{aligned} w_\phi &= \lim_{k\to\infty}\|(\Delta-\rho^2)^k\phi(x)\|^{1/2k}_{L^2((0,\infty),A(x)dx)}\\ &\le \liminf_{k\to\infty}\|(\Delta-\rho^2)^{2k}\phi(x)\|^{1/4k}_{L^q((0,\infty),A(x)dx)}.\end{aligned} \tag{2.3}$$

Also, for every $k \in \mathbb{N}$, we have

$$\begin{aligned}&\|(\Delta-\rho^2)^{k+1}\phi(x)\|^2_{L^2((0,\infty),A(x)dx)}\\ &\le \|(\Delta-\rho^2)\phi(x)\|_{L^{q'}((0,\infty),A(x)dx)}\|(\Delta-\rho^2)^{2k+1}\phi(x)\|_{L^q((0,\infty),A(x)dx)}.\end{aligned}$$

Note that $(\Delta-\rho^2)\phi \neq 0$. Indeed, if $(\Delta-\rho^2)\phi = 0$ then $y^2\mathcal{F}(\phi)(y)=0$, $y\in(0,\infty)$. This implies that $\mathcal{F}(\phi)=0$ and, therefore, $\phi=0$.

Hence, we can write

$$\begin{aligned} w_\phi &= \lim_{k\to\infty} \|(\Delta-\rho^2)^{k+1}\phi(x)\|_{L^2((0,\infty),A(x)dx)}^{1/2(k+1)} \\ &= \lim_{k\to\infty} \|(\Delta-\rho^2)^{k+1}\phi(x)\|_{L^2((0,\infty),A(x)dx)}^{1/2k+1} \\ &\leq \liminf_{k\to\infty} \|(\Delta-\rho^2)^{2k+1}\phi(x)\|_{L^q((0,\infty),A(x)dx)}^{1/2(2k+1)}. \end{aligned} \tag{2.4}$$

From (2.3) and (2.4) it follows

$$w_\phi \leq \liminf_{k\to\infty} \|(\Delta-\rho^2)^{k}\phi(x)\|_{L^q((0,\infty),A(x)dx)}^{1/2k}.$$

(v) Finally, by combining the above results we conclude always that

$$w_\phi = \lim_{k\to\infty} \|(\Delta-\rho^2)^{k}\phi(x)\|_{L^q((0,\infty),A(x)dx)}^{1/2k}.$$

Thus the proof is finished. □

By using the relation of the generalized Fourier transform $\mathcal{F}$ with the classical Euclidean Fourier transform on $\mathbb{R}$ we can establish the following result that can be seen as a version for the Chébli–Trimèche transform of [13, Theorem 3] (see also [1, Theorem 1]).

Proposition 2.3. *Assume that $\phi=\mathcal{F}(\Phi)$, where $\Phi\in\mathcal{D}(\mathbb{R})=\bigcup_{a>0}\mathcal{D}_a(\mathbb{R})$. Then, for every $1\leq q\leq\infty$, we have*

$$\lim_{k\to\infty}\left\|\frac{d^k}{dx^k}\phi\right\|_{L^q((0,\infty),dx)}^{1/k} = \sigma_\phi,$$

where $\sigma_\phi=\sup\{y\in(0,\infty) : y\in \operatorname{supp}\Phi)\}$, when $\phi\neq 0$, and $\sigma_\phi=0$, when $\phi=0$.

Proof. According to [11, Proposition 7.1, (2)], [12, (III,3)] and [3, Lemma 4.11] we can write

$$\mathcal{F}\Phi = \mathcal{F}_0(\mathbb{A}\Phi),$$

where $\mathcal{F}_0$ is the classical Fourier transform on $\mathbb{R}$ and $\mathbb{A}$ represents the Abel transformation defined, for every $f \in \mathcal{D}(\mathbb{R})$, by

$$\mathbb{A}(f)(x) = \int_x^\infty f(y)K(y,x)A(y)dy, \quad x \in (0,\infty).$$

Here, for each $y \in (0,\infty)$, $K(y,\,.\,)$ is a nonnegative even continuous function that is supported in $[-y,y]$, and such that the following representation for the function ψ_y

$$\psi_y(x) = \int_0^x K(x,t)\cos(yt)dt, \quad x \in (0,\infty) \text{ and } y \in \mathbb{C},$$

holds ([12, Théorème 4.1], [12, (I.2)] and [3, p. 92]).

By [12, Theorem III.1, (iii)] and [3, Lemma 4.10], $\Phi \in \mathcal{D}_a(\mathbb{R})$, with $a > 0$, if and only if $\mathbb{A}\Phi \in \mathcal{D}_a(\mathbb{R})$.

Hence, according to [1, Theorem 1], we find that

$$\sigma_\phi = \lim_{k\to\infty} \left\| \frac{d^k}{dx^k}\phi \right\|_{L^q((0,\infty),dx)}^{1/k}. \qquad \square$$

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