Publ. Math. Debrecen
60 / 3-4 (2002), 359–371

Common fixed point theorems for multi-valued non-self mappings

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Abstract. In this paper we prove three common fixed point theorems for a pair of multi-valued non-self mappings in metrically convex metric spaces. Our results generalize and extend the main theorem of Assad and Kirk [2] and the theorems of Assad [1], Itoh [4] and Khan [5].

1. Introduction

Markin [6] and Nadler [7] initiated the study of fixed point theorems for multi-valued mappings. There are many fixed point theorems for multi-valued mappings of a closed subset K of a complete metric space (X, d) into a class of subsets of K. However, in many applications the contractive mappings occur in convex setting and involved mapping is not a self-mapping of K. Recently some authors ([1], [2], [4], [5], [8]) gave sufficient conditions for some multi-valued mappings from K into a class of closed bounded subset of X to have a fixed point in K. In this paper we prove two main common fixed point theorems for a pair of multi-valued non-self mappings. We use a more effective method of a proof in both theorems and obtain theorems which generalize the main theorem of Assad and Kirk [2] and the theorems of Assad [1], Itoh [4] and Khan [5].

Mathematics Subject Classification: 54C60, 47H10, 54H25.
Key words and phrases: multi-valued non-self mapping, metrically convex metric space.
The second author was supported by Korea Research Foundation Grant (KRF-2001-015-DP0025).

2. Main results

Let (X, d) be a metric space and let $CB(X)$ denote the family of all nonempty bounded and closed subsets of X. For $A, B \in CB(X)$, let $H(A, B)$ denote the distance of subsets A and B in the Hausdorff metric introduced by d on $CB(X)$, i.e.

$$H(A, B) = \max\big(\sup\{D(a, B) : a \in A\},\ \sup\{D(A, b) : b \in B\}\big),$$

where

$$D(x, A) = \inf\{d(x, a) : a \in A\}.$$

It is known that $CB(X)$ is a metric space with the distance function H.

In the theorem below we assume X is a complete metric space which is convex in the sense of Menger, that is, X has the property that for each x, y in X with $x \neq y$ there exists z in X, $x \neq z$, $y \neq z$, such that

$$d(x, z) + d(z, y) = d(x, y).$$

Further (see [2], [3]), if K is a closed subset of X and if $x \in K$ and $y \notin K$, then there exists a point z in ∂K, $\partial K =$ the boundary of K, such that

$$d(x, z) + d(z, y) = d(x, y).$$

We prove the following simple lemma which enables to make more effective the proof of the theorems related to multi-valued mappings on metric spaces.

Lemma 1. *If $A, B \in CB(X)$ and $a \in A$, then for any positive number $q < 1$ there exists $b = b(a)$ in B such that*

$$qd(a, b) \leq H(A, B). \tag{2.1}$$

Proof. If $H(A, B) = 0$, then $A = B$ and (2.1) trivially holds for $b(a) = a$.

Suppose now that $H(A, B) > 0$. By definition of $D(a, B)$ and $H(A, B)$, for any positive number ε there exists $b \in B$ such that

$$d(a, b) \leq D(a, B) + \varepsilon \leq H(A, B) + \varepsilon. \tag{2.2}$$

Let $0 < q < 1$. Then $q^{-1} - 1 > 0$. Since $H(A, B) > 0$,

$$\varepsilon = (q^{-1} - 1)H(A, B) > 0.$$

By inserting this ε in (2.2), we get (2.1). □

Now, we prove the following:

Theorem 2.1. *Let (X, d) be a complete metrically convex metric space and K a nonempty closed subset of X. Let S, T be mappings of K into $CB(X)$ such that*

$$H(Sx, Ty) \leq \alpha d(x, y) + \beta \max\{D(x, Sx) + D(y, Ty), D(x, Ty) + D(y, Sx)\}, \tag{2.3}$$

where α, β are nonnegative real numbers satisfying

$$\lambda = \alpha + 3\beta + \alpha\beta < 1. \tag{2.4}$$

If $Sx \subseteq K$ and $T \subseteq K$ for each $x \in K$ then there exists an $u \in K$ such that $u \in Su$, $u \in Tu$ and $Su = Tu$.

Proof. We select two sequences $\{x_n\}$ and $\{y_n\}$ in K and X, respectively, in the following way:

Let x_0 in ∂K and $x_1 = y_1 \in Sx_0$ be arbitrary. Let a be any fixed number such that $0 < a < \frac{1}{2}$. Put $q = \lambda^a$.

Then from (2.4), $q < 1$. By Lemma 1.1, we can choose $y_2 \in Tx_1$ such that

$$qd(y_1, y_2) \leq H(Sx_0, Tx_1).$$

If $y_2 \in K$, put $x_2 = y_2$. If $y_2 \notin K$, then, as X is convex, we can choose $x_2 \in \partial K$ such that

$$d(x_1, x_2) + d(x_2, y_2) = d(x_1, y_2).$$

Let $y_3 \in Sx_2$ be such that

$$qd(y_2, y_3) \leq H(Tx_1, Sx_2).$$

By induction we may obtain sequences $\{x_n\}$ and $\{y_n\}$ such that for $n = 1, 2, \ldots$

(i) $y_n \in Sx_{n-1}$, if n is odd and
$y_n \in Tx_{n-1}$, if n is even;

(ii) $qd(y_n, y_{n+1}) \leq H(Sx_{n-1}, Tx_n)$, if n is odd and
$qd(y_n, y_{n+1}) \leq H(Tx_{n-1}, Sx_n)$, if n is even;

(iii) $y_{n+1} = x_{n+1}$, if $y_{n+1} \in K$ for all n, or

(iv) $d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) = d(x_n, y_{n+1})$ and $x_{n+1} \in \partial K$,
if $y_{n+1} \notin K$ for all n.

Define

$$P = \{x_i \in \{x_n\} : x_i = y_i\}, \quad Q = \{x_i \in \{x_n\} : x_i \neq y_i\}.$$

Observe that if $x_n \in Q$ for some n, then x_{n-1} and x_{n+1} belong to P, as two consecutive terms of $\{x_n\}$ cannot be in Q.

We wish to estimate $d(x_n, x_{n+1})$. Three cases need to be considered.

Case 1. $x_n \in P$ and $x_{n+1} \in P$. If n is odd, then from (ii) and (2.3) we have

$$\begin{aligned} qd(x_n, x_{n+1}) &= qd(y_n, y_{n+1}) \leq H(Sx_{n-1}, Tx_n) \leq \alpha d(x_{n-1}, x_n) \\ &+ \beta \max\{D(x_{n-1}, Sx_{n-1}) + D(x_n, Tx_n), D(x_{n-1}, Tx_n) + D(x_n, Sx_{n-1})\} \\ &\leq \alpha d(x_{n-1}, x_n) + \beta \max\{d(x_{n-1}, x_n) + d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1})\}. \end{aligned}$$

Hence, using the triangle inequality for $d(x_{n-1}, x_{n+1})$,

$$qd(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n) + \beta\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big). \tag{2.5}$$

From (2.5) we get

$$d(x_n, x_{n+1}) \leq \left(\frac{\alpha + \beta}{q - \beta}\right) d(x_{n-1}, x_n). \tag{2.6}$$

We obtain a similar inequality for n even.

Case 2. $x_n \in P$ and $x_{n+1} \in Q$. Then by (iv)

$$d(x_n, x_{n+1}) \leq d(x_n, y_{n+1}) = d(y_n, y_{n+1}).$$

By the same method, as in Case 1, we have for odd and even n

$$d(x_n, x_{n+1}) \leq \left(\frac{\alpha+\beta}{q-\beta}\right) d(x_{n-1}, x_n). \tag{2.7}$$

Case 3. $x_n \in Q$ and $x_{n+1} \in P$. By the triangle inequality we have

$$d(x_n, x_{n+1}) \leq d(x_n, y_n) + d(y_n, x_{n+1}) = d(x_n, y_n) + d(y_n, y_{n+1}).$$

Let n be odd. Then from (ii) and (2.3) we have

$$\begin{aligned} qd(x_n, x_{n+1}) &\leq qd(x_n, y_n) + qd(y_n, y_{n+1}) \leq qd(x_n, y_n) + H(Sx_{n-1}, Tx_n) \\ &\leq qd(x_n, y_n) + \alpha d(x_{n-1}, x_n) + \beta \max\{D(x_{n-1}, Sx_{n-1}) \\ &\quad + D(x_n, Tx_n), D(x_{n-1}, Tx_n) + D(x_n, Sx_{n-1})\} \\ &\leq qd(x_n, y_n) + \alpha d(x_{n-1}, x_n) + \beta \max\{d(x_{n-1}, y_n) \\ &\quad + d(x_n, y_{n+1}), d(x_{n-1}, x_{n+1}) + d(x_n, y_n)\}. \end{aligned} \tag{2.8}$$

Since two consecutive terms of $\{x_n\}$ cannot be in Q, $x_{n-1} \in P$. Then by (iv), $d(x_n, y_n) + d(x_{n-1}, x_n) = d(x_{n-1}, y_n)$ and hence, as $\alpha \leq \lambda < \lambda^a = q$, we have

$$qd(x_n, y_n) + \alpha d(x_{n-1}, x_n) \leq qd(x_{n-1}, y_n).$$

Also, by the triangle inequality,

$$\begin{aligned} d(x_{n-1}, x_{n+1}) + d(x_n, y_n) &\leq d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_n, y_n) \\ &= d(x_{n-1}, y_n) + d(x_n, x_{n+1}). \end{aligned}$$

Thus from (2.8) we get

$$qd(x_n, x_{n+1}) \leq qd(x_{n-1}, y_n) + \beta d(x_{n-1}, y_n) + \beta d(x_n, x_{n+1})$$

and hence

$$d(x_n, x_{n+1}) \leq \left(\frac{q+\beta}{q-\beta}\right) d(x_{n-1}, y_n). \tag{2.9}$$

We obtain a similar inequality for an even n.

Since by the above observation $x_{n-1} \in P$, we have $d(x_{n-1}, y_n) = d(y_{n-1}, y_n)$ and so by the same method as in Case 1 we get

$$d(x_{n-1}, y_n) \leq \left(\frac{\alpha+\beta}{q-\beta}\right) d(x_{n-2}, x_{n-1}). \tag{2.10}$$

By (2.9) and (2.10) we obtain

$$d(x_n, x_{n+1}) \leq \left(\frac{\alpha+\beta}{q-\beta}\right)\left(\frac{q+\beta}{q-\beta}\right) d(x_{n-2}, x_{n-1}). \tag{2.11}$$

Since $q = \lambda^a < 1$, we have

$$\begin{aligned} h &= \left(\frac{\alpha+\beta}{q-\beta}\right)\left(\frac{q+\beta}{q-\beta}\right) \\ &= 1 + \frac{(\alpha+\beta)(q+\beta) + 2q\beta - \beta^2 - q^2}{q^2 - 2q\beta + \beta^2} \\ &\leq 1 + \frac{(\alpha+\beta)(1+\beta) + 2\beta - \beta^2 - q^2}{q^2 - 2q\beta + \beta^2} \\ &= 1 - \frac{\lambda^{2a} - (\alpha + 3\beta + \alpha\beta)}{(\lambda^a - \beta)^2} = 1 - \frac{\lambda^{2a} - \lambda}{(\lambda^a - \beta)^2}. \end{aligned} \tag{2.12}$$

Since $\lambda^{2a} > \lambda$, we conclude that $h < 1$.

By (2.6), (2.7) and (2.11), we conclude that in all cases

$$d(x_n, x_{n+1}) \leq h \max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\} \tag{2.13}$$

for all $n \geq 2$, where h is given by (2.12).

Now it is easily shown by induction that from (2.13) we have

$$d(x_n, x_{n+1}) \leq h^{(n-1)/2} \max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}.$$

For $m > n > N$,

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=N}^{\infty} d(x_i, x_{i+1}) \\ &\leq \left(\frac{h^{N/2}}{h^{1/2} - h}\right) \max\left\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\right\}. \end{aligned}$$

Hence we conclude that $\{x_n\}$ is a Cauchy sequence, hence convergent. Call the limit u. From the way in which the $\{x_n\}$ were chosen, there exists an infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \in P$. Then for an odd $n_k = m$, we have

$$\begin{aligned}
&D(x_{n_k}, Tu) \le H(Sx_{m-1}, Tu) \\
&\le \alpha d(x_{m-1}, u) \\
&\quad + \beta \max\big\{D(x_{m-1}, Sx_{m-1}) + D(u, Tu), D(x_{m-1}, Tu) + D(u, Sx_{m-1})\big\} \\
&\le \alpha d(x_{m-1}, u) \\
&\quad + \beta \max\big\{d(x_{m-1}, x_m) + D(u, Tu), D(x_{m-1}, Tu) + d(u, x_m)\big\}.
\end{aligned}$$

Taking the limit as $k \to \infty$ yields

$$D(u, Tu) \le \beta D(u, Tu),$$

which implies, as $\beta < 1$, that $D(u, Tu) = 0$. Since Tu is closed, $u \in Tu$. Similarly, we can show that $u \in Su$. Thus u is a common fixed point of S and T.

From (2.3)

$$\begin{aligned}
H(Su, Tu) &\le \alpha d(u, u) \\
+\beta \max\big\{D(u, Su) + D(u, Tu), &D(u, Tu) + D(u, Su)\big\} = 0,
\end{aligned}$$

which implies $Su = Tu$. □

From Theorem 2.1 we have the main result of Khan [5] as corollary.

Corollary 2.1 ([5, Theorem 3.1])**.** *Let (X, d) be a complete metrically convex metric space and K a nonempty closed subset of X. Let S, T be mappings of K into $CB(X)$ such that*

$$\begin{aligned}
H(Sx, Ty) \le \alpha d(x, y) &+ \beta\big(D(x, Sx) + D(y, Ty)\big) \\
&+ \gamma\big(D(x, Ty) + D(y, Sx)\big),
\end{aligned} \tag{2.14}$$

where α, β, γ are nonnegative real numbers satisfying

$$\frac{(\alpha + \beta + \gamma)(1 + \beta + \gamma)}{(1 - \beta - \gamma)^2} < 1. \tag{2.15}$$

If $S(x) \subseteq K$ and $T(x) \subseteq K$ for any $x \in \partial K$, then there exists $u \in K$ such that $u \in Su$ and $u \in Tu$.

Proof. It is clear that (2.14) implies

$$H(Sx, Ty) \leq \alpha d(x, y)$$
$$+(\beta + \gamma) \max\{D(x, Sx) + D(y, Ty), D(x, Ty) + D(y, Sx)\}.$$

Since

$$\frac{(\alpha + \beta + \gamma)(1 + \beta + \gamma)}{(1 - \beta - \gamma)^2}$$
$$= \frac{\alpha + 3(\beta + \gamma) + \alpha(\beta + \gamma) - 2(\beta + \gamma) + (\beta + \gamma)^2}{1 - 2(\beta + \gamma) + (\beta + \gamma)^2},$$

we see that (2.15) implies (2.4):

$$\lambda = \alpha + 3(\beta + \gamma) + \alpha(\beta + \gamma) < 1.$$

Thus all assumptions of Theorem 2.1 are fulfiled, where now $(\beta + \gamma)$ has the same significance as β in Theorem 2.1.

As a slightly generalization of Theorem 2.1 we have the following result.

Theorem 2.2. *Let (X, d) be a complete metrically convex metric space, K a nonempty closed subset of X and let $F = \{T_j\}_{j \in J}$ be a family of multi-valued mappings of K into $CB(X)$. Suppose that there exists some $T_i \in F$ such that for each $T_j \in F$*

$$H(T_i x, T_j y) \leq \alpha_j d(x, y)$$
$$+\beta_j \max\{D(x, T_i x) + D(y, T_j y), D(x, T_j y) + D(y, T_i x)\}, \tag{2.16}$$

where α_j, β_j are nonnegative real numbers satisfying

$$\lambda_j = \alpha_j + 3\beta_j + \alpha_j \beta_j < 1.$$

If $T_j(\partial K) \subseteq K$ for each $T_j \in F$ then a family F has a common fixed point in K, i.e. there exists some $u \in K$ such that $u \in T_j u$ for all $T_j \in F$.

Proof. Let T_{j_0} be an arbitrary but fixed member of F. Then from Theorem 2.1 with $S = T_i$ and $T = T_{j_0}$, there exists a point in K, say u, which is a common fixed point of T_i and T_{j_0}. Let $T_j \in F$, $T_j \neq T_{j_0}$, be arbitrary. Then from (2.16) we have

$$D(u, T_j u) \leq H(T_i u, T_j u) \leq \beta_j \max\{0 + D(u, T_j u), D(u, T_j u) + 0\},$$

and hence

$$(1 - \beta_j) D(u, T_j u) \leq 0.$$

Since $\beta_j \leq \lambda_j < 1$, we have $D(u, T_j u) = 0$. Hence $u \in T_j u$, which completes the proof. □

Now we shall give a common fixed point theorem for a pair of continuous multi-valued mappings weaking the condition (2.3), not requiring that the constant λ be less than 1. We need the following.

Definition 1. Let K be a nonempty subset of a metric space (X, d). A mapping $T : K \to CB(X)$ is said to be *continuous at* $x_0 \in K$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $H(Tx, Tx_0) < \varepsilon$, whenever $d(x, x_0) < \delta$. If T is continuous at each point of K, then T is said to be *continuous on* K.

Theorem 2.3. *Let (X, d) be a complete and metrically convex metric space, K a nonempty compact subset of X. Let S, T be continuous mappings of K into $CB(X)$ such that for all $x, y \in K$ with $x \neq y$,*

$$\begin{aligned} H(Sx, Ty) &< \alpha d(x, y) \\ &+ \beta \max\{D(x, Sx) + D(y, Ty), D(x, Ty) + D(y, Sx)\}, \end{aligned} \tag{2.17}$$

where $\alpha, \beta \geq 0$ and such that

$$\alpha + 3\beta + \alpha\beta \leq 1. \tag{2.18}$$

If $Sx \subset K$ and $Tx \subset K$ for each $x \in \partial K$, then there exists an $u \in K$ such that $u \in Su$, $u \in Tu$ and $Su = Tu$.

PROOF. Let $f(x) = D(x, Sx)$ for each $x \in K$. Since for each $x, y \in K$

$$D(x, Sx) \leq d(x, y) + D(y, Sx); \quad D(y, Sx) \leq D(y, Sy) + H(Sy, Sx),$$

we have

$$\begin{aligned} |f(x) - f(y) &\leq |D(x, Sx) - D(y, Sx)| + |D(y, Sx) - D(y, Sy)| \\ &\leq d(x, y) + H(Sx, Sy). \end{aligned}$$

Hence, as S is continuous, $f(x)$ is continuous. Similarly, the function $g(x) = D(x, Tx)$ is continuous. Thus, the function

$$h(x) = \min\{D(x, Sx), D(x, Tx)\}$$

is continuous, and since K is compact, there exists a $z \in K$ such that $h(z) = \min\{h(x) : x \in K\}$. Without loss of generality we may suppose that $h(z) = D(z, Sz)$, i.e. that

$$D(z, Sz) \leq \min\{D(x, Sx), D(x, Tx)\}$$

for each $x \in K$. We shall show that $D(z, Sz) = 0$. Assume the contrary that $h(x) > 0$ for all $x \in K$. Let $\{x_n\}$ be a sequence in Sz such that

$$\lim_{n\to\infty} d(z, x_n) = D(z, Sz). \tag{2.19}$$

If there exists an infinite subsequence of $\{x_n\}$ which is contained in a compact subset K, then there exists a subsequence $\{x_{n_i}\}$ which converges to some x_0. Since Sz is closed, $x_0 \in Sz$. Thus $d(z, x_0) = D(z, Sz)$. From (2.17) we obtain

$$\begin{aligned} D(x_0, Tx_0) &\leq H(Sz, Tx_0) \\ &< \alpha d(z, x_0) + \beta \max\{D(z, Sz) + D(x_0, Tx_0), D(z, Tx_0)\} \end{aligned}$$

and hence, as $D(z, Tx_0) \leq d(z, x_0) + D(x_0, Tx_0) = D(z, Sz) + D(x_0, Tx_0)$, we have

$$D(x_0, Tx_0) < \alpha D(z, Sz) + \beta\big(D(z, Sz) + D(x_0, Tx_0)\big).$$

Now, using that $D(z, Sz) \leq D(x_0, Tx_0)$ and $\alpha + 2\beta \leq 1$, we have

$$D(x_0, Tx_0) < (\alpha + 2\beta) D(x_0, Tx_0) \leq D(x_0, Tx_0),$$

a contradiction.

Suppose now that $x_n \notin K$ for all sufficiently large n. Since X is convex and $z \in K$, for each such x_n there exists $y_n \in \partial K$ such that

$$d(z, y_n) + d(y_n, x_n) = d(z, x_n). \tag{2.20}$$

Since ∂X is compact, we may suppose, for the sake of convenience, that $\{y_n\}$ converges to some $y_0 \in \partial K$. Since g is continuous,

$$\lim_{n\to\infty} D(y_n, Ty_n) = D(y_0, Ty_0). \tag{2.21}$$

By the triangle inequality, (2.20) and (2.17) we have

$$\begin{aligned}
D(y_n, Ty_n) &\le d(y_n, x_n) + D(x_n, Ty_n) \\
&\le d(z, x_n) - d(z, y_n) + H(Sz, Ty_n) < d(z, x_n) - d(z, y_n) \\
&\quad + \alpha d(z, y_n) + \beta \max\{D(z, Sz) + D(y_n, Ty_n), D(z, Ty_n) + D(y_n, Sz)\} \\
&\le d(z, x_n) \\
&\quad + \beta \max\{D(z, Sz) + D(y_n, Ty_n), d(z, y_n) + D(y_n, Ty_n) + d(y_n, x_n)\} \\
&= d(z, x_n) + \beta \max\{D(z, Sz) + D(y_n, Ty_n), d(z, x_n) + D(y_n, Ty_n)\}.
\end{aligned}$$

Taking the limit when n tends to infinity and considering (2.19) and (2.21) we get

$$D(y_0, Ty_0) \le D(z, Sz) + \beta\big(D(z, Sz) + D(y_0, Ty_0)\big).$$

Hence

$$D(y_0, Ty_0) \le \frac{1+\beta}{1-\beta} D(z, Sz). \tag{2.22}$$

Since $y_0 \in \partial K$, $Ty_0 \subset K$. Thus Ty_0 is compact and so there exists $u \in Ty_0$ such that $d(y_0, u) = D(y_0, Ty_0)$.

From (2.17),

$$\begin{aligned}
D(u, Su) &\le H(Ty_0, Su) \\
&< \alpha d(u, y_0) + \beta \max\{D(u, Su) + D(y_0, Ty_0), D(y_0, Su)\}.
\end{aligned}$$

Since $D(y_0, Su) \le d(y_0, u) + D(u, Su) = D(y_0, Ty_0) + D(u, Su)$, we have

$$D(u, Su) < \alpha D(y_0, Ty_0) + \beta\big(D(u, Su) + D(y_0, Ty_0)\big)$$

and hence

$$D(u, Su) < \frac{\alpha+\beta}{1-\beta} D(y_0, Ty_0).$$

So by (2.22) we have

$$D(u, Su) < \frac{\alpha+\beta}{1-\beta} \cdot \frac{1+\beta}{1-\beta} D(z, Sz). \tag{2.23}$$

Since

$$\frac{(\alpha+\beta)(1+\beta)}{(1-\beta)^2} = \frac{\alpha+3\beta+\alpha\beta-2\beta+\beta^2}{1-2\beta+\beta^2},$$

by (2.18) we get $(\alpha+\beta)(1+\beta)/(1-\beta)^2 \le 1$. Thus by (2.23) we have

$$D(u, Su) < D(z, Sz),$$

a contradiction with definition of $D(z, Sz)$. Therefore, $D(z, Sz) = 0$. Hence, as Sz is closed, $z \in Sz$. Further, if we suppose that $z \notin Tz$, then by (2.17) we have

$$D(z, Tz) \le H(Sz, Tz) < \beta D(z, Tz) \le D(z, Tz),$$

a contradiction. Therefore, z is a common fixed point of S and T. This completes the proof. □

Remark. Theorem 2.3 is a generalization of Theorem 3.4 of Kahn [5] and a generalization and an extension of Theorem 2 of Itoh [4] and Theorem of Assad [1]. The presented method of proof gives a simplification of the corresponding proofs given by Itoh and Khan.

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(Received April 17, 2001; revised June 15, 2001)