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Three theorems connected with δ-quasi monotone sequences and their application to an integrability theorem

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Abstract. Three theorems of R. P. Boas Jr. and one of Č. V. Stanojevič and V. B. Stanojevič are generalized. All of the theorems utilize the benifit of the δ-quasi monotone sequences, comparing them to the monotone or quasi-monotone sequences.

1. Introduction

In [1] R. P. Boas, Jr. defined the notions of δ-quasi-monotonic and δ-quasi-positive sequences and showed that they are as useful as the conventional ones for one kind of theorem about trigonometric series. Later several authors have used these definitions and the relevant theorems proved by Boas in [1] at different other topics.

In the present paper we are going to generalize three theorems of Boas [1] and to exhibit here only one application of our results, but we are convinced that our generalizations will be applicable at several subjects, e.g. at summability theory.

As an application we shall generalize an interesting integrability theorem Č. V. Stanojevič and V. B. Stanojevič [5] which itself is an extension of the well-known Sidon–Telyakovskii theorem [4], [6].

A sequence $\{a_n\}$ is called *δ-quasi-monotonic* if $a_n \to 0$, $a_n > 0$ ultimately, and $\Delta a_n \geq -\delta_n$. Here $\{\delta_n\}$ is a sequence of positive numbers whose properties will be selected appropriately in different contexts.

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A sequence $\{a_n\}$ is called δ-*quasi-positive* if it is the sequence of differences of a δ-quasi-monotonic sequence $\{A_n\}$.

K and K_i will denote positive constants, not necessarily the same at each occurrence. Furthermore, sums without limits are over $1 \le n < \infty$.

If $\beta_n \ge 0$ and a sequence $\{d_n\}$ are δ-quasi – monotonic or quasi – positive then the convergence of $\sum \beta_n d_n$ will be denoted by $\sum \beta_n d_n < \infty$.

Now we recall the theorems of Boas to be generalized here.

Theorem A. *If $\{a_n\}$ is δ-quasi-monotonic with $\sum n^\gamma \delta_n < \infty$ then $\sum n^{\gamma-1} a_n < \infty$ implies that $n^\gamma a_n \to 0$ $(\gamma \ne 0)$; if $\gamma = 0$, and $\sum \delta_n \log n < \infty$, the conclusion is that $a_n \log n \to 0$.*

Theorem B. *If $\{a_n\}$ is δ-quasi-monotonic with $\sum n^\gamma \delta_n < \infty$ $(\gamma \ge 0)$, and $\sum n^{\gamma-1} a_n < \infty$, then $\sum n^\gamma |\Delta a_n| < \infty$ $(\gamma > 0)$, $\sum |\Delta a_n| \log n < \infty$ $(\gamma = 0)$.*

Theorem C. *If $\{a_n\}$ is δ-quasi-positive with $\sum n^\gamma \delta_n < \infty$ $(\gamma \ge 0)$, and $\sum n^\gamma a_n < \infty$, then $\sum n^\gamma |a_n| < \infty$.*

Remark. It seems to me that the statement of Theorem B in the special case $\gamma = 0$ requires some additional conditions. Namely the sequence

$$a_n := \begin{cases} 2/n \log^2 n, & \text{if } n = 2k, \\ 1/n \log^2 n, & \text{if } n = 2k+1, \end{cases}$$

is δ-quasi-monotonic with $\sum \delta_n < \infty$ if

$$\delta_n := 4/n \log^2 n,$$

furthermore $\sum a_n/n$ clearly converges, but $\sum |\Delta a_n| \log n = \infty$.

Our Theorem 2 will show that a sufficient additional condition is $\sum \delta_n \log n < \infty$.

2. Theorems

Our theorems read as follows.

Theorem 1. *Let $\{\alpha_n\}$ be a positive sequence with the property*

$$|\Delta \alpha_n| = O\left(\frac{\alpha_n}{n}\right). \tag{2.1}$$

If $\{a_n\}$ *is* δ*-quasi-monotonic with*

$$\sum n\alpha_n\delta_n < \infty, \tag{2.2}$$

then

$$\sum \alpha_n a_n < \infty \tag{2.3}$$

implies that

$$n\alpha_n a_n \to 0. \tag{2.4}$$

If we assume that $\alpha_n = \frac{\gamma_n}{n}$, *where* $\{\gamma_n\}$ *is an increasing sequence of positive numbers satisfying the condition*

$$\gamma_n = O(n\rho_n|\Delta\gamma_n|) \tag{2.5}$$

with a certain increasing sequence $\{\rho_n\}$ *of positive numbers, for which*

$$\sum 1/n\rho_n = \infty, \tag{2.6}$$

then with the condition

$$\sum(\gamma_{n+1} - \gamma_n)a_n < \infty \tag{2.7}$$

in place of (2.3), *we also have the conclusion* (2.4).

Remark. By (2.1) we always have $\gamma_{n+1} - \gamma_n = O(\alpha_n)$, but at several case $\gamma_{n+1} - \gamma_n = o(\alpha_n)$ also holds, that is, (2.7) claims less than (2.3), in general.

It is easy to see that if $\alpha_n = n^{\gamma-1}$, $\gamma \neq 0$, then Theorem 1 reduces to the first part of Theorem A; and its second part with $\rho_n = \gamma_n = \log n$ includes the special case $\gamma = 0$ of Theorem A.

Theorem 2. *Let* $\{\lambda_n\}$ *be a positive sequence with the property*

$$|\Delta\lambda_n| = O\left(\frac{\lambda_n}{n}\right). \tag{2.8}$$

If $\{a_n\}$ is δ-quasi-monotonic with

$$\sum n\lambda_n\delta_n < \infty, \tag{2.9}$$

then

$$\sum \lambda_n a_n < \infty \tag{2.10}$$

implies that

$$\sum n\lambda_n|\Delta a_n| < \infty. \tag{2.11}$$

If $\{\lambda_n\}$ is decreasing and there exists a sequence $\{\rho_n\}$ satisfying (2.6) *for which*

$$\sum_{i=1}^{n} \lambda_i = O(\rho_n n\lambda_n), \tag{2.12}$$

also holds, then with

$$\sum \left(\sum_{i=1}^{n} \lambda_i\right) \delta_n < \infty \tag{2.13}$$

in place of (2.9),

$$\sum \left(\sum_{i=1}^{n} \lambda_i\right) |\Delta a_n| < \infty \tag{2.14}$$

also maintains.

In the case $\lambda_n = n^{\gamma-1}$, $\gamma > 0$, Theorem 2 withholds Theorem B, and if $\gamma = 0$ then its corrected version.

Theorem 3. *If $\{a_n\}$ is δ-quasi positive with* (2.2) *and*

$$\sum n\alpha_n a_n < \infty \tag{2.15}$$

then

$$\sum n\alpha_n |a_n| < \infty. \tag{2.16}$$

It is clear that Theorem 3 with $\alpha_n = n^{\gamma-1}$, $\gamma \geq 0$ reduces to Theorem C.

3. Proofs

Proof of Theorem 1. Our proof follows similar lines as that of Boas. Suppose that $0 < m < n$. If we add the inequalities

$$n\alpha_n \Delta a_{n-1} \geq -n\alpha_n \delta_{n-1}, \quad (n-1)\alpha_{n-1}\Delta a_{n-2} \geq -(n-1)\alpha_{n-1}\delta_{n-2},$$
$$\dots, (m+1)\alpha_{m+1}\Delta a_m \geq -(m+1)\alpha_{m+1}\delta_m,$$

we achieve

$$-n\alpha_n a_n + \sum_{k=m+1}^{n-1} a_k((k+1)\alpha_{k+1} - k\alpha_k) + (m+1)\alpha_{m+1}\delta_m$$
$$\geq -\sum_{k=m}^{n-1}(k+1)\alpha_{k+1}\delta_k.$$

By (2.1)

$$|(k+1)\alpha_{k+1} - k\alpha_k| = |k(\alpha_k - \alpha_{k+1}) - \alpha_{k+1}| \leq K\alpha_k,$$

this, (2.2) and (2.3) imply that

$$n\alpha_n a_n - m\alpha_m a_m \leq o(1) \quad (m, n \to \infty). \tag{3.1}$$

We cannot have $\liminf n\alpha_n a_n > 0$, since this leads to a contraction of (2.3). Next we show that

$$\limsup n\alpha_n a_n = 0 \tag{3.2}$$

also holds. Namely (2.3) implies that there exists for each positive ε an infinite sequence of indices m for which

$$m\alpha_m a_m < \varepsilon. \tag{3.3}$$

Now suppose that $\limsup n\alpha_n a_n > 0$. Then there exists an infinite sequence of indices n such that

$$n\alpha_n a_n > 3\varepsilon. \tag{3.4}$$

For each m satisfying (3.3) take a large n, $n > m$, satisfying (3.4) we get a contradiction of (3.1).

This proves (3.2), and herewith (2.4) as well.

To prove the special case $\alpha_n = \gamma_n/n$ we have only to observe that then

$$(k+1)\alpha_{k+1} - k\alpha_k = \gamma_{k+1} - \gamma_k,$$

and then (2.2) and (2.7) (instead of (2.3)) imply (3.1). Hence the proof follows the same lines as above with (2.7) in place of (2.3), naturally we have to utilize the hypotheses (2.5) and (2.6) assumed on $\{\rho_n\}$ and $\{\gamma_n\}$.

The proof is complete. $\square$

PROOF of Theorem 2. By partial summation we have

$$\sum_{k=1}^{n} \lambda_k a_k = \sum_{k=1}^{n-1} k\Delta(\lambda_k a_k) + n\lambda_n a_n.$$

Theorem 1 with $\alpha_n = \lambda_n$ implies that $n\lambda_n a_n \to 0$, thus, by (2.10),

$$\sum \lambda_k a_k = \sum k\Delta(\lambda_k a_k) < \infty. \tag{3.5}$$

Since

$$k\Delta(\lambda_k a_k) = k\lambda_k \Delta a_k + k a_{k+1}\Delta\lambda_k, \tag{3.6}$$

furthermore, by (2.8) and (2.10),

$$\sum k a_{k+1}|\Delta\lambda_k| < \infty,$$

thus (3.5) and (3.6) imply that

$$\sum k\lambda_k \Delta a_k < \infty. \tag{3.7}$$

Finally, by (2.9) and (3.7), we get that

$$\begin{aligned} \sum k\lambda_k|\Delta a_k| &= \sum k\lambda_k|\Delta a_k + \delta_k - \delta_k| \\ &\le \sum k\lambda_k(\Delta a_k + \delta_k) + \sum k\lambda_k\delta_k \\ &\le \sum k\lambda_k\Delta a_k + 2\sum k\lambda_k\delta_k < \infty, \end{aligned} \tag{3.8}$$

and this proves (2.11).

To prove (2.14) we have only to observe that

$$\sum_{k=1}^{n} \lambda_k a_k = \sum_{k=1}^{n-1} (\Delta a_k) \sum_{i=1}^{k} \lambda_i + a_n \sum_{i=1}^{n} \lambda_i. \tag{3.9}$$

Now put $\gamma_1 = \lambda_1/2$, $\gamma_n = \sum_{i=1}^{n-1} \lambda_i$, $n \geq 2$ and $\alpha_n = \gamma_n/n$. Then, by (2.13) and (2.10), the conditions (2.2) and (2.7) of Theorem 1 hold, thus the result of Theorem 1 conveys that

$$\left(\sum_{i=1}^{n} \lambda_i \right) a_n = o(1). \tag{3.10}$$

Considering the hypotheses (2.8), the relations (3.9) and (3.10) imply that

$$\sum \gamma_n \Delta a_n < \infty.$$

With this in place of (3.7), the argument used in (3.8) presents the statement (2.14).

We have completed the proof. □

Proof of Theorem 3. Since $\{a_n\}$ is δ-quasi-positive, thus

$$A_n - A_{n+1} = a_n \geq -\delta_n$$

holds. Hence it clearly follows that

$$|a_n| = |a_n + \delta_n - \delta_n| \leq |a_n + \delta_n| + |\delta_n| = a_n + 2\delta_n.$$

Thus (2.2) and (2.15) plainly imply (2.16); and this completes the proof. □

We remark that our proof is essentially shorter than that of Theorem C.

4. Application

As we have written in the Introduction, as an application of our theorems we shall generalize an interesting integrability theorem of Č. V. Stanojevič and V. B. Stanojevič [5]. We observe again that their following theorem is an extension of the well-known SIDON–TELYAKOVSKII theorem [4], [6].

First we recall some definitions and notations.

A complex null sequence $\{c_n\}$ satisfying $\sum|\Delta(c_n - c_{-n})|\log n < \infty$ is called *weakly even.*

The partial sums of the complex trigonometric series $\sum_{n=-\infty}^{\infty} c_n e^{int}$ will be denoted by $S_n(c) = S_n(c,t) = \sum_{k=-n}^{n} c_k e^{ikt}$, $t \in T$. If a trigonometric series is the Fourier series of some $f \in L^1(T)$, then the notations $c_n = \hat{f}_n$ and $S_n(c,t) = S_n(f,t)$ will be used.

A weakly even null sequence $\{c_n\}$ of complex number belongs to the *class* S_p^* if for some $1 < p \le 2$ and some monotone sequence $\{A_n\}$ such that $\sum A_n < \infty$, the condition

$$\frac{1}{n}\sum_{k=1}^{n}\frac{|\Delta c_k|^p}{A_k^p} = O(1) \tag{4.1}$$

holds.

Now we can recall the theorem of STANOJEVIČ's [5].

Theorem D. *Let* $\{c_n\} \in S_p^*$. *Then*

(i) *for* $t \neq 0$, $\lim_{n\to\infty} S_n(c,t) = f(t)$ *exists;*

(ii) $f \in L^1(T)$*;*

(iii) $\|S_n(f) - f\| = o(1)$ *is equivalent to* $\hat{f}_n \log|n| = o(1)$.

Now we define a new class $S_p^*(\delta, \alpha_n)$ wider than S_p^*.

A weakly even null sequence $\{c_n\}$ of complex numbers belongs to the *class* $S_p^*(\delta, \alpha_n)$ if $1 < p \le 2$, $\{\alpha_n\}$ is a positive sequence fulfilling (2.1), furthermore there exists a δ-quasi-monotone sequence $\{A_n\}$ of positive numbers with (2.2), plus satisfying the following two conditions $\sum \alpha_n A_n < \infty$ and

$$\sum_{k=1}^{n}\frac{|\Delta c_k|^p}{A_k^p} \le K n \alpha_n^p. \tag{4.2}$$

It is easy to see that if for every n $\delta_n = 0$ and $\alpha_n = 1$ then $S_p^*(\delta, \alpha_n) \equiv S_p^*$.

Our result reads as follows.

Theorem 4. *Let* $\{c_n\} \in S_p^*(\delta, \alpha_n)$. *Then*

(i) *for* $t \neq 0$, $\lim_{n\to\infty} S_n(c,t) = f(t)$ *exists;*

(ii) $f \in L^1(T)$*;*

(iii) $\|S_n(f) - f\| = o(1)$ *is equivalent to* $\hat{f}_n \log|n| = o(1)$.

Proof of Theorem 4. Our proof follows similar lines as that of its forerunner in [5].

First we show that $\{c_n\}$ is of bounded variation whence (i) follows. An Abel rearrangement and Hölder inequality give that

$$\begin{aligned}\sum_{k=1}^{n}|\Delta c_k| &= \sum_{k=1}^{n-1}|\Delta A_k|\sum_{i=1}^{k}\frac{|\Delta c_i|}{A_i} + A_n\sum_{i=1}^{n}\frac{|\Delta c_i|}{A_i}\\ &\le \sum_{k=1}^{n-1}k|\Delta A_k|\left(\frac{1}{k}\sum_{i=1}^{k}\frac{|\Delta c_i|^p}{A_i^p}\right)^{1/p} + nA_n\left(\frac{1}{n}\sum_{i=1}^{n}\frac{|\Delta c_i|^p}{A_i^p}\right)^{1/p}\\ &\le \sum_{k=1}^{n-1}k\alpha_k|\Delta A_k|\left(\frac{1}{k\alpha_k^p}\sum_{i=1}^{k}\frac{|\Delta c_i|^p}{A_i^p}\right)^{1/p} + n\alpha_nA_n\left(\frac{1}{n\alpha_n^p}\sum_{i=1}^{n}\frac{|\Delta c_i|^p}{A_i^p}\right)^{1/p}.\end{aligned}$$

Hence, by (4.2),

$$\sum_{k=1}^{n}|\Delta c_k| \le K\left(\sum_{k=1}^{n-1}k\alpha_k|\Delta A_k| + n\alpha_nA_n\right). \tag{4.3}$$

By Theorem 1 with $a_n = A_n$ and Theorem 2 with $a_n = A_n$ and $\lambda_n = \alpha_n$ the right-hand side of (4.3) is uniformly bounded, thus

$$\sum|\Delta c_k| < \infty,$$

as we have stated.

In order to prove (ii) we use the so-called modified trigonometric sums introduced by J. W. Garrett and Č. V. Stanojevič [3]. Let

$$D_n(t) = \sin(n+1/2)t/\sin t/2 \quad \text{and} \quad E_n(f) = \sum_{k=0}^{n} e^{ikt}.$$

Then

$$S_n(c,t) - c_n E_n(t) + c_{-n} E_{-n}(t) = g_n(c,t)$$
$$= \sum_{k=1}^{n-1} (\Delta(c_{-k} - c_k))(E_{-k}(t) - 1) - c_{-n} + \sum_{k=0}^{n+1} (\Delta c_k) D_k(t).$$

From (i) it follows that for $t \neq 0$

$$f(t) - g_n(c,t) = \sum_{k=n}^{\infty} (\Delta c_k) D_k(t) + \sum_{k=n}^{\infty} (\Delta(c_{-k} - c_k)) E_{-k}(t).$$

Hence

$$\|f - g_n(c)\| \leq \int_T \Big|\sum_{k=n}^{\infty} (\Delta c_k) D_k(t)\Big| dt + K \sum_{k=n}^{\infty} |\Delta(c_{-k} - c_k)| \log k. \tag{4.4}$$

Since $\{c_n\}$ is weakly even it remains to show that the integral tends to zero as $n \to \infty$.

If $t \neq 0$ we use the identity

$$\sum_{k=n}^{\infty} (\Delta c_k) D_k(f) = \sum_{k=n-1}^{\infty} \Delta A_k \sum_{i=1}^{k} \frac{\Delta c_i}{A_i} D_i(t) - A_n \sum_{i=1}^{n-1} \frac{\Delta c_i}{A_i} D_i(t)$$

and get that

$$\begin{aligned} \int_0^\pi \Big|\sum_{k=n}^{\infty} (\Delta c_k) D_k(t)\Big| dt \leq \sum_{k=n-1}^{\infty} |\Delta A_k| \int_0^\pi \Big|\sum_{i=1}^{k} \frac{\Delta c_i}{A_i} D_i(t)\Big| dt \\ + A_n \int_0^\pi \Big|\sum_{i=1}^{n} \frac{\Delta c_i}{A_i} D_i(t)\Big| dt. \end{aligned} \tag{4.5}$$

In [5] it is proved that for any N

$$\int_0^\pi \left|\sum_{i=1}^{N} \frac{\Delta c_i}{A_i} D_i(t)\right| dt \leq K N^{1-\frac{1}{p}} \left(\sum_{i=1}^{N} \frac{|\Delta c_i|^p}{A_i^p}\right)^{1/p}. \tag{4.6}$$

Using (4.4), (4.5), (4.6) and finally (4.2), we have that

$$\begin{aligned}
\|f-g_n(c)\| &\le K\sum_{k=n-1}^{\infty}|\Delta A_k|k^{1-\frac{1}{p}}\left(\sum_{i=1}^{k}\frac{|\Delta c_i|^p}{A_i^p}\right)^{1/p}\\
&\quad + KA_n n^{1-\frac{1}{p}}\left(\sum_{i=1}^{n}\frac{|\Delta c_i|^p}{A_i^p}\right)^{1/p}+o(1)\\
&\le K\sum_{k=n-1}^{\infty}k\alpha_k|\Delta A_k|+Kn\alpha_n A_n+o(1).
\end{aligned} \tag{4.7}$$

Since $\{c_n\}\in S_p^*(\delta,\alpha_n)$ thus the assumptions of Theorems 1 and 2 are satisfied with $a_n=A_n$ and $\alpha_n=\lambda_n$, therefore the right-hand side of (4.7) tends to zero, that is,

$$\|f-g_n(c)\|=o(1) \tag{4.8}$$

holds, and since $g_n(c)$ is a polynomial, it follows that f is integrable. Herewith (ii) is also proved.

Finally the proof of (iii) follows from the inequality

$$\Big|\|f-S_n(f)\|-\|\hat{f}_nE_n+\hat{f}_{-n}E_{-n}\|\Big|\le\|f-g_n(c)\|=o(1)$$

and from the fact, proved in W. O. Bray and Č. V. Stanojevič [2], that

$$\|\hat{f}_nE_n+\hat{f}_{-n}E_{-n}\|=o(1)$$

holds if and only if

$$\hat{f}_n\log n=o(1).$$

Herewith the proof of Theorem 4 is complete. □

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