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Points of monotonicity in Musielak–Orlicz function spaces endowed with the Orlicz norm

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Abstract. Points of lower monotonicity, upper monotonicity, lower local uniform monotonicity and upper local uniform monotonicity in Musielak–Orlicz function spaces L_M^0 endowed with the Orlicz norm are characterized. Criteria for lower and upper local uniform monotonicities of L_M^0 are deduced.

1. Introduction

It is well known that various monotonicity properties are important in applications to the approximation theory and ergodic theory in Banach lattices (see [1]–[3], [11] and [13]). Roughly speaking, monotonicity properties of Banach lattices play similar role as rotundity properties of Banach spaces. Monotonicity properties are restrictions of respective rotundity properties to the set of the couples of comparable elements in the positive cone of a Banach lattice (see [10]). Such properties can also be used to prove monotonicity and rotundity properties of Calderón–Lozanovskiĭ spaces (see [5] and [10]). But sometime we only need to know whether a certain (fixed) point is a point of suitable monotonicity and we need not to know if the whole lattice is suitable monotone. Various monotonicity points in Banach lattices play similar rule as various rotundity points (extreme points, exposed points, strongly extreme points, denting points,

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H-points, points of local uniform rotundity, etc.) in Banach spaces. Therefore, we discuss in this paper criteria in order that a fixed element of the unit sphere of L^0_M be suitable monotone.

Let $(X, \| \ \|)$ be a Banach lattice with a partial order "$\leq$" and let $S(X)$ denote its unit sphere of X. Let X^+ denote the positive cone in X. A point $x \in S(X)$ is said to be upper monotone (or a point of upper monotonicity) if for any $y \in X^+ \setminus \{0\}$ there holds $\|x + y\| > 1$. We write then $x \in$ UM for short. A point $x \in S(X^+)$ is said to be lower monotone (or a point of lower monotonicity) if for any $y \in X^+ \setminus 0$ satisfying $y \leq x$ there holds $\|x - y\| < 1$. We write then $x \in$ LM for short. It is obvious that X is strictly monotone (STM for short) if and only if every point $x \in S(X^+)$ is a UM-point (equivalently, every point $x \in S(X^+)$ is an LM-point). However, as we will see below the notions of a UM-point and an LM-point are different.

A point $x \in S(X^+)$ is called upper locally uniformly monotone (ULUM for short) or a point of upper local uniform monotonicity if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $y \in X^+$ and $\|y\| \geq \varepsilon$, then $\|x + y\| \geq 1 + \delta$. A point $x \in S(X^+)$ is said to be lower locally uniformly monotone (LLUM for short) or a point of lower local uniform monotonicity if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for $y \in X^+$ with $y \leq x$ and $\|y\| \geq \varepsilon$ there holds $\|x - y\| \leq 1 - \delta$.

Obviously, X is ULUM (resp. LLUM) if and only if every point $x \in S(X^+)$ is ULUM (resp. LLUM). Strict monotonicity and uniform monotonicity were defined in [2]. Lower and upper local uniform monotonicity were defined in [11]. Although these notions were used already, they were not distinguished.

Let (G, Σ, μ) be a monotonic, complete and σ-finite measure space, $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+$ be a set of positive numbers from $\mathbb{R}$ and $\mathbb{N}$ be the set of natural numbers. A mapping $M : G \times \mathbb{R} \to [0, +\infty]$ is said to be a Musielak–Orlicz function if there is a set $F \in \Sigma$ with $\mu(F) = 0$ such that for any $t \in G \setminus F$, the function $M(t, \cdot)$ is convex, even, continuous at zero and left-hand side continuous on $\mathbb{R}_+$ (infinite left limits are not excluded here), and for any $u \in \mathbb{R}$ the function $M(\cdot, u)$ is Σ-measurable. We denote by $p(t, u)$ the right derivative of $M(t, \cdot)$ at u and by $N(t, v)$ the function complementary to $M(t, u)$ in the sense of Young. We define

$$e(t) = \sup\{u \geq 0 : M(t, u) = 0\}, \quad B(t) = \sup\{u > 0 : M(t, u) < \infty\},$$

$$\tilde{e}(t) = \sup\{v \geq 0 : N(t, v) = 0\}, \quad \tilde{B}(t) = \sup\{v > 0 : N(t, v) < \infty\}.$$

For a set $G_0 \subset G$ we write $M \in \Delta_2(G_0)$ if there exist $K > 0$ and a Σ-measurable function $\delta : G \to \mathbb{R}_+$ such that $\int_{G_0} \delta(t)d\mu < \infty$ and for μ-a.e. $t \in G_0$ and all $u \in \mathbb{R}$, we have

$$M(t, 2u) \leq KM(t, u) + \delta(t).$$

In place of $M \in \Delta_2(G)$ we write shortly $M \in \Delta_2$. We define on the space $L^0 = L^0(G, \Sigma, \mu)$ of all (equivalence classes of) Σ-measurable real functions on G the convex modular (see [1])

$$\varrho_M(x) = \int_G M(t, x(t))d\mu.$$

The Musielak–Orlicz space L_M generated by a Musielak–Orlicz function M is defined by

$$L_M = \{x \in L^0 : \varrho_M(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

Its subspace of order continuous elements is defined by

$$E_M = \{x \in L^0 : \varrho_M(\lambda x) < \infty \text{ for any } \lambda > 0\}.$$

Both spaces L_M and E_M are endowed in this paper with the Amemiya–Orlicz norm (we say Orlicz norm for short)

$$\|x\|^0 = \inf_{k>0} \frac{1}{k}(1 + \varrho_M(kx)).$$

Under this norm the spaces L_M and E_M are denoted by L^0_M and E^0_M, respectively. The Luxemburg norm is defined in L_M by the formula

$$\|x\| = \inf\{\lambda > 0 : \varrho_M(x/\lambda) \leq 1\}.$$

This norm is equivalent to the Orlicz norm, namely $\|x\| \leq \|x\|^0 \leq 2\|x\|$ for any $x \in L_M$. For the theory of Musielak–Orlicz spaces we refer to [6] and [18] and for the theory of Orlicz spaces to [6], [12] and [17]–[19].

For any $x \in L^0_M$ we define

$$\xi_M(x) = \inf\{c > 0 : \varrho_M(x/c) < \infty\},$$

$$\begin{aligned} G_x &= \operatorname{supp} x = \{t \in G : x(t) \neq 0\}, \\ K(x) &= [k^*_x, k^{**}_x] \quad \text{if} \quad k^{**}_x < \infty \text{ and} \\ K(x) &= [k^*_x, k^{**}_x) \quad \text{if} \quad k^{**}_x = \infty, \end{aligned}$$

where

$$k_x^* = \inf\{k > 0 : \varrho_N(p \circ k|x|) \geq 1\},$$

$$k_x^{**} = \sup\{k > 0 : \varrho_N(p \circ k|x|) \leq 1\},$$

$(p \circ k|x|)(t) = p(t, k|x(t)|)$ for $t \in G$.

It is well known (see [6] and [19]) that $\xi_M(x) = d(x, E_M^0)$. The situation that $k_x^* = \infty$ is possible. It appears for example if $M(t,u) = |u|$ for $t \in G$ and $u \in \mathbb{R}$. If $k_x^{**} < \infty$, then $k^{-1}(1 + \varrho_M(kx)) = \|x\|^0$ for any $k \in K(x)$. If $k_x^* < \infty$ but $k_x^{**} = \infty$, then $k^{-1}(1 + \varrho_M(kx)) = \|x\|^0$ for any $k \in [k_x^*, k_x^{**})$ and $\|x\|^0 = \lim_{k\to\infty} k^{-1}(1 + \varrho_M(kx))$ (see [6], [19] and [21]). Monotonicity properties of Orlicz spaces, Musielak–Orlicz spaces and Lorentz spaces were considered in [4], [8], [9], [11], [13]–[15] and [20].

2. Results

We start with the following

Proposition 1. *Let $x \in L_M^0 \setminus \{0\}$. Then:*

(i) *If $\int_{G_x} N(t, \tilde{B}(t))d\mu > 1$, then $K(x) \neq \emptyset$ and*

$$\|x\|^0 = \frac{1}{k}(1 + \varrho_M(kx)) \quad \text{if and only if } k \in K(x).$$

(ii) *If $\int_{G_x} N(t, \tilde{B}(t))d\mu \leq 1$, then*

$$\|x\|^0 = \int_G |x(t)|\tilde{B}(t)d\mu.$$

Proof. (i). Note that $\tilde{B}(t) = \lim_{n\to\infty}(M(t,u)/u) = \lim_{n\to\infty} p(t,u)$. Threfore, by the left continuity of $N(t,\cdot)$ for μ-a.e. $t \in G$ and by the Beppo–Levi theorem, we get

$$\int_{G_x} N(t, \tilde{B}(t))d\mu = \lim_{k\to\infty} \int_{G_x} N(t, p(t, k|x(t)|))\, d\mu.$$

Thus, the assumption from (i) implies that $\varrho_N(p \circ k|x|) > 1$ for some $k > 0$. Consequently, $k_x^{**} < \infty$ and so, by the facts presented at the end of the introduction, the thesis of (i) follows.

The proof of (ii) is the same as in the case of Orlicz spaces in [7], so we omit it here.

Theorem 1. *If $x \in S(L_M^0)$, $x \geq 0$ and $K(x) = \emptyset$, then x is both a* ULUM*-ponit and an* LLUM*-point.*

PROOF. On the basis of Proposition 1, we have $\int_{G_x} N(t, \tilde{B}(t)) d\mu \leq 1$ and $\|x\|^0 = \int_G |x(t)|\tilde{B}(t)\, d\mu$. For any $y \in (L_M^0)^+$ with $y \leq x$ and $\|y\|^0 \geq \varepsilon$, we have $\int_{G_{x-y}} N(t, \tilde{B}(t))\, d\mu \leq 1$. Therefore, $\|x-y\|^0 = \int_G (x(t) - y(t))\tilde{B}(t) d\mu$. Hence

$$\|x-y\|^0 = \int_G x(t)\tilde{B}(t) d\mu - \int_G y(t)\tilde{B}(t) d\mu = \|x\|^0 - \|y\|^0 \leq 1 - \varepsilon.$$

This means that x is an LLUM-point.

Assume for the contrary that x is not a ULUM-point. Then, there exists a sequence (x_n) in $(L_M^0)^+$ such that $\|x_n\|^0 \geq \varepsilon > 0$ and $\|x + x_n\|^0 \to 1$.

We consider few cases.

I. There exists an infinite number of n such that $K(x_n + x) = \emptyset$. In virtue of Proposition 1, we have

$$\begin{aligned}\|x_n + x\|^0 &= \int_G (x(t) + x_n(t))\tilde{B}(t) d\mu = \int_G x(t)\tilde{B}(t) d\mu + \int_G x_n(t)\tilde{B}(t) d\mu \\ &\geq \|x\|^0 + \|x_n\|^0 \geq 1 + \varepsilon,\end{aligned}$$

for infinitely many $n \in \mathbb{N}$, a contradiction.

II. There exists an infinite number of n such that $K(x_n + x) \neq \emptyset$. In this case we may assume without loss of generality that $K_n(x_n + x) \neq \emptyset$ for any $n \in \mathbb{N}$. Let $k_n \in K(x_n + x)$, $n = 1, 2, \dots$.

We consider two subcases.

II 1. $k_n \to k_0 < \infty$. Then

$$\|x_n + x\|^0 = \frac{1}{k_n}(1 + \varrho_M(k_n(x_n + x))) \geq \frac{1}{k_n}(1 + \varrho_M(k_n x)).$$

Therefore, by $K(x) = \emptyset$, $\lim_{n\to\infty} \|x_n + x\|^0 = 1$ and the Fatou lemma, we get

$$\begin{aligned}1 = \lim_{n\to\infty} \|x_n + x\|^0 &\geq \frac{1}{k_0}(1 + \varrho_M(k_0 x)) \\ &> \inf_{k>0} k^{-1}(1 + \varrho_M(kx)) = \|x\|^0 = 1,\end{aligned}$$

a contradiction.

II 2. $k_n \to \infty$. Then, by superadditivity of M on $\mathbb{R}_+$, we get

$$\begin{aligned} 1 &= \underline{\lim}_{n\to\infty}\|x_n + x\|^0 = \underline{\lim}_{n\to\infty} k_n^{-1}(1+\varrho_M(k_n(x_n+x))) \\ &\geq \underline{\lim}_{n\to\infty} k_n^{-1}(1+\varrho_M(k_n x_n)) + \varrho_M(k_n x)) \\ &= \underline{\lim}_{n\to\infty} [k_n^{-1}(1+\varrho_M(k_n x_n)) + k_n^{-1}(1+\varrho_M(k_n x))] \\ &\geq \underline{\lim}_{n\to\infty} (\|x_n\|^0 + \|x\|^0) \geq 1+\varepsilon, \end{aligned}$$

a contradiction. This finishes the proof. □

Theorm 2. *A point $x \in S((L_M^0)^+)$ is upper monotone if and only if $K(x) = \emptyset$ or*

$$kx(t) \geq e(t) \quad (\mu - \text{a.e.})$$

for any $k \in K(x)$ whenever $K(x) \neq \emptyset$.

Proof. Necessity. Otherwise, there is $k \in K(x)$ such that $A := \{t \in G : kx(t) < e(t)\}$ has positive measure. Let $y(t) = (\frac{e(t)}{k} - x(t))\chi_A$. Then $y > 0$ and

$$\begin{aligned} \|x+y\|^0 &\leq \frac{1}{k}(1+\varrho_M(1+(k(x+y)))) \\ &\leq \frac{1}{k}(1+\varrho_M(kx)) + \frac{1}{k}\int_A M(t, k(x(t)+y(t)))d\mu \\ &= \|x\|^0 + \frac{1}{k}\int_A M(t, e(t))d\mu \\ &= \|x\|^0 = 1, \end{aligned}$$

which means that x is not a UM-point.

Sufficiency. If $K(x) = \emptyset$, then x is a UM-point by Theorem 1. Assume that $K(x) \neq \emptyset$, $k \in K(x)$ and $y > 0$. Since $G_{x+y} \supset G_x$, we easily deduce that $K(x+y) \neq \emptyset$, i.e. there is $h > 0$ such that

$$\|x+y\|^0 = \frac{1}{h}(1+\varrho_M(h(x+y))).$$

If $h \notin K(x)$, then

$$\begin{aligned} \|x+y\|^0 &= \frac{1}{h}(1+\varrho_M(h(x+y))) \geq \frac{1}{h}(1+\varrho_M(hx)) \\ &> \frac{1}{k}(1+\varrho_M(kx)) = 1. \end{aligned}$$

If $h \in K(x)$, then by the assumption we get that $hx(t) \geq e(t)$ μ-a.e., whence

$$\begin{aligned}\|x+y\|^0 &= \frac{1}{h}(1+\int_G M(t,h(x(t)+y(t)))\,d\mu \\ &\geq \frac{1}{h}\left(1+\int_G M(t,hx(t))\,d\mu+\int_G M(t,e(t)+hy(t))\,d\mu\right) \\ &> \frac{1}{h}(1+\varrho_M(hx)) = \|x\|^0 = 1.\end{aligned}$$

□

Corollary 1. *The space* L^0_M *is* STM *if and only if* $\varrho_N(\tilde{B}) \leq 1$ *or* $e(t)=0$ *μ-a.e.*.

PROOF. Sufficiency. If $\varrho_N(\tilde{B}) \leq 1$, then by Proposition 1, $\|x\|^0 = \int_G |x(t)|\tilde{B}(t)d\mu$ for any $x \in L^0_M$. Therefore L^0_M is even uniformly monotone. Assume now that $e(t)=0$ μ-a.e.. Take $x \in S(L^0_M)$, $x>0$ and $y>0$. If $K(x)=\emptyset$, then x is UM-monotone by Theorem 1. If $K(x) \neq \emptyset$, then the condition $kx(t) \geq e(t)$ holds for μ-a.e. $t \in G$ and for any $k \in K(x)$. Therefore, by Theorem 2, x is a UM-point. We proved that, under the assumptions, any point $x \in S((L^0_M)^+)$ is a UM-point. Therefore L^0_M is STM.

Necessity. Assume that $\varrho_N(\tilde{B}) > 1$ and $e(t)>0$ for $t \in A$, where $A \in \Sigma$ and $\mu(A)>0$. Then there is $B \in \Sigma$ such that $G \setminus B \subset A$, $\mu(G\setminus B)>0$ and $\varrho_N(\tilde{B}\chi_B) > 1$.

Consequently, for any $x \in L^0_M$ with $\|x\|^0=1$ and $G_x = B$ there is $k>0$ such that $\varrho_N(p\circ kx) > 1$. This yields $K(x) \neq \emptyset$ for such x. Let $k \in K(x)$. Defining $y = x + \frac{e(t)}{k}\chi_{G\setminus B}$, we get

$$\|y\|^0 \leq \frac{1}{k}(1+\varrho_M(ky)) = \frac{1}{k}(1+\varrho_M(kx)) = 1.$$

Since $0<x<y$, this means that $L^0_M \notin$ STM. □

Remark. The proof of the necessity of Corollary 1 can be found in [8]. We presented it here for the sake of completeness only.

Theorem 3. *A point* $x \in S((L^0_M)^+)$ *with* $K(x) \neq \emptyset$ *is a lower monotone point if and only if whenever* $k \in K(x)$ *then*

(i) $\mu\{t \in G : 0 < kx(t) < e(t)\} = 0$,

(ii) *If* $A \subset \{t \in G : 0 < kx(t) = e(t)\}$ *and* $\mu(A) > 0$, *then there exists* $s \in (0,1)$ *such that* $\int_{G\setminus A} N(t, p(t, \frac{kx(t)}{1-s}))d\mu < 1$.

Proof. If (i) does not hold, then $\mu(\{t \in G : 0 < kx(t) < e(t)\}) > 0$. Hence, there is $\varepsilon_0 > 0$ such that the set

$$C := \{t \in G : 0 < (1+\varepsilon_0)kx(t) < e(t)\}$$

has positive measure. For any $\varepsilon \in (0, \varepsilon_0)$ there holds

$$\varrho_N((p \circ (1-\varepsilon)kx)\chi_{G\setminus C}) \le \varrho_N(p \circ (1-\varepsilon)kx) \le 1$$

and

$$\varrho_N((p \circ (1+\varepsilon)kx)\chi_{G\setminus C}) = \varrho_N(p \circ (1+\varepsilon)kx) \ge 1,$$

so $k \in K(x\chi_{G\setminus C})$, i.e.

$$\|x\chi_{G\setminus C}\|^0 = k^{-1}(1 + \varrho_M(kx\chi_{G\setminus C})) = k^{-1}(1 + \varrho_M(kx)).$$

This means that $\|x - x\chi_C\|^0 = 1$. Since $x\chi_C > 0$, this contradicts the fact that x is an LM-point.

If (ii) does not hold, there exist $A \subset \{t \in G : 0 < kx(t) = e(t)\}$ with $\mu(A) > 0$ such that $\varrho_N((p \circ (1+\varepsilon)kx)\chi_{G\setminus A}) \ge 1$ for any $\varepsilon > 0$. This yields that $k^*_{x\chi_{G\setminus A}} = k$, whence $k \in K(x\chi_G)$.

So,

$$\|x\chi_{G\setminus A}\|^0 = k^{-1}(1 + \varrho_M(kx\chi_{G\setminus A})) = k^{-1}(1 + \varrho_M(kx)) = \|x\|^0 = 1.$$

i.e. $\|x - x\chi_A\|^0 = 1$. Since $x\chi_A > 0$, this means that x is not an LM-point.

Sufficiency. Denote $A = \{t \in G_x : 0 < kx(t) = e(t)\}$. Then $G_x \setminus A = \{t \in G_x : kx(t) > e(t)\}$. Assume that $0 \le y \le x$ and $y \ne 0$. Since $G_y \subset G_x$ and $\mu(G_y) > 0$, we have $\mu(G_y \cap (G_x \setminus A)) > 0$ or $\mu(G_y \cap A) > 0$. We will consider two cases.

Case I. $\mu(G_y \cap (G_x \setminus A)) > 0$. For $t \in G_y \cap (G_x \setminus A)$ we have $M(t, k(x(t) - y(t))) < M(t, kx(t))$.

So,

$$\int_{G_y \cap (G_x \setminus A)} M(t, k(x(t) - y(t)))d\mu < \int_{G_y \cap (G_x \setminus A)} M(t, kx(t))d\mu,$$

whence

$$\|x-y\|^0 \le k^{-1}(1+\varrho_M(k(x-y))) < k^{-1}(1+\varrho_M(kx)) = \|x\|^0 = 1.$$

Case II. $\mu(G_y \cap A) > 0$. For $t \in G_y \cap A$ we have $kx(t) - ky(t) < kx(t) = e(t)$. There exists $\theta \in (0,1)$ such that the set

$$\Omega = \left\{ t \in G_y \cap A : \frac{k(x(t)-y(t))}{1-\theta} < e(t) \right\}$$

has positive measure. By condition (ii) there is $s \in (0,\theta]$ such that $\varrho_N(p \circ \frac{kx}{1-s}\chi_{G\setminus\Omega}) < 1$. Hence

$$\varrho_N\left(p\circ\frac{k}{1-s}(x-y)\right) \le \varrho_N\left(p\circ\frac{kx}{1-s}\chi_{G\setminus\Omega}\right) + \varrho_N\left(p\circ\frac{k(x-y)}{1-\theta}\chi_\Omega\right) < 1.$$

This means that $k^*_{x-y} \ge \frac{k}{1-s}$, whence $k \notin K(x-y)$. Consequently,

$$\begin{aligned}\|x-y\|^0 &= (k^*_{x-y})^{-1}(1+\varrho_M(k^*_{x-y}(x-y)) < k^{-1}(1+\varrho_M(k(x-y)))\\ &\le k^{-1}(1+\varrho_M(kx)) = \|x\|^0 = 1.\end{aligned}$$ □

Theorem 4. *If $x \in S((L^0_M)^+)$ and $K(x) \neq \emptyset$, then x is an* LLUM*-point if and only if whenever $k \in K(x)$ then*

(i) $\xi_M(x) = 0$,

(ii) $\mu(\{t \in G : 0 < ke(t) < e(t)\}) = 0$,

(iii) *For any $\varepsilon > 0$ there is $s \in (0,1)$ such that if $A \subset \{t \in G : 0 < kx(t) = e(t)\}$ and $\mu(A) \ge \varepsilon$, then $\int_{G\setminus A} N(t, p\circ\frac{kx}{1-s}(t))d\mu \le 1-s$.*

Proof. Necessity. If (i) does not hold, then $\xi_M(x) = \varepsilon > 0$. Choose $(G_n) \subset \Sigma$ such that $\mu(G_n) \to 0$ and $\xi_M(x\chi_{G_n}) = \xi_M(x)$ $(n = 1,2,\dots)$. Then $\|x\chi_{G_n}\|^0 \ge \varepsilon$ for any $n \in \mathbb{N}$ and

$$\|x - x\chi_{G_n}\|^0 = \|x\chi_{G\setminus G_n}\|^0 \to \|x\|^0 = 1$$

because L^0_M has the Fatou property. This means that x is not an LLUM-point.

The necessity of condition (ii) follows by Theorem 3. If condition (iii) is not true, then there exist $\varepsilon > 0$ and a sequence (A_n) of measurable subsets of the set $\{t \in G : 0 < kx(t) = e(t)\}$ such that $\mu(A_n) \geq \varepsilon$ and

$$\int_{G\setminus A_n} N\left(t, p \circ \frac{kx(t)}{1-\frac{1}{n}}\right) d\mu > 1 - \frac{1}{n} \quad (n = 1, 2, \dots).$$

We will concider two cases.

Case I. For an infinite number of $n \in \mathbb{N}$ (say for any $n \in \mathbb{N}$ without loss of generality) there holds

$$\int_{G\setminus A_n} N\left(t, p \circ \frac{kx(t)}{1-\frac{1}{n}}\right) d\mu > 1.$$

This yields that $k^*_{x\chi_{G\setminus A_n}} \leq \frac{k}{1-\frac{1}{n}}$ for those $n \in \mathbb{N}$. The inequality $k^*_{x\chi_{G\setminus A_n}} \geq k$ follows by the fact that $k \in K(x)$ and $\varrho_N(p \circ kx\chi_{A_n}) = 0$. So, $k^*_{x\chi_{G\setminus A_n}} \to k$ as $n \to \infty$. Denote $k_n = k^*_{x\chi_{G\setminus A_n}}$. Then

$$\|x\chi_{G\setminus A_n}\|^0 = k_n^{-1}\big(1 + \varrho_M(k_n x\chi_{G\setminus A_n})\big) \geq k_n^{-1}\big(1 + \varrho_M(k_n x\chi_{G\setminus A})\big),$$

where $A = \{t \in G : 0 < kx(t) = e(t)\}$. Hence, we get by the Fatou lemma

$$\lim_{n\to\infty} \|x\chi_{G\setminus A_n}\|^0 \geq k^{-1}\big(1 + \varrho_M(kx\chi_{G\setminus A})\big) = k^{-1}\big(1 + \varrho_M(kx)\big) = 1,$$

whence it follows that $\|x\chi_{G\setminus A_n}\|^0 \to 1$.

Notice that $e(t) > 0$ implies $M(t, 2e(t)) > 0$. There is $a > 0$ such that $\mu(\{t \in A : M(t, 2e(t)) < a\}) < \frac{\varepsilon}{2}$. Hence

$$\varrho_M(2kx\chi_{A_n}) = \int_{A_n} M(t, 2e(t))d\mu \geq \frac{\varepsilon}{2}a$$

and consequently $\|2kx\chi_{A_n}\|^0 \geq \min(\frac{\varepsilon}{2}a, 1)$ or equivalently $\|x\chi_{A_n}\|^0 \geq \frac{1}{k}\min(\frac{\varepsilon}{4}a, \frac{1}{2})$ $(n = 1, 2, \dots)$. Combining this with $\|x - x\chi_{A_n}\|^0 = \|x\chi_{G\setminus A_n}\|^0 \to 1$, we get that x is not an LLUM-point.

Case II. The inequality

$$\int_{G\setminus A_n} N\left(t, p\left(t, \frac{kx(t)}{1-\frac{1}{n}}\right)\right) d\mu \leq 1$$

holds for an infinite number of $n \in \mathbb{N}$ (we can assume that it holds for any $n \in \mathbb{N}$). Then

$$\begin{aligned}\|x\chi_{G\setminus A_n}\|^0 &\geq \int_G x\chi_{G\setminus A_n}(t)p\left(t, \frac{kx(t)}{1-\frac{1}{n}}\chi_{G\setminus A_n}\right)d\mu \\ &= \int_{G\setminus A_n} x(t)p\left(t, \frac{kx(t)}{1-\frac{1}{n}}\right)d\mu \\ &= \frac{1-\frac{1}{n}}{k}\left[\int_{G\setminus A_n} M\left(t, \frac{kx(t)}{1-\frac{1}{n}}\right) + N\left(t, p\left(t, \frac{kx(t)}{1-\frac{1}{n}}\right)\right)d\mu\right] \\ &\geq \frac{1-\frac{1}{n}}{k}\left(\int_{G\setminus A} M\left(t, \frac{kx(t)}{1-\frac{1}{n}}\right)d\mu + 1 - \frac{1}{n}\right) \\ &\to \frac{1}{k}\big(1+\varrho_M(kx\chi_{G\setminus A})\big) = \frac{1}{k}(1+\varrho_M(kx)) = \|x\|^0 = 1.\end{aligned}$$

Since $\|x\chi_{G\setminus A_n}\|^0 \to 1$, this yields that x is not an LLUM-point.

Sufficiency. If x is not an LLUM-point, then there is a sequence (x_n) in $(L^0_M)^+$ such that $0 \leq x_n \leq x$, $\|x_n\|^0 \geq \varepsilon > 0$ for $n = 1, 2, \ldots$. and $\|x - x_n\|^0 \to 1$ as $n \to \infty$.

It is easy to prove that $x_n \overset{\mu}{\nrightarrow} 0$ (otherwise we can deduce that $\|x_n\|^0 \to 0$). So, we can assume without loss of generality that there are $\sigma > 0$ and $\delta > 0$ such that $\mu(E_n) \geq \delta$ for any $n \in \mathbb{N}$, where $E_n = \{t \in G : kx_n(t) \geq \sigma\}$. Denote $A = \{t \in G_x : kx(t) \leq e(t)\}$. We will consider two cases.

Case I. $\mu((G_x \setminus A) \cap E_n) \geq \frac{\delta}{2}$ $(n = 1, 2, \ldots)$. If $t \in G_x \setminus A$, then $kx(t) > e(t)$. So, $M(t, kx(t) - \sigma) < M(t, kx(t))$. There is $a > 0$ such that

$$\mu\big(\{t \in G \setminus A : M(t, kx(t) - \sigma) > M(t, kx(t)) - a\}\big) < \frac{\delta}{4}.$$

Denote

$$B_n = \{t \in (G \setminus A) \cap E_n : M(t, kx(t) - \sigma) \leq M(t, kx(t) - a\}.$$

Then $\mu(B_n) \geq \frac{\delta}{4}$ for any $n \in \mathbb{N}$, whence

$$\begin{aligned}\|x - x_n\|^0 &\leq k^{-1}\big(1 + \varrho_M(k(x - x_n))\big) \\ &\leq k^{-1}(1 + \varrho_M(kx\chi_{G\setminus B_n}) + \varrho_M\big((kx - \sigma)\chi_{B_n}\big)\big) \\ &\leq k^{-1}\big(1 + \varrho_M(kx\chi_{G\setminus B_n}) + \varrho_M(kx\chi_{B_n}) - a\mu(B_n)\big)\end{aligned}$$

$$\leq k^{-1}\left(1+\varrho_M(kx)-\frac{a\delta}{4}\right)=1-\frac{a\delta}{4k},$$

which is a contradiction.

Case II. $\mu(A\cap E_n)\geq\frac{\delta}{2}$ for any $n\in\mathbb{N}$. There is $\theta\in(0,1)$ such that $\mu(\{t\in A: e(t)-\sigma\geq(1-\theta)e(t)\})<\frac{\delta}{4}$. Denoting

$$H_n=\{t\in A\cap E_n: e(t)-\sigma\leq(1-\theta)e(t)\},$$

we have $\mu(H_n)\geq\frac{\delta}{4}$. By condition (iii), there is $0<s\leq\theta$ satisfying

$$\int_{G\setminus H_n}N\left(t,p\left(t,\frac{kx(t)}{1-s}\right)\right)d\mu\leq 1-s\quad(n=1,2,\dots).$$

Hence

$$\begin{aligned}\varrho_N\left(p\left(\frac{k}{1-s}(x-x_n)\right)\right)&\leq\int_{G\setminus H_n}N\left(t,p\left(t,\frac{k}{1-s}x(t)\right)\right)d\mu\\&\quad+\int_{H_n}N\left(t,p\left(t,\frac{kx(t)-\sigma}{1-s}\right)\right)d\mu\\&\leq 1-s+\int_{H_n}N\left(t,p\left(t,\frac{1-\theta}{1-s}e(t)\right)\right)d\mu=1-s.\end{aligned}$$

This means that $k_n:=k^*_{x-x_n}\geq\frac{k}{1-s}$. So, we have

$$\begin{aligned}1-\|x-x_n\|^0&=k^{-1}(1+\varrho_M(kx))-k_n^{-1}(1+\varrho_M(k_n(x-x_n)))\\&\geq k^{-1}(1+\varrho_M(k(x-x_n)))-(1-s)k^{-1}(1+\varrho_M(k(1-s)^{-1}(x-x_n)))\\&\geq sk^{-1}\left\{1-\int_G\left(\frac{k}{1-s}(x(t)-x_n(t))p\left(t,\frac{k}{1-s}(x(t)-x_n(t))\right)\right.\right.\\&\qquad\left.\left.-M\left(t,\frac{k}{1-s}(x(t)-x_n(t))\right)\right)d\mu\right\}\\&=sk^{-1}\left\{1-\int_G\left(\frac{k}{1-s}(x(t)-x_n(t))p\left(t,\frac{k}{1-s}(x(t)-x_n(t))\right)\right.\right.\\&\qquad\left.\left.-M\left(t,\frac{k}{1-s}(x(t)-x_n(t))\right)\right)d\mu\right\}\\&=sk^{-1}\left\{1-\int_G N\left(t,\frac{k}{1-s}(x(t)-x_n(t))\right)d\mu\right\}\end{aligned}$$

$$\geq sk^{-1}\{1-(1-s)\} = s^2k^{-1},$$

whence $\|x-x_n\|^0 \leq 1-s^2k^{-1}$, which is a contradiction finisfing the proof. □

By Theorem 4 we can easily deduce the following result proved originally in [8].

Corollary 2. *The space L^0_M is* LLUM *if and only if $\varrho_N(\tilde{B}) \leq 1$ or $e(t)=0$ μ-a.e. in G and $M \in \Delta_2$.*

Theorem 5. *A point $x \in S((L^0_M)^+)$ with $K(x) \neq \emptyset$ is a* ULUM*-point if and only if whenever $k \in K(x)$, then*

(i) *$kx(t) \geq e(t)$ μ-a.e. in G,*

(ii) *If $G_0 \subset G$, $G_0 \in \Sigma$, $s \in (0,1)$ and $\varrho_M(\frac{kx}{1-s}\chi_{G_0}) < \infty$, then $M \in \Delta_2(G_0)$.*

Proof. Necessity. The necessity of (i) follows by Theorem 2. If (ii) is not true there are $G_0 \subset G$, $G \in \Sigma$, and $s \in (0,1)$ satisfying $\varrho_M(\frac{kx}{1-s}\chi_{G_0}) < \infty$, and $M \notin \Delta_2(G_0)$. There are $x_n \in L^0_M(G_0)$ such that $x_n = x\chi_{G_n}$, $\|x_n\|^0 \geq \varepsilon > 0$ for any $n \in \mathbb{N}$, $\mu(G_n) \to 0$ and $\varrho_M(x_n) \to 0$. Hence

$$\begin{aligned}\|x+sk^{-1}x_n\|^0 &\leq k^{-1}\big(1+\varrho_M(k(x+sk^{-1}x_n))\big)\\ &= k^{-1}\left(1+\varrho_M(kx\chi_{G\setminus G_n}) + \int_{G_n} M\Big(t,(1-s)\frac{kx(t)}{1-s}+sx_n(t)\Big)d\mu\right)\\ &\leq k^{-1}(1+\varrho_M(kx)+(1-s)\varrho_M\left(\frac{kx}{1-s}\chi_{G_n}\right)+s\varrho_M(x_n)) \to 1.\end{aligned}$$

But $\|sk^{-1}x_n\|^0 \geq sk^{-1}\varepsilon$ for any $n \in \mathbb{N}$, so x is not a ULUM-point. □

Before proving the sufficiency we prove the following

Lemma 1. *If $M \in \Delta_2$, then for any $\varepsilon > 0$ there is $\delta > 0$ such that $x \in (L_M)^+$ and $\|x\|^0 \geq \varepsilon$ imply $\varrho_M(e+x) \geq \delta$.*

Proof. Otherwise, there exists a sequence (x_n) in $(L^0_M)^+$ with $\|x_n\|^0 \geq \varepsilon$ for any $n \in \mathbb{N}$ such that $\varrho_M(e+x_n) \to 0$ as $n \to \infty$. From $\int_G M(t,e(t)+x_n(t))d\mu \to 0$ we deduce that $M(t,e(t)+x_n(t)) \xrightarrow{\mu} 0$. Hence it follows that $e(t)+x_n(t) \xrightarrow{\mu} e(t)$ and consequently $x_n(t) \xrightarrow{\mu} 0$.

Since $M \in \Delta_2$, for any $\lambda > 0$ there are $K > 0$ and $\delta_0 \in L_1$ satisfying the inequality

$$M(t, \lambda u) \leq KM(t, u) + \delta_0(t)$$

for all $u \in \mathbb{R}$ and μ-a.e. $t \in G$. There exists $A \in \Sigma$ such that $\mu(A) < \infty$ and $\int_{G \setminus A} \delta_0(t) d\mu < \frac{1}{6}$. Further, by the Jegorov theorem, we can choose $E \subset A$, $E \in \Sigma$ such that $\int_E \delta_0(t) d\mu < \frac{1}{6}$ and $x_n \to 0$ uniformly in $A \setminus E$. Then for n large enough (say $n \geq m$) we have

$$\varrho_M(x_n) \leq \varrho_M(e + x_n) < \frac{1}{3K} \quad \text{and} \quad \int_{A \setminus E} M(t, \lambda x_n(t)) d\mu < \frac{1}{3}.$$

Therefore we have for $n \geq m$,

$$\begin{aligned} \varrho_M(t, \lambda x_n) &= \int_G M(t, \lambda x_n(t)) d\mu \\ &= \int_{A \setminus E} M(t, \lambda x_n(t)) d\mu + \int_{G \setminus (A \setminus E)} M(t, \lambda x_n(t)) d\mu \\ &\leq \int_{A \setminus E} M(t, \lambda x_n(t)) d\mu + K \int_{G \setminus (A \setminus E)} (M(t, x_n(t)) + \delta_0(t)) d\mu \\ &< \frac{1}{3} + K \frac{1}{3K} + \int_{G \setminus (A \setminus E)} \delta_0(t) d\mu \\ &< \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1. \end{aligned}$$

Consequently, $\|x_n\|^0 \leq 2\|x_n\| \leq \frac{2}{\lambda}$. By the arbitrariness of $\lambda > 0$, this contradicts the condition $\|x_n\|^0 \geq \varepsilon > 0$ for any $n \in \mathbb{N}$. So, the lemma is proved.

Now we give a proof of the sufficiency of Theorem 5. Assume the assumptions are satisfied. If x is not a ULUM-point, there exists a sequence (x_n) in $(L_M^0)^+$ such that $\|x_n\|^0 \geq 4\varepsilon > 0$ for any $n \in \mathbb{N}$ and $\|x + x_n\|^0 \to 1$ as $n \to \infty$. We may assume that $k := k_x^* < \infty$ (otherwise, $k(x) = \emptyset$ and so, by Theorem 1, x is a ULUM-point). Take $k_n := k_{x_n+x}^*$ ($n \in \mathbb{N}$). It is easy to see that $k_n \leq k$ (since $x(t) + x_n(t) \geq x(t)$). We may assume without loss of generality (passing to a subsequence if necessary) that $k_n \to k_0$ as $n \to \infty$. We will consider four cases.

Case I. $k_n \to k_0 < k$. Then

$$\begin{aligned} 1 &= \lim_{n\to\infty} \|x + x_n\|^0 = \lim_{n\to\infty} k_n^{-1}(1 + \varrho_M(k_n(x + x_n))) \\ &\geq \lim_{n\to\infty} k_n^{-1}(1 + \varrho_M(k_n x)) \\ &\geq k_0^{-1}(1 + \varrho_M(k_0 x)) > k^{-1}(1 + \varrho_M(kx)) = \|x\|^0 = 1, \end{aligned}$$

which is a contradiction.

Therefore, we may assume in the remaining part of the proof that $k_n \to k$. Denote

$$G_\infty = \{t \in G : kx(t) = B(t)\}.$$

Take a sequence (G_n) of measurable sets in $G \setminus G_\infty$ such that $G_1 \subset G_2 \subset \dots$,

$$\varrho_M\left(k\left(1 - \frac{1}{n}\right)^{-1} x\chi_{G_n}\right) < \infty$$

and $\mu(G_n) \to \mu(G \setminus G_\infty)$. Denote

$$B_n = \{t \in G : x_n(t) \geq \varepsilon x(t)\}.$$

From

$$4\varepsilon \leq \|x_n\|^0 \leq \|x_n\chi_{G\setminus B_n}\|^0 + \|x_n\chi_{B_n}\|^0 < \varepsilon + \|x_n\chi_{B_n}\|^0,$$

we obtain that $\|x_n\chi_{B_n}\|^0 > 3\varepsilon$ for any $n \in \mathbb{N}$.

Case II. There is an infinite number of $n \in \mathbb{N}$ satisfying the inequality $\|x_n\chi_{B_n\cap G_\infty}\|^0 \geq \varepsilon$. Then clearly $\mu(B_n \cap G_\infty) > 0$ for an infinite sequence of n. We may assume without loss of generality that $\|x_n\chi_{B_n\cap G_\infty}\|^0 \geq \varepsilon$ for any $n \in \mathbb{N}$. Therefore

$$\begin{aligned} 1 &\leftarrow \|x + x_n\|^0 = k_n^{-1}(1 + \varrho_M(k_n(x + x_n))) \\ &\geq k^{-1}\int_{B_n\cap G_\infty} M(t, k_n(x(t) + x_n(t)))d\mu \\ &\geq k^{-1}\int_{B_n\cap G_\infty} M(t, k_n(1+\varepsilon)x(t))d\mu \\ &\geq k^{-1}\int_{B_n\cap G_\infty} M\left(t, \left(1 + \frac{\varepsilon}{2}\right)kx(t)\right)d\mu \\ &= k^{-1}\int_{B_n\cap G_\infty} M\left(t, \left(1 + \frac{\varepsilon}{2}\right)B(t)\right)d\mu = k^{-1}\cdot\infty\cdot\mu(B_n \cap G_\infty) = \infty, \end{aligned}$$

which is a contradiction.

So, in the further part of the proof we may assume (and we do this) that $\|x_n\chi_{B_n\cap G_\infty}\|^0 < \varepsilon$, i.e. $\|x_n\chi_{B_n\cap(G\setminus G_\infty)}\|^0 \geq 2\varepsilon$ for any $n \in \mathbb{N}$.

Case III. There is $m \in \mathbb{N}$ such that $\|x_n\chi_{B_n\cap G_m}\|^0 \geq \varepsilon$ for any $n \in \mathbb{N}$.

By condition (ii), $M \in \Delta_2(G_m)$. By Lemma 1, there is $\delta > 0$ such that $\varrho_M((e+kx_n)\chi_{B_n\cap G_m}) > \delta$ for any $n \in \mathbb{N}$. Noticing that $kx(t) \geq e(t)$, we have for n large enough,

$$\begin{aligned}
\|x+x_n\|^0 &= k_n^{-1}(1+\varrho_M(k_n(x+x_n))) \\
&\geq k_n^{-1}(1+\varrho_M(k_nx\chi_{G\setminus(B_n\cap G_m)}) + \varrho_M(k_n(x+x_n)\chi_{B_n\cap G_m})) \\
&\geq k_n^{-1}\left(1+\varrho_M(k_nx\chi_{G\setminus(B_n\cap G_m)}) + \varrho_M(k(x+x_n)\chi_{B_n\cap G_m}) - \frac{\delta}{2}\right) \\
&\geq k_n^{-1}\left(1+\varrho_M(k_nx\chi_{G\setminus(B_n\cap G_m)}) + \varrho_M(kx\chi_{B_n\cap G_m}) + \varrho_M((e+kx_n)\chi_{B_n\cap G_m}) - \frac{\delta}{2}\right) \\
&\geq k_n^{-1}\left(1+\varrho_M(k_nx) + \frac{\delta}{2}\right) \to k^{-1}(1+\varrho_M(kx)) + \delta(2k)^{-1} = 1+\delta(2k)^{-1},
\end{aligned}$$

which is a contradiction. So, we may assume (and we do this) in the further part of the proof that for any $m \in \mathbb{N}$, the inequality $\|x_n\chi_{B_n\cap G_m}\|^0 \geq \varepsilon$ holds for at most finite number of $n \in \mathbb{N}$. Therefore, we may assume without loss of generality that there holds the following

Case IV. $\|x_n\chi_{B_n\cap(G\setminus G_\infty\setminus G_n)}\|^0 \geq \varepsilon$ for any $n \in \mathbb{N}$. Denote $B_n \cap (G \setminus G_\infty \setminus G_n) = \Omega_n$ $(n = 1, 2, \dots)$. We consider two subcases.

IV.1.

$$\inf_n \varrho_M\left(\left(1+\frac{\varepsilon}{2}\right)kx\chi_{\Omega_n}\right) = ka > 0.$$

Take n_0 large enough satisfying

$$k^{-1}(1+\varrho_M(kx\chi_{G_\infty\cup G_{n_0}})) > 1-\frac{a}{4}.$$

For n large enough,

$$k_n^{-1}\big(1+\varrho_M(k_nx\chi_{G_\infty\cup G_{n_0}})\big) > 1-\frac{a}{4}-\frac{a}{4} = 1-\frac{a}{2}.$$

Hence, we get for n large enough,

$$\begin{aligned}\|x+x_n\|^0 &\geq k_n^{-1}\big(1+\varrho_M(k_n x\chi_{G_\infty\cup G_{n_0}})+\varrho_M(k_n(x+x_n)\chi_{G\setminus(G_\infty\cup G_{n_0})})\big)\\ &\geq 1-\frac{a}{2}+k_n^{-1}\varrho_M(k_n(x+x_n)\chi_{\Omega_n})\\ &\geq 1-\frac{a}{2}+\frac{1}{k}\varrho_M(k_n(1+\varepsilon)x\chi_{\Omega_n})\\ &\geq 1-\frac{a}{2}+k^{-1}\varrho_M\left(\left(1+\frac{\varepsilon}{2}\right)kx\chi_{\Omega_n}\right)\geq 1-\frac{a}{2}+a=1+\frac{a}{2},\end{aligned}$$

which is a contradiction.

Threfore we need only to consider the subcase

IV.2.

$$\inf_n \varrho_M\left(\left(1+\frac{\varepsilon}{2}\right)kx\chi_{\Omega_n}\right)=0.$$

We may assume (passing to a subsequence if necessary) that

$$\sum_{n=1}^{\infty}\varrho_M\left(\left(1+\frac{\varepsilon}{2}\right)kx\chi_{\Omega_n}\right)<\infty.$$

Let $\Omega=\bigcup_{n=1}^{\infty}\Omega_n$. Then $\varrho_M((1+\frac{\varepsilon}{2})kx\chi_\Omega)<\infty$. By condition (ii), $M\in\Delta_2(\Omega)$.

From $\|x_n\chi_{\Omega_n}\|^0\geq\varepsilon$ and $x_n\chi_{\Omega_n}\in L^0_M(\Omega)$ for any $n\in\mathbb{N}$ it follows that there is $\delta>0$ such that $\varrho_M((e+kx_n)\chi_{\Omega_n})\geq\delta$. Hence we get for n large enough,

$$\begin{aligned}\|x+x_n\|^0 &= k_n^{-1}(1+\varrho_M(k_n(x+x_n)\chi_{G\setminus\Omega_n})+\varrho_M(k_n(x+x_n)\chi_{\Omega_n}))\\ &\geq k_n^{-1}\left(1+\varrho_M(k_n x\chi_{G\setminus\Omega_n})+\varrho_M(k(x+x_n)\chi_{\Omega_n})-\frac{\delta}{2}\right)\\ &\geq k_n^{-1}\left(1+\varrho_M(k_n x\chi_{G\setminus\Omega_n})+\varrho_M(kx\chi_{\Omega_n})+\varrho_M((e+kx_n)\chi_{\Omega_n})-\frac{\delta}{2}\right)\\ &\geq k_n^{-1}\left(1+\varrho_M(k_n x)+\frac{\delta}{2}\right)\to 1+\delta(2k)^{-1},\end{aligned}$$

which is a contradiction. The proof is completed. □

From Theorem 5 we can obtain the following result from [8].

Corollary 3. *The space* L_M^0 *is* ULUM *if and only if* $\varrho_M(\tilde{B}) \leq 1$ *or* $e(t) = 0$ *for* μ*-a.e.* $t \in G$ *and* $M \in \Delta_2$.

Criteria for points of monotonicity of various kinds in E_M^0 we present below. Their proofs are quite similar to the respective proofs for Theorems 2, 3, 4 and 5. So, we omit them in this paper.

Theorem 6. *A point* $x \in S((E_M^0)^+)$ *with* $K(x) \neq \emptyset$ *is upper monotone if and only if whenever* $k \in K(x)$, *then* $kx(t) \geq e(t)$ *for* μ*-a.e.* $t \in G$.

Corollary 4. *The space* E_M^0 *is* STM *if and only if* $e(t) = 0$ *for* μ*-a.e.* $t \in G$ *or* $\varrho_M(\tilde{B}) \leq 1$.

Theorem 7. *A point* $x \in S((E_M^0)^+)$ *with* $K(x) \neq \emptyset$ *is lower monotone if and only if whenever* $k \in K(x)$, *then*

(i) $\mu(\{t \in G_x : kx(t) < e(t)\}) = 0$;

(ii) *If* $A \subset \{t \in G_x : kx(t) = e(t)\}$ *and* $\mu(A) > 0$, *then there is* $s \in (0,1)$ *such that* $\varrho_N((p \circ \frac{kx}{1-s})\chi_{G\setminus A}) < 1$.

Theorem 8. *A point* $x \in S((E_M^0)^+)$ *with* $K(x) \neq \emptyset$ *is lower locally uniformly monotone if and only if whenever* $k \in K(x)$, *then*

(i) $\mu(\{t \in G_x : kx(t) < e(t)\}) = 0$;

(ii) *For any* $\varepsilon > 0$ *there is* $s \in (0,1)$ *such that if* $A \subset \{t \in G_x : kx(t) = e(t)\}$ *and* $\mu(A) \geq \varepsilon$, *then* $\varrho_N(p \circ \frac{kx}{1-s}\chi_{G\setminus A}) \leq 1 - s$.

Corollary 5. *The space* E_M^0 *is* LLUM *if and only if* $\varrho_N(\tilde{B}) \leq 1$ *or* $e(t) = 0$ *for* μ*-a.e.* $t \in G$.

Theorem 9. *A point* $x \in S((E_M^0)^+)$ *with* $K(x) \neq \emptyset$ *is upper locally uniformly monotone if and only if whenever* $k \in K(x)$, *then*

(i) $kx(t) \geq e(t)$ *for* μ*-a.e.* $t \in G$ *and* (ii) $M \in \Delta_2$.

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