

## A conditional Cauchy equation on normed spaces

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*Dedicated to Professor Lajos Tamássy on his 70th birthday*

**Abstract.** Mappings, defined on normed vector spaces and additive on all pairs of vectors of equal norm, have been studied before only in certain special cases. Here we derive the additivity of such mappings on spaces of dimension  $\geq 3$  with the aid of an interesting connectivity theorem. As a consequence, the additivity of an odd, isosceles orthogonally additive mapping is proved.

### 1. Introduction

In a recent paper [2], C. ALSINA and J-L. GARCIA-ROIG considered the conditional functional equation

$$(1) \quad f(x + y) = f(x) + f(y), \quad \text{whenever} \quad \|x\| = \|y\|,$$

where the unknown function  $f : X \rightarrow Y$  is a continuous mapping from a real inner product space  $X$  of dimension  $\geq 2$  into a real topological vector space  $Y$ . As a main result, they derived the linearity of such a mapping  $f$ . Also they recognized the close connection between this equation and orthogonally additive mappings: a solution is necessarily odd and additive on orthogonal pairs of vectors. This was proved for  $Y = \mathbf{R}^n$  without using the continuity, but in fact the proof works also for an arbitrary Abelian group  $Y$ . Now the theory says (see e.g. RÄTZ [6]) that an odd, orthogonally additive mapping from an inner product space (real or complex) into an Abelian group is necessarily additive. When in addition it is continuous with values in a real topological vector space, then of course it is linear as well.

A much more interesting question arises when we drop the inner product and consider equation (1) on a real normed linear space  $X$ . Namely it was just J-L. GARCIA-ROIG [4], who investigated this rather difficult problem for the first. He was interested in the positively homogeneous and norm preserving solution of (1) from a two dimensional real normed vector space into itself (see also ALSINA [1]). Such a solution is always linear, i.e. a linear isometry. With the aid of a recent description [3] of the isometry group of a real normed plane, a complete solution was given.

Here we pay attention to equation (1) on higher dimensional normed linear spaces. As a main result we prove that any solution of equation (1) on a real normed vector space of dimension  $\geq 3$  with values in an Abelian group is necessarily additive. We can perform this using particular vectors of equal norm. The existence of these vectors can be guaranteed by certain continuous functions defined on the connected intersection of spheres of equal radii. Finally we apply the results to mappings additive with respect to the JAMES' [5] isosceles orthogonality.

## 2. Connectivity theorems

**Lemma 2.1.** *Let  $F \subset V \times W$  be a relation between two metric spaces  $V$  and  $W$ . Assume that  $F$  is defined on the whole  $V$  and*

- (i)  $V$  is connected;
- (ii)  $F(v)$  is connected for every  $v \in V$ ;
- (iii) for any  $v_0 \in V$  and for all sequences  $v_n \in V$  such that  $v_n \rightarrow v_0$ , there is a subsequence  $v_{n_k}$  and a convergent sequence of values  $w_k \in F(v_{n_k})$  with  $w_k \rightarrow w_0 \in F(v_0)$ .

Then  $F$  is connected.

PROOF. First observe that  $\{v\} \times F(v) \subset V \times W$  is connected for all  $v \in V$ . Now, on contrary, assume that  $F$  fails to be connected. Then there exist disjoint open subsets of  $V \times W$ ,  $D_1$  and  $D_2$  such that  $F \subset D_1 \cup D_2$  and  $F \cap D_1 \neq \emptyset \neq F \cap D_2$ . This means that for some  $v'_i \in V$  and  $w'_i \in F(v'_i)$  we have  $(v'_i, w'_i) \in D_i$  ( $i = 1, 2$ ). With respect to the above observation, it follows that for all  $v \in V$  either  $\{v\} \times F(v) \subset D_1$  or  $\{v\} \times F(v) \subset D_2$ . Now using the notation  $D'_i = \{v \in V \mid \{v\} \times F(v) \subset D_i\}$  ( $i = 1, 2$ ), we have  $V = D'_1 \cup D'_2$  and  $v'_i \in D'_i$  ( $i = 1, 2$ ).

We are going to derive a contradiction to the connectedness of  $V$  by showing the sets  $D'_1, D'_2$  to be open as well. For this reason take a point  $v_0 \in D'_1$  and any sequence  $v_n \in V$  converging to  $v_0$ . Now by hypothesis (iii), there is a subsequence  $v_{n_k}$  and a convergent sequence  $w_k \in F(v_{n_k})$  with  $w_k \rightarrow w_0 \in F(v_0)$ . Then  $(v_{n_k}, w_k)$  converges to the point  $(v_0, w_0) \in D_1$  and because of the openness of  $D_1$ ,  $(v_{n_k}, w_k) \in D_1$  for sufficiently large  $k$ 's. This means that  $v_{n_k} \in D'_1$  for these sufficiently large  $k$ 's, and because of the arbitrary choice of the sequence  $v_n$ ,  $v_0$  is an interior point of  $D'_1$ ,

i.e. it is an open set, and by the same argument, so is  $D'_2$ . This completes the proof.

**Corollary 2.2.** *Let  $V$  be a connected while  $W$  be a compact metric space and  $F \subset V \times W$  be a relation. If  $F$  is closed and  $F(v)$  is non-empty, connected set for all  $v \in V$ , then  $F$  is itself connected and so is  $F(V)$ .*

PROOF. We have only to show that the assumption (iii) of the previous lemma holds true. For this reason take any sequence  $v_n \in V$  with  $v_n \rightarrow v_0 \in V$ . Now choose an arbitrary sequence of values  $w_n \in F(v_n)$ . Since  $W$  is compact, the Bolzano-Weierstrass Theorem ensures the existence of a convergent selection  $w_{n_k} \rightarrow w_0 \in W$ . This means that  $F \ni (v_{n_k}, w_{n_k}) \rightarrow (v_0, w_0)$  and since  $F$  is a closed set,  $w_0 \in F(v_0)$ , what was to be proved.

Finally, observe that  $F(V)$  is a projective image of the connected set  $F$ .

In what follows we apply this general result to the intersection of spheres in normed spaces. Here and further on  $(X, \|\cdot\|)$  stands for a real normed linear space of dimension  $\geq 3$  and for a vector  $a \in X$  and scalar  $\rho > 0$  let  $S_a(\rho)$  denote the sphere  $\{x \in X \mid \|x - a\| = \rho\}$ . The special values  $a = 0$  or  $\rho = 1$  are omitted. Thus for  $S_0(1)$  simply write  $S$ . Now we are going to show that the intersection  $S \cap S_a$  is connected whenever  $0 < \|a\| \leq 2$ . For this reason let  $u = a/\|a\|$  and choosing a vector  $v \in S \setminus \text{lin}\{u\}$ , consider the closed halfplane  $P_v^+ = \{\alpha u + \beta v \mid \alpha, \beta \in \mathbf{R}, \beta \geq 0\}$  in the plane  $P_v$  spanned by  $u$  and  $v$ . Then we have

**Lemma 2.3.** *The set  $K_v = S \cap S_a \cap P_v^+$  is non-empty, closed and convex (in fact an interval) and so it is connected.*

PROOF.  $K_v$  is obviously closed. On the other hand, the mapping  $\varphi : P_v \rightarrow \mathbf{R}$ ,  $\varphi(x) = \|x - a\|$  is continuous on the connected set  $S \cap P_v^+$  containing  $-u, u$  and because of  $\varphi(-u) = 1 + \|a\| \geq 1 \geq |1 - \|a\|| = \varphi(u)$ , there exists a vector  $x \in S \cap P_v^+$  with  $\varphi(x) = 1$ , i.e.  $x \in K_v$ .

To prove the convexity of  $K_v$ , take vectors  $x, y \in K_v$ . Then all the vectors  $x' = x - a$ ,  $y' = y - a$ ,  $x'' = -x$ ,  $y'' = -y$ ,  $x''' = -x'$ ,  $y''' = -y'$  are in  $S$ . We have to show that the line segment  $[x, y]$  is contained in  $K_v$ . For  $x = y$  there is nothing to prove, while for  $b = y - x \neq 0$  the following cases have to be dealt with:

*Case I.*  $b = \beta a$ . Changing the role of  $x$  and  $y$  if it is necessary, we may assume that  $\beta > 0$ . Let  $z \in ]x, y[$ . Then  $z = \lambda x + (1 - \lambda)y$  for some  $0 < \lambda < 1$ . Thus  $\|z\| \leq \lambda\|x\| + (1 - \lambda)\|y\| = 1$ . On the other hand, a simple computation shows that  $x = \mu z + (1 - \mu)x'$  with  $0 < \mu = 1/[1 + \beta(1 - \lambda)] < 1$ . Hence  $1 = \|x\| \leq \mu\|z\| + (1 - \mu)\|x'\|$  which implies that  $\|z\| \geq 1$ , i.e.  $z \in S$ .

Similarly,  $z' = z - a = \lambda x' + (1 - \lambda)y'$  and so  $\|z'\| \leq \lambda\|x'\| + (1 - \lambda)\|y'\| = 1$ . Moreover, as it can easily be seen  $y' = \mu z' + (1 - \mu)y$  with

$0 < \mu = 1/[1 + \beta\lambda] < 1$ . Therefore  $1 = \|y'\| \leq \mu\|z'\| + (1 - \mu)\|y\|$  whence  $\|z'\| \geq 1$  holds true as well. This means that  $\|z - a\| = 1$ , i.e.  $z \in S_a$  and so  $z \in K_v$  proving the desired inclusion  $[x, y] \subset K_v$ .

*Case II.*  $a$  and  $b$  are linearly independent. Then  $x$  and  $y$  can be expressed as  $x = \alpha a + \beta b$  and  $y = \alpha a + (\beta + 1)b$ . Without loss of generality, we may assume that  $\beta \geq 0$ . We proceed on by showing that  $\alpha = 1/2$ . On contrary assume that

- (i)  $\alpha < 0$ : Then for  $0 < \mu = -\alpha/[\beta - \alpha] < 1$  and  $\sigma = 1 + 1/[\beta - \alpha]$  we have  $\sigma x = \mu x' + (1 - \mu)y$  which yields the contradiction

$$1 < \sigma = \|\sigma x\| \leq \|\mu x'\| + \|(1 - \mu)y\| = 1.$$

- (ii)  $\alpha > 1$ : Then for  $0 < \mu = [\alpha - 1]/[\alpha - 1 + \beta] < 1$  and  $\sigma = 1 + 1/[\alpha - 1 + \beta]$  we have  $\sigma x' = \mu x + (1 - \mu)y'$  which yields the contradiction

$$1 < \sigma = \|\sigma x'\| \leq \|\mu x\| + \|(1 - \mu)y'\| = 1.$$

- (iii)  $0 < \alpha < 1/2$ : Then for  $0 < \mu = \alpha/[\alpha + \beta] \leq 1$  and  $\sigma = 1 + [1 - 2\alpha]/[\alpha + \beta]$  we have  $\sigma x'' = \mu x' + (1 - \mu)y''$  which yields the contradiction

$$1 < \sigma = \|\sigma x''\| \leq \|\mu x'\| + \|(1 - \mu)y''\| = 1.$$

- (iv)  $1/2 < \alpha < 1$ : Then for  $0 < \mu = [1 - \alpha]/[1 - \alpha + \beta] \leq 1$  and  $\sigma = 1 + [2\alpha - 1]/[1 - \alpha + \beta]$  we have  $\sigma x' = \mu x'' + (1 - \mu)y'$  which yields the contradiction

$$1 < \sigma = \|\sigma x'\| \leq \|\mu x''\| + \|(1 - \mu)y'\| = 1.$$

- (v)  $\alpha = 0$  or  $\alpha = 1$  is impossible because of  $1 = \|x\| = \beta\|b\| < (\beta + 1)\|b\| = \|y\| = 1$  or  $1 = \|x'\| = \beta\|b\| < (\beta + 1)\|b\| = \|y'\| = 1$ , respectively.

Now let  $z \in ]x, y[$ . Then  $z = \lambda x + (1 - \lambda)y$  for some  $0 < \lambda < 1$ . Thus  $\|z\| \leq \lambda\|x\| + (1 - \lambda)\|y\| = 1$ . On the other hand, a simple computation shows that  $x = \mu z + (1 - \mu)y'''$  with  $0 < \mu = [2\beta + 1]/[2\beta + 2 - \lambda] < 1$ . Hence  $1 = \|x\| \leq \mu\|z\| + (1 - \mu)\|y'''\|$  which implies that  $\|z\| \geq 1$ , i.e.  $z \in S$ .

Similarly,  $z' = z - a = \lambda x' + (1 - \lambda)y'$  and so  $\|z'\| \leq \lambda\|x'\| + (1 - \lambda)\|y'\| = 1$ . Moreover, as it can easily be seen  $x' = \mu z' + (1 - \mu)y''$  with  $0 < \mu = [2\beta + 1]/[2\beta + 2 - \lambda] < 1$ . Therefore  $1 = \|x'\| \leq \mu\|z'\| + (1 - \mu)\|y''\|$  whence  $\|z'\| \geq 1$  holds true as well. This means that  $\|z - a\| = 1$ , i.e.  $z \in S_a$  and so  $z \in K_v$  proving the desired inclusion  $[x, y] \subset K_v$ .

**Theorem 2.4.** *In a real normed vector space of dimension  $\geq 3$  the intersection of spheres of equal radii is connected (or empty).*

**PROOF.** Since any sphere is a continuous image of one of unit radius, we may and do assume that the spheres are  $S$  and  $S_a$  with  $0 < \|a\| \leq 2$ . We are going to show that  $K = S \cap S_a$  is connected.

Let  $b \in K$  be an arbitrarily fixed vector and for any  $x \in K$  choose vectors  $v_{x1}, v_{x2} \in S$  such that  $a, b, x \in \text{lin}\{u, v_{x1}, v_{x2}\} = M_x$  with  $\dim M_x = 3$ . Then  $W_x = S \cap M_x$  is a closed, bounded sphere in the finite dimensional subspace  $M_x$  and so it is compact while in the two dimensional subspace  $L_x = \text{lin}\{v_{x1}, v_{x2}\}$ , the set  $V_x = S \cap L_x$  is a connected circle. Now consider the relation  $F_x \subset V_x \times W_x$  defined for any  $v \in V_x$  by  $F_x(v) = K_v$ , the non-empty, connected set given in the previous lemma. As soon as it will have been shown that  $F_x$  is closed, Corollary 2.2 implies the connectivity of

$$F_x(V_x) = \bigcup_{v \in V_x} (K \cap P_v^+) = K \cap M_x \ni x, b.$$

To do this, take a sequence  $(v_n, w_n) \in F_x$  converging to some  $(v_0, w_0) \in V_x \times W_x$ . Since  $w_n \in K_{v_n} \subset P_{v_n}^+$  and  $v_n \rightarrow v_0$ , we have  $w_0 \in P_{v_0}^+$ . Also  $w_n \in K_{v_n} \subset K$  and because of  $K$  is closed,  $w_0 \in K$ , i.e.  $w_0 \in K \cap P_{v_0}^+ = K_{v_0} = F_x(v_0)$ . Thus  $(v_0, w_0) \in F_x$  and therefore  $F_x$  is closed.

Finally  $\bigcap_{x \in K} F_x(V_x)$  is non-empty (namely it contains  $b$ ), whence

$$K = \bigcup_{x \in K} F_x(V_x)$$

is connected.

### 3. The main result

**Theorem 3.1.** *Suppose that  $(X, \|\cdot\|)$  is a real normed linear space of dimension  $\geq 3$  and  $(Y, +)$  is an Abelian group. If a mapping  $f : X \rightarrow Y$  satisfies the conditional Cauchy equation (1), then it is additive.*

PROOF. Let  $x, y \in X$  be arbitrarily given. We may and do assume that  $\|x\| < \|y\| = \rho$ . Next we claim that there exist vectors  $x_1, x_2 \in X$  such that

- (a)  $2x = x_1 + x_2$
- (b)  $\|x_1\| = \|x_2\| = \rho$
- (c)  $\|x_1 + y\| = \|x_2 + y\|$ .

Indeed, let  $K$  denote the connected intersection of spheres  $S_0(\rho)$  and  $S_{2x}(\rho)$ . Then for any  $z \in K$  we have  $z' = 2x - z \in K$ . Consider the continuous function  $\varphi : K \rightarrow \mathbf{R}$  defined by  $\varphi(z) = \|y + z\| - \|y + z'\|$ . Since  $\varphi(z') = -\varphi(z)$ ,  $z \in K$ , the function  $\varphi$  changes its sign on  $K$  and taking into account the connectivity of  $K$ , there is a vector  $z_0 \in K$  with  $\varphi(z_0) = 0$ . Then  $x_1 = z_0$  and  $x_2 = z'_0$  satisfy all the requirements (a), (b) and (c).

Finally, we can compute by equation (1) as follows:

$$f(2x + 2y) = f([x_1 + y] + [x_2 + y]) = f(x_1 + y) + f(x_2 + y) =$$

$$\begin{aligned}
&= f(x_1) + f(y) + f(x_2) + f(y) = f(x_1 + x_2) + 2f(y) = \\
&= f(2x) + f(2y)
\end{aligned}$$

which was to be proved.

In the rest of the paper we apply this main theorem to orthogonally additive mappings. Namely consider the James' *isosceles orthogonality*  $\perp$  on  $X$  defined by  $x \perp y \iff \|x+y\| = \|x-y\|$  ( $x, y \in X$ ). It is known (see JAMES [5]) that the homogeneity of  $\perp$  characterizes inner product spaces. Now a mapping  $f : X \rightarrow Y$  with values in an Abelian group  $(Y, +)$  is said to be (*isosceles*) *orthogonally additive*, if it satisfies the conditional equation

$$(2) \quad f(x+y) = f(x) + f(y), \quad \text{whenever } x \perp y.$$

For large classes of homogeneous orthogonality relation  $\perp$ , the odd solutions are additive (see RÄTZ-SZABÓ [7]). Here we derive the same for the non-homogeneous isosceles orthogonality.

**Theorem.** *Suppose that  $(X, \|\cdot\|)$  is a real normed linear space of dimension  $\geq 3$  and  $(Y, +)$  is an Abelian group. If a mapping  $f : X \rightarrow Y$  is odd and isosceles orthogonally additive, then it is additive.*

PROOF. First observe that  $0 \perp 0$  whence  $f(0) = 0$ . Moreover  $f(2x) = 2f(x)$  holds for all  $x \in X$ . Indeed, as it can easily be shown, there exists  $y \in X$  with  $\|x\| = \|y\|$  and  $\|x+y\| = \|x-y\|$  (just take the continuous function  $\varphi : S(\|x\|) \rightarrow \mathbf{R}$  defined by  $\varphi(u) = \|x+u\| - \|x-u\|$ ; since its domain is connected and  $\varphi(-u) = -\varphi(u)$ ,  $\varphi(y) = 0$  for some  $y \in S(\|x\|)$ ). Then  $(x+y) \perp (x-y)$ ,  $x \perp \pm y$  and so using the orthogonal additivity and oddness of  $f$ , we have

$$\begin{aligned}
f(2x) &= f([x+y] + [x-y]) = f(x+y) + f(x-y) = \\
&= [f(x) + f(y)] + [f(x) + f(-y)] = 2f(x).
\end{aligned}$$

In what follows we are going to show that  $f$  satisfies equation (1). Let  $x, y \in X$  with  $\|x\| = \|y\|$ . Then for  $u = x+y$  and  $v = x-y$  we have  $u \perp \pm v$  and so

$$\begin{aligned}
f(2x) + f(2y) &= f(u+v) + f(u-v) = [f(u) + f(v)] + [f(u) + f(-v)] = \\
&= 2f(u) = f(2u) = f(2x+2y).
\end{aligned}$$

This completes the proof.

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