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Lattice of generalized neighbourhood sequences in nD and ∞D

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To the memory of Professor Péter Kiss

Abstract. In this paper we generalize the concept of neighbourhood sequences introduced by Das *et al.* [2]. We extend the natural ordering relation given in [2] to the set of generalized nD-neighbourhood sequences, and investigate the structure obtained. As we do not always get nice properties, another ordering relation is introduced which behaves better. We also involve the abstract digital plane $\mathbb{Z}^\infty$ into our analysis, and extend our results to this case. Our investigations generalize previous results of Das [1] and Fazekas [4] in 2D and 3D, respectively.

1. Introduction

Rosenfeld and Pfaltz gave two types of motions in two-dimensional digital geometry (see [7]). The cityblock motion is restricted to horizontal and vertical movements only. That is, two points on the digital plane $\mathbb{Z}^2$ are neighbours, if one of their coordinate values coincide, while the others differ at most by 1. The chessboard motion beside horizontal and vertical steps, also allows diagonal movements. In this case two points of $\mathbb{Z}^2$ are neighbours when both of their coordinate values differ at most by 1. The so called octagonal distance can be obtained by the alternating use of

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these motions. (The exact concept of distance will be given in the following chapter.) More detailed description about these and other concepts of digital topology can be found in [6] and [8].

Das, Chakrabarti and Chatterji (see [2]) extended the definition of the ordinary octagonal distance, allowing arbitrarily long periodic sequences of cityblock and chessboard motions, called neighbourhood sequences. Moreover, they established a formula for calculating the distance of two points in the nD digital space, determined by such a neighbourhood sequence. Using this formula, Das in [1] showed that on the set of periodic 2D-neighbourhood sequences a natural partial ordering relation can be introduced. Furthermore, he investigated the structure of this set and some of its subsets with respect to this ordering. More precisely, he proved that under this ordering, the set of l-periodic 2D-neighbourhood sequences forms a distributive lattice. Das also claimed the same for the set of sequences with a period at most l (see Theorem 4 of [1]), and that the set of the periodic 2D-neighbourhood sequences forms a complete compact distributive lattice with respect to the relation mentioned above (cf. Corollary 1 of [1]). However, these two results of Das [1] are false. This follows from our Propositions 3.8 and 3.7, respectively. Recently, Fazekas proved that a similar partial ordering can also be introduced for neighbourhood sequences in 3D (see [4]).

In this paper we generalize the concept of neighbourhood sequences, allowing not periodic sequences only. We show that the results of Das [1] and Fazekas [4] about ordering the set of periodic neighbourhood sequences, can be extended to arbitrary dimension, even in case of generalized neighbourhood sequences. We also prove that in 2D the set of such sequences forms a complete distributive lattice under this relation. Moreover, we extend our investigations to ∞D, which is the most interesting case theoretically. We give a formula for the calculation of the distance of two points in $\mathbb{Z}^\infty$ with respect to a generalized neighbourhood sequence. By the help of this result we generalize the natural ordering relation to ∞D. The lattice obtained in ∞D under this ordering relation, in a certain sense is the closure of the union of the finite dimensional lattices; this shows the significance of such investigations.

We also study the structure of some subsets of the generalized neighbourhood sequences in nD and ∞D under the ordering mentioned. We involve into our investigations all types of subsets which were studied by

Das [1] in the periodic case. Unfortunately, in several cases we obtain negative results: some of the studied structures do not have nice properties. Thus we consider another relation, which is in close connection with the original one. More precisely, the natural ordering is a refinement of the relation introduced here. We show that under this new ordering, the examined sets (with one exception) form lattices, with certain further properties in some cases.

Finally, to support our investigations we note that neighbourhood sequences may have many applications not only in 2D, but also from 3D on. In 3D one can think of three-dimensional pictures, for instance in medical applications (see e.g. [3] and the references given there). Interestingly, even in 2D higher dimensional neighbourhood sequences can be useful. For example, in case of colour images (when three additional colour parameters appear), or during tracking motion (when time can be consdiered as a third dimension). The motivation and theoretical background of such investigations can be found e.g. in [5].

2. Basic concepts

In order to reach the aims formulated in the introduction we would like to give the basic definitions and notation in this chapter. From now on, n will denote an arbitrary positive integer.

Definition 2.1. Let p and q be two points in $\mathbb{Z}^n$. The ith coordinate of the point p is indicated by $\mathrm{Pr}_i(p)$. Let N be an integer with $0 \leq N \leq n$. The points p and q are N-neighbours, if the following two conditions hold:

- $|\mathrm{Pr}_i(p) - \mathrm{Pr}_i(q)| \leq 1$ for $1 \leq i \leq n$,
- $\sum_{i=1}^{n} |\mathrm{Pr}_i(p) - \mathrm{Pr}_i(q)| \leq N$.

Definition 2.2. The infinite sequence $B = \{b(i) : i \in \mathbb{N}$ and $b(i) \in \{1, 2, \ldots, n\}\}$ is called a generalized nD-neighbourhood sequence. If for some $l \in \mathbb{N}$, $b(i) = b(i + l)$ holds for every $i \in \mathbb{N}$, then B is called periodic, with a period l, or simply l-periodic. In this case we will use the abbreviation $B = \{b(1), \ldots, b(l)\}$.

Remark 2.3. We note that the above concept of the generalized nD-neighbourhood sequences is actually a generalization of the notion of neighbourhood sequences introduced in [2]. The authors in [1], [2] and [4] dealt only with periodic sequences.

The simple distances introduced by Rosenfeld and Pfaltz [7] can also be given by (periodic) neighbourhood sequences. Namely, the city-block-, chessboard- and octagonal distances can be generated by the sequences $\{1\}$, $\{2\}$ and $\{1,2\}$, respectively.

Definition 2.4. Let p and q be two points in $\mathbb{Z}^n$ and $B = \{b(i) : i \in \mathbb{N}\}$ a generalized nD-neighbourhood sequence. The point sequence $\Pi(p,q;B)$ – having the form $p = p_0,\ p_1,\dots,\ p_m = q$, where p_{i-1} and p_i are $b(i)$-neighbours for $1 \le i \le m$ – is called a path from p to q determined by B. The length $|\Pi(p,q;B)|$ of the path $\Pi(p,q;B)$ is m. Clearly, there always exist paths from p to q, determined by B. The distance between p and q is defined as the length of a shortest path, and is denoted by $d(p,q;B)$.

Using the above distance we cannot obtain a metric in $\mathbb{Z}^n$ for every nD-neighbourhood sequence. In order to prove this, consider the following simple example. Let $B = \{2,1\}$, $n = 2$, $p = (0,0)$, $q = (1,1)$ and $r = (2,2)$. In this case $d(p,q;B) = 1$, $d(q,r;B) = 1$, but $d(p,r;B) = 3$.

We have the following natural question: knowing B, how can we decide whether the distance function related to B is a metric on the n-dimensional digital plane, or not? The answer in the periodic case can be found in [2].

For later use we need to introduce some further notation.

Notation 2.5. Let p and q be two points in $\mathbb{Z}^n$, and $B = \{b(i) : i \in \mathbb{N}\}$ a generalized nD-neighbourhood sequence. Let

$$x = (x(1), x(2), \dots, x(n)),$$

where x is the nonincreasing ordering of $|\operatorname{Pr}_i(p) - \operatorname{Pr}_i(q)|$, that is, $x(i) \ge x(j)$ if $i < j$. For $k = 1,\dots,n$, and $i \in \mathbb{N}$ put

$$a_k = \sum_{j=1}^{n-k+1} x(j),$$

$$b_k(i) = \begin{cases} b(i), & \text{if } b(i) < n-k+2, \\ n-k+1, & \text{otherwise,} \end{cases}$$

$$f_k(i) = \sum_{j=1}^{i} b_k(j).$$

Furthermore, set $f_k(0) = 0$.

The following result of Das *et al.* (cf. [2]) provides an algorithm for the calculation of the distance $d(p, q; B)$, defined in Definition 2.4.

Theorem 2.6 (see [2]). *Let p and q be two points in $\mathbb{Z}^n$, and $B = \{b(i) : i \in \mathbb{N}\}$ a periodic nD-neighbourhood sequence with period l. Using the above notation, for $i = 1, \dots, n$ put*

$$g_k(i) = f_k(l) - f_k(i-1) - 1, \quad 1 \le i \le l.$$

The length of the shortest paths from p to q determined by B is given by the following formula:

$$d(p, q; B) = \max_{k=1}^{n} d_k(p, q),$$

$$\text{where} \quad d_k(p, q) = \sum_{i=1}^{l} \left\lfloor \frac{a_k + g_k(i)}{f_k(l)} \right\rfloor.$$

Now we recall some definitions and remarks from lattice theory that we need to analyze the lattices of the generalized neighbourhood sequences.

As usual, let $(P, \le)$ denote a partially ordered set. An element $a \in P$ is the least upper bound (greatest lower bound) of a subset $S \subseteq P$ if for all $x \in S$, $a \ge x$ ($a \le x$), and $b \ge a$ ($b \le a$) for every upper bound (lower bound) b of S. Moreover, if every pair of elements $\{(x, y) : x, y \in P\}$ has a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$ then $(P, \le)$ is called a lattice. The lattice $(P, \le)$ is distributive if for all $x, y, z \in P$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Clearly, $(P, \le)$ is distributive if and only if

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

The lattice $(P, \le)$ is complete if its every subset $S \subseteq P$ has a least upper bound $\bigvee S$ and a greatest lower bound $\bigwedge S$. Let $(P, \le)$ be a complete lattice and $S \subseteq P$. The set $S^c = \{x \in P : x \le \bigvee S\}$ is called the closure of S.

Remark 2.7. It is well known that $(P, \le)$ is complete if its every subset has a least upper bound.

3. Neighbourhood sequences in nD

It is a natural question that what kind of relation exists between the distance functions generated by two given neighbourhood sequences B_1 and B_2. The complexity of the problem can be characterized by the following 2D periodic example from [1]. Let $B_1 = \{1, 1, 2\}$, $B_2 = \{1, 1, 1, 2, 2, 2\}$. Choose the points $o = (0, 0)$, $p = (3, 1)$ and $q = (6, 3)$. In this case we obtain that $d(o, p; B_1) = 3 < 4 = d(o, p; B_2)$, but $d(o, q; B_1) = 7 > 6 = d(o, q; B_2)$. So the distances generated by B_1 and B_2 cannot be compared.

In [1] it is shown that using the functions $f_k(i)$ defined in Notation 2.5, a nice ordering relation can be established for periodic neighbourhood sequences in 2D. A similar result was proved by Fazekas in 3D (see [4]). Now we extend these results to nD with arbitrary $n \in \mathbb{N}$, to generalized nD-neighbourhood sequences. Clearly, this case also includes the periodic one. Especially, we note that our result is new even for $n = 2$ and 3.

First we need the following simple lemma, which is, however of great importance, because it shows that Theorem 2.6 can be used to compute the distance of two points concerning an arbitrary generalized nD-neighbourhood sequence.

Lemma 3.1. *Let p and q be two points in $\mathbb{Z}^n$ with $\sum_{i=1}^{n} |\operatorname{Pr}_i(p) - \operatorname{Pr}_i(q)| = c$. Let $A = \{a(i) : i \in \mathbb{N}\}$ and $B = \{b(i) : i \in \mathbb{N}\}$ be two generalized nD-neighbourhood sequences, with $a(i) = b(i)$ for $i \leq c$. Then $d(p, q; A) = d(p, q; B)$.*

Proof. First, it is clear that $d(p, q; A) \leq c$. Let $d(p, q; A) = h$, and let $p = p_0, p_1, \ldots, p_h = q$ be a path from p to q determined by A in $\mathbb{Z}^n$. However, by $h \leq c$ and $a(i) = b(i)$ for $1 \leq i \leq c$, we obtain that p_{i-1} and p_i are $b(i)$-neighbours for $i = 1, \ldots, h$, hence $d(p, q; B) \leq h = d(p, q; A)$. The opposite inequality can be proved in a similar way, and the lemma follows. $\square$

Theorem 3.2. *Using the notation introduced in 2.5, for any generalized nD-neighbourhood sequences $B_1 = \{b^{(1)}(i) : i \in \mathbb{N}\}$ and $B_2 = \{b^{(2)}(i) : i \in \mathbb{N}\}$*

$$d(p, q; B_1) \leq d(p, q; B_2), \quad \text{for all } p, q \in \mathbb{Z}^n$$

if and only if

$$f_k^{(1)}(i) \geq f_k^{(2)}(i), \quad \text{for all } i \in \mathbb{N},\ k \in \{1, \ldots, n\},$$

where $f_k^{(1)}(i)$ *and* $f_k^{(2)}(i)$ *correspond to* B_1 *and* B_2*, respectively.*

PROOF. First we prove that if $d(p,q;B_1) \leq d(p,q;B_2)$ for any p,q, then $f_k^{(1)}(i) \geq f_k^{(2)}(i)$ for every $i \in \mathbb{N}$, $k \in \{1,\dots,n\}$. The proof is indirect. Assume that there are such $i \in \mathbb{N}$ and $k \in \{1,\dots,n\}$ for which $f_k^{(1)}(i) < f_k^{(2)}(i)$ holds. Put

$$u_j = \begin{cases} |\{b^{(2)}(t) : 1 \leq t \leq i, b^{(2)}(t) = j\}|, & \text{for } 1 \leq j < k, \\ |\{b^{(2)}(t) : 1 \leq t \leq i, b^{(2)}(t) \geq j\}|, & \text{for } j = k, \end{cases}$$

let $p = (0,0,\dots,0)$ and

$$\mathrm{Pr}_h(q) = \begin{cases} \sum_{j=1}^{h} u_j, & \text{for } h \leq k, \\ 0, & \text{for } h > k. \end{cases}$$

Using the definition of $d(p,q;B)$, it is clear that $d(p,q;B_2)$ is equal to i. On the other hand, by the assumption $f_k^{(2)}(i) > f_k^{(1)}(i)$, and by the definition of p and q, we have $d(p,q;B_1) > i$, which is a contradiction.

Conversely, suppose that $f_k^{(1)}(i) \geq f_k^{(2)}(i)$ for every $i \in \mathbb{N}$, $k \in \{1,\dots,n\}$. Let p and q be two points in $\mathbb{Z}^n$, and put $c = \sum_{h=1}^{n} |\mathrm{Pr}_h(p) - \mathrm{Pr}_h(q)|$. Without loss of generality we may assume that $c \geq 1$. To derive $d(p,q;B_1) \leq d(p,q;B_2)$, by Theorem 2.6 and Lemma 3.1, it is sufficient to show that for $k \in \{1,\dots,n\}$

$$d_k^{(1)}(p,q) = \sum_{j=1}^{c} \left\lfloor \frac{a_k + g_k^{(1)}(j)}{f_k^{(1)}(c)} \right\rfloor \leq \sum_{j=1}^{c} \left\lfloor \frac{a_k + g_k^{(2)}(j)}{f_k^{(2)}(c)} \right\rfloor = d_k^{(2)}(p,q)$$

holds. For this we prove that for any fixed k with $k \in \{1,\dots,n\}$

$$\left\lfloor \frac{a_k + g_k^{(1)}(j)}{f_k^{(1)}(c)} \right\rfloor \leq \left\lfloor \frac{a_k + g_k^{(2)}(j)}{f_k^{(2)}(c)} \right\rfloor \quad \text{for } 1 \leq j \leq c.$$

Using the definition of $g_k(j)$, the above inequalities are equivalent to the following ones:

$$\left\lfloor \frac{a_k + f_k^{(1)}(c) - f_k^{(1)}(j-1) - 1}{f_k^{(1)}(c)} \right\rfloor \leq \left\lfloor \frac{a_k + f_k^{(2)}(c) - f_k^{(2)}(j-1) - 1}{f_k^{(2)}(c)} \right\rfloor,$$

$$1 \leq j \leq c,$$

which is the same as

$$1+\left\lfloor\frac{(a_k-1)-f_k^{(1)}(j-1)}{f_k^{(1)}(c)}\right\rfloor\leq 1+\left\lfloor\frac{(a_k-1)-f_k^{(2)}(j-1)}{f_k^{(2)}(c)}\right\rfloor,\quad 1\leq j\leq c.$$

If $(a_k-1)-f_k^{(2)}(j-1)\geq 0$, then we even have

$$\frac{(a_k-1)-f_k^{(1)}(j-1)}{f_k^{(1)}(c)}\leq\frac{(a_k-1)-f_k^{(2)}(j-1)}{f_k^{(2)}(c)}.$$

Indeed, this inequality is equivalent to

$$f_k^{(2)}(c)(a_k-1-f_k^{(1)}(j-1))\leq f_k^{(1)}(c)(a_k-1-f_k^{(2)}(j-1)),$$

which clearly holds because of our assumption $f_k^{(2)}(i)\leq f_k^{(1)}(i)$, $i\in\mathbb{N}$.

In the case of $(a_k-1)-f_k^{(2)}(j-1)<0$, by the definitions of f_k and a_k, we obviously have

$$\left\lfloor\frac{(a_k-1)-f_k^{(2)}(j-1)}{f_k^{(2)}(c)}\right\rfloor=-1.$$

However, using again $f_k^{(2)}(i)\leq f_k^{(1)}(i)$, $i\in\mathbb{N}$, now the equality

$$\left\lfloor\frac{(a_k-1)-f_k^{(1)}(j-1)}{f_k^{(1)}(c)}\right\rfloor=-1$$

also holds, which completes the proof of the theorem. □

Definition 3.3. Let S_n, S'_n, $S'_n(l_{\geq})$ and $S'_n(l)$ be the sets of generalized, periodic, at most l-periodic and l-periodic ($l\in\mathbb{N}$) nD-neighbourhood sequences, respectively. For any $B_1,B_2\in S_n$ we define the relation $\sqsupseteq^*$ in the following way:

$$B_1\sqsupseteq^* B_2\quad\Longleftrightarrow\quad f_k^{(1)}(i)\geq f_k^{(2)}(i)$$

for all $i\in\mathbb{N}$ and $k\in\{1,\dots,n\}$.

Remark 3.4. Using the previous theorem, it is evident that $\sqsupseteq^*$ is a partial ordering relation in S_n, hence also all in its subsets. Moreover, this relation $\sqsupseteq^*$ in 2D and 3D, is clearly identical to those introduced by Das [1] and Fazekas [4], respectively.

Beside S_n, we investigate the structure of all those sets which were studied by Das [1] in the periodic case, like S'_n, $S'_n(l_{\geq})$ and $S'_n(l)$ under $\sqsupseteq^*$. Unfortunately, in most cases the above sets with respect to this relation does not form a nice structure. Although some of the forthcoming results are very easy to prove, they are of certain importance because provide some information about the differences of the sets involved. Our only "positive" result in this direction is the following.

Theorem 3.5. *$(S_2, \sqsupseteq^*)$ is a complete distributive lattice.*

Proof. Let $S \subseteq S_2$, and for every $C \in S_2$, for $k \in \{1,2\}$ denote by $f_k^{(C)}(i)$ the corresponding functions defined in Notation 2.5. Put $b(1) = \max_{A\in S}\left(f_2^{(A)}(1)\right)$ and inductively, for $i \in \mathbb{N}$, $i \geq 2$ let $b(i)$ be the minimum of those $j \in \{1,2\}$ for which $\sum_{h=1}^{i} b(h) \geq f_2^{(A)}(i)$ holds for all $A \in S$, and let $B = \{b(i) : i \in \mathbb{N}\}$. By Theorem 3.2, and by the definition of B we have $B \sqsupseteq^* A$ for every $A \in S$. (Clearly, $f_1^{(B)}(i) = f_1^{(A)}(i) = i$ for $i \in \mathbb{N}$.) Let B' be an arbitrary upper bound of S, and let $j \in \mathbb{N}$ be arbitrary, but fixed. The definition of B implies that $f_2^{(B)}(j) = f_2^{(A)}(j)$ for some $A \in S$. Hence, as B' is an upper bound of S, $f_2^{(B')}(j) \geq f_2^{(A)}(j) \geq f_2^{(B)}(j)$, which proves the minimality of B. Now, by Remark 2.7 we have that $(S_2, \sqsupseteq^*)$ is a complete lattice.

Let A_1, A_2, A_3 be arbitrary elements of S_2 and $i \in \mathbb{N}$. Now

$$\begin{aligned}
f_2^{(A_1\wedge(A_2\vee A_3))}(i) &= \min\left(f_2^{(A_1)}(i), \max\left(f_2^{(A_2)}(i), f_2^{(A_3)}(i)\right)\right)\\
&= \max\Big(\min\left(f_2^{(A_1)}(i), f_2^{(A_2)}(i)\right), \min\left(f_2^{(A_1)}(i), f_2^{(A_3)}(i)\right)\Big)\\
&= f_2^{((A_1\wedge A_2)\vee(A_1\wedge A_3))}(i),
\end{aligned}$$

which proves the distributive property of the lattice $(S_2, \sqsupseteq^*)$. The proof of the theorem is complete. □

Now we show that the above Theorem does not hold in higher dimensions. Roughly speaking, the reason of this phenomenon is that from 3D on, we have to deal with $n-1 \geq 2$ "non-trivial" $f_k(i)$, when $k > 1$.

Proposition 3.6. *$(S_n, \sqsupseteq^*)$ is not a lattice for $n \geq 3$.*

Proof. Put $A_1 = \{3,1\}$, $A_2 = \{2\}$, $B_1 = \{2,1,3,1,3,1,3,1,3,\dots\}$ and $B_2 = \{1,3,1,1,1,1,\dots\}$. Clearly, $A_1 \sqsupseteq^* B_1$ and $A_2 \sqsupseteq^* B_1$. On the other hand, if $B_3 \in S_n$ then $A_1 \sqsupseteq^* B_3$ and $A_2 \sqsupseteq^* B_3$ implies $B_3 \not\sqsupseteq^* B_1$. Indeed, the first element of B_3 can be at most 2. Suppose that the first i elements of B_1 and B_3 are identical, but the $(i+1)$th elements are different. Now $B_3 \sqsupseteq^* B_1$ would yield that $b^{(3)}(i+1) > b^{(1)}(i+1)$, where these numbers are the corresponding elements of B_3 and B_1, respectively. If $b^{(1)}(i+1) = 1$ then $b^{(3)}(i+1) \geq 2$, which implies $f_2^{(B_3)}(i+1) > f_2^{(A_1)}(i+1)$, contradicting $A_1 \sqsupseteq^* B_3$. (Here for $C \in S_n$ and $i \in \{1,\dots,n\}$, $f_k^{(C)}(i)$ denotes the functions defined in Notation 2.5.) Otherwise, when $b^{(1)}(i+1) = 3$ then $b^{(3)}(i+1) \geq 4$, whence $f_4^{(B_3)}(i+1) > f_4^{(A_2)}(i+1)$, which contradicts $A_2 \sqsupseteq^* B_3$. (The latter case can occur only when $n \geq 4$.) On the other hand, it is clear that $A_1 \sqsupseteq^* B_2, A_2 \sqsupseteq^* B_2$, but $B_1 \not\sqsupseteq^* B_2$. Hence, A_1 and A_2 have no greatest lower bound, thus S_n is not a lattice for $n \geq 3$. $\square$

Concerning some special sets of periodic sequences, we show that similar unpleasant properties of $\sqsupseteq^*$ also occur. In what follows we list these "negative" results. We note that Propositions 3.7 (case $n = 2$) and 3.8 disprove Corollary 1 and Theorem 4 of Das [1], respectively.

Proposition 3.7. *$(S'_n, \sqsupseteq^*)$ is not a lattice for $n \geq 2$.*

Proof. Let $A_1 = \{2,1,1\}$ and $A_2 = \{1,2,2\}$. The least upper bound of these sequences in $(S_n, \sqsupseteq^*)$ is obviously $B = \{2,1,2,1,2,2,1,2,2,1,2,2,\dots\}$, which is clearly not in S'_n. Suppose that $B' = \{b'(1),\dots,b'(l)\}$, and $B' = A_1 \vee A_2$ in S'_n. However, in this case $B' \sqsupseteq^* B$ in S_n, but $B' \neq B$, hence for some $i \in \mathbb{N}$, $b'(i) > b(i)$ must hold; suppose that i is the least number with this property. Now putting $b''(j) = b(j)$ for $1 \leq j \leq 3i$ and $B'' = \{b''(1),\dots,b''(3i)\}$, we have $B'' \sqsupseteq^* A_1, A_2$, but $B'' \not\sqsupseteq^* B'$, which contradicts $B' = A_1 \vee A_2$ in S'_n. The proof of the proposition is complete. $\square$

Proposition 3.8. *$(S'_2(l_{\geq}), \sqsupseteq^*)$ is not a lattice for any $l \geq 5$.*

Proof. First let $A_1 = \{1,2,2\}$ and $A_2 = \{1,2,2,2,1\}$. One can readily verify that A_1 and A_2 have no least upper bound in $S'_2(5_{\geq})$, thus the statement holds for $l = 5$.

Let now $l \geq 6$, and choose any even number k with $\frac{l}{6} < k \leq \frac{l}{3}$. Put $X = \{x_1, \ldots, x_k\} = \{1, 2, 2, 1, 2, 1, 2, 1, \ldots, 2, 1\}$ and $Y = \{y_1, \ldots, y_k\} = \{2, 1, 1, 2, 2, 1, 2, 1, 2, 1, \ldots, 2, 1\}$, i.e. $x_1 = 1, x_2 = 2$ and $x_i = (i \mod 2)+1$ for $3 \leq i \leq k$ while $y_1 = 2, y_2 = 1, y_3 = 1, y_4 = 2$ and $y_i = (i \mod 2) + 1$ for $5 \leq i \leq k$. Let $A_1 = \{X, Y\}$ be the sequence of period $2k$, obtained by writing Y after X, i.e. the elements of the $2k$ long period of A_1 with odd indices equal to 2 and with even indices equal to 1, except for $a^{(1)}(1) = 1, a^{(1)}(2) = 2, a^{(1)}(k + 3) = 1, a^{(1)}(k + 4) = 2$. Similarly, let $A_2 = \{X, Y, X\}$ be a sequence of period $3k$. Furthermore, set $B_1 = \{2, 1\}$ and $B_2 = \{1, 2, 2\}$. Observe that in S_2 we have $A_1 \vee A_2 = A = \{X, Y, X, 2, 1, 2, 1, \ldots, 2, 1\}$ of length $6k$ and $B_1 \wedge B_2 = B = \{1, 2, 2, 1, 2, 1, 2, 1, \ldots\}$, which is not periodic. We claim that $A_1 \vee A_2$ does not exist in $S'_2(l_{\geq})$. Indeed, if such a $C = \{c(i) : i \in \mathbb{N}\} \in S'_2(l_{\geq})$ exists, then in S_2, $A \sqsubseteq^* C \sqsubseteq^* B$ must hold. However, every period of C should be even, moreover, $c(2i - 1) + c(2i) = 3$ should hold for every $i \in \mathbb{N}$. Furthermore, $c(1) = 1$ should also be valid, which in view of $l < 6k$ and $A \sqsubseteq^* C$ is impossible, and the Proposition is proved. $\square$

Remark 3.9. It is easy to check that $(S'_2(l_{\geq}), \sqsupseteq^*)$ is a distributive lattice if $1 \leq l \leq 4$. Hence the statement of Theorem 4 of Das [1] is valid in these special cases only.

Proposition 3.10. *$(S'_n(l_{\geq}), \sqsupseteq^*)$ is not a lattice for any $l \geq 2$, $n \geq 3$.*

Proof. Let $l \geq 5$. It is clear that the sequences A_1 and A_2, defined in the proof of the previous proposition, have no least upper bound in $S'_n(l_{\geq})$.

Let now $2 \leq l \leq 4$, and let $A_1 = \{a^{(1)}(1), \ldots, a^{(1)}(l)\}$ and $A_2 = \{2\}$, where A_1 is defined in the following way: $a^{(1)}(1), \ldots, a^{(1)}(l)$ is the first l elements of $\{3, 1, 3, 1\}$. It is easy to check that these sequences have no least upper bound in $S'_n(l_{\geq})$, which completes the proof. $\square$

As we mentioned above, $S'_2(l)$ is a distributive lattice for every $l \in \mathbb{N}$ (see [1]). The following result shows that this statement cannot be generalized to nD with $n \geq 3$.

Proposition 3.11. *$(S'_n(l), \sqsupseteq^*)$ is not a lattice for any $l \geq 2$, $n \geq 3$.*

Proof. Let $l \geq 2$ and $n \geq 3$ be arbitrary, but fixed integers. We use the same sequences as in Proposition 3.6. Namely, in S_n choose the following sequences: $A_1 = \{3, 1\}$, $A_2 = \{2\}$, $B_1 = \{2, 1, 3, 1, 3, 1, 3, 1, 3, \ldots\}$

and $B_2 = \{1,3,1,1,1,1,\dots\}$. Define $A'_j = \{{a^{(j)}}'(1),\dots,{a^{(j)}}'(l)\}$, and $B'_j = \{{b^{(j)}}'(1),\dots,{b^{(j)}}'(l)\}$ for $j = 1,2$ in $S'_n(l)$ in the following way: ${a^{(j)}}'(i) = a^{(j)}(i)$, ${b^{(j)}}'(i) = b^{(j)}(i)$, $i = 1,\dots,l$. Just as in the proof of Proposition 3.6, it is easy to show that if $A'_1 \wedge A'_2$ exists in $S'_n(l)$, then it must be B'_1. However, clearly $B'_2 \sqsubseteq^* A'_1, A'_2$, but $B'_2 \not\sqsubseteq^* B'_1$, which completes the proof. $\square$

The above results show that under the relation $\sqsupseteq^*$ we cannot obtain a nice structure neither in S_n, nor in various subsets of it. Now we introduce a new ordering relation, which is in close connection with $\sqsupseteq^*$. Moreover, S_n and its subsets considered above, will form much nicer structures under this new relation.

Definition 3.12. For any $B_1 = \{b^{(1)}(i) : i \in \mathbb{N}\}$, $B_2 = \{b^{(2)}(i) : i \in \mathbb{N}\} \in S_n$ we define the relation $\sqsupseteq$ in the following way:

$$B_1 \sqsupseteq B_2 \iff b^{(1)}(i) \geq b^{(2)}(i), \quad \text{for every } i \in \mathbb{N}.$$

Remark 3.13. It is clear that $\sqsupseteq^*$ is a proper refinement of $\sqsupseteq$ in S_n, S'_n, $S'_n(l_\geq)$ and $S'_n(l)$.

We examine the structure of S_n, S'_n, $S'_n(l_\geq)$ and $S'_n(l)$ with respect to $\sqsupseteq$. As we will see, the structures we get will be much nicer than in the case of $\sqsupseteq^*$.

Proposition 3.14. *$(S_n, \sqsupseteq)$ is a complete distributive lattice with greatest lower bound $\bigwedge S_n = \{1\}$ and least upper bound $\bigvee S_n = \{n\}$.*

Proof. From the definition of $\sqsupseteq$ it follows that this relation is reflexive, antisymmetric and transitive on S_n. Thus $(S_n, \sqsupseteq)$ is a partially ordered set.

It is clear that for every $B_1, B_2 \in S_n$, $B_1 \wedge B_2$ and $B_1 \vee B_2$ exist in S_n, and we have

$$B_1 \wedge B_2 = \{\min(b^{(1)}(i), b^{(2)}(i)) : i \in \mathbb{N}\},$$

$$B_1 \vee B_2 = \{\max(b^{(1)}(i), b^{(2)}(i)) : i \in \mathbb{N}\}.$$

Thus $(S_n, \sqsupseteq)$ is a lattice. Let now $S = \{B_\gamma : B_\gamma \in S_n,\ \gamma \in \Gamma\}$ with some index set Γ, and put $b(i) = \max_{\gamma\in\Gamma}\left(b^{(\gamma)}(i)\right)$, where $b^{(\gamma)}(i)$ is the ith element of B_γ. Clearly, $B = \{b(i) : i \in \mathbb{N}\}$ is the least upper bound of S, hence by Remark 2.7 $(S_n, \sqsupseteq)$ is a complete lattice.

The statements that this lattice is distributive and $\bigwedge S_n = \{1\}$, $\bigvee S_n = \{n\}$ are trivial. $\square$

Proposition 3.15. $(S_n', \sqsupseteq)$ *is a distributive lattice with greatest lower bound* $\bigwedge S_n' = \{1\}$ *and least upper bound* $\bigvee S_n' = \{n\}$.

Proof. It is clear that if $B_1, B_2 \in S_n'$, then $B_1 \wedge B_2$ and $B_1 \vee B_2$ – given in the proof of the previous proposition – are also in S_n'. Thus, as $S_n' \subseteq S_n$ and $\{1\}, \{n\} \in S_n'$, we immediately obtain the statement. □

However, the ordering relation $\sqsupseteq$ has worse properties in S_n' than in S_n. This is shown by the following "negative" result.

Proposition 3.16. *For* $n \geq 2$, $(S_n', \sqsupseteq)$ *is not a complete lattice.*

Proof. To prove the statement, we give a counterexample. To find such an example, we construct a monotonously increasing and a monotonously decreasing sequence in S_n', such that their "limit" sequence is the same, however, this "limit" is in $S_n \setminus S_n'$. This will show that $(S_n', \sqsupseteq)$ is not complete.

For the precise formulation of the above idea we need some notation. If $A \in S_n'$ is l-periodic for some $l \in \mathbb{N}$, i.e. $A = \{a(1), \ldots, a(l)\}$, then put $A^2(l) = \{a(1), \ldots, a(l), a(1), \ldots, a(l)\}$. (That is, we write A into a $2l$-periodic form.) Moreover, if $A \in S_2'$ and A is l-periodic, $A = \{a(1), \ldots, a(l)\}$, then for $i \in \{1, \ldots, l\}$ let $A^{\langle i \rangle}(l) = \{a'(1), \ldots, a'(l)\}$ such that $a'(j) = a(j)$ for $j \neq i$ and $a'(i) = 3 - a(i)$. (That is, all the elements of A remains the same, except for the ith which is changed.) Finally, let u_k and v_k be integer sequences defined by $u_0 = v_0 = 0$, $u_k = u_{k-1} + k$, $v_k = v_{k-1} + k + 1$ for $k \geq 1$. Clearly, the length of the closed interval $I_t = [u_t, v_t]$ is t, and these intervals for $t \geq 0$ provide a partition of $\mathbb{N} \cup \{0\}$.

Now let $A_{-1} = \{1\}$, $B_{-1} = \{2\}$, and define the sequences A_k and B_k in S_n' in the following way. If $k \in I_t$ where t is even, then let $A_k = A_{k-1}^2(2^k)$, $B_k = \left(B_{k-1}^2(2^k)\right)^{\langle 2^k \rangle}(2^{k+1})$, and if t is odd then put $A_k = \left(A_{k-1}^2(2^k)\right)^{\langle 2^k \rangle} \times (2^{k+1})$, $B_k = B_{k-1}^2(2^k)$. (That is, to obtain A_k and B_k, we write the 2^k long periods of A_{k-1} and B_{k-1} after A_{k-1} and B_{k-1}, respectively, and then we change exactly one of the 2^{k+1}th long periods obtained, at the 2^kth place; the parity of t determines which period is to be modified. For the first few values of k we get $A_{-1} = \{1\}$, $A_0 = \{1, 1\}$, $A_1 = \{1, 2, 1, 1\}$, $A_2 = \{1, 2, 1, 2, 1, 2, 1, 1\}$, $A_3 = \{1, 2, 1, 2, 1, 2, 1, 1, 1, 2, 1, 2, 1, 2, 1, 1\}$ and $B_{-1} = \{2\}$, $B_0 = \{1, 2\}$, $B_1 = \{1, 2, 1, 2\}$, $B_2 = \{1, 2, 1, 2, 1, 2, 1, 2\}$, $B_3 = \{1, 2, 1, 2, 1, 2, 1, 1, 1, 2, 1, 2, 1, 2, 1, 2\}$.) From the definition of the sequences

A_k and B_k immediately follows that $A_{k-1} \sqsubseteq A_k$ and $B_{k-1} \sqsupseteq B_k$ for every $k \in \mathbb{N} \cup \{0\}$, with strict inequalities in both sequences infinitely often. Moreover, observe that for every $k \in \mathbb{N} \cup \{0, -1\}$ for the 2^{k+1} long periods $\{a^{(k)}(1), \dots, a^{(k)}(2^{k+1})\}$ and $\{b^{(k)}(1), \dots, b^{(k)}(2^{k+1})\}$ of A_k and B_k, respectively, we have $a^{(k)}(i) = b^{(k)}(i)$ for $1 \le i < 2^{k+1}$ and $a^{(k)}(2^{k+1}) < b^{(k)}(2^{k+1})$; this fact is just a straightforward consequence of the shape of A_{-1} and B_{-1}, and of the definition of the sequences. Now for every $i \in \mathbb{N}$ choose a $k \in \mathbb{N}$ with $2^{k+1} > i$, and put $c(i) = a^{(k)}(i)$ and $C = \{c(i) \ : \ i \in \mathbb{N}\}$. By the above mentioned property of A_k and B_k, C is well-defined. We show that $C \in S_n \setminus S'_n$. Indeed, from the definition of A_k and B_k it follows that for every $t \in \mathbb{N} \cup \{0\}$ we have $c(i) = c(i + ms)$ with $1 \le i \le 2^{u_t}$, $s = 2^{u_t}$ and $0 \le m \le 2^{v_t - u_t + 1} - 1$, but $c(2^{u_t}) \ne c(2^{v_t + 1}) = c(2^{u_{t+1}})$. However, it is impossible for any periodic sequence to have this property.

Let now $S = \{A_k \ : \ k \in \mathbb{N} \cup \{0, -1\}\}$, and suppose that $B = \bigvee S$ in S'_n for some $B = \{b(1), \dots, b(l)\}$. However, then in S_n $B \sqsupseteq C$, but $B \ne C$, hence for some $i \in \mathbb{N}$, $b(i) > c(i)$. But this implies $B_k \not\sqsupseteq B$ in S'_n, if $2^{k+1} > i$, which by $B_k \sqsupseteq A_j$, $j \in \mathbb{N} \cup \{0, -1\}$, contradicts $B = \bigvee S$ in S'_n. Hence S has no least upper bound in S'_n, which proves that $(S'_n, \sqsupseteq)$ is not complete. $\square$

The forthcoming proposition shows that $S'_n(l_{\ge})$ is not a "good" subset of S_n, in the sense that it does not form a nice structure even under $\sqsupseteq$. Of course, it is not surprising in view of the following observation: if A_1 and A_2 are in $S'_n(l_{\ge})$, then $A_1 \vee A_2$ and $A_1 \wedge A_2$ defined in S_n, does not belong to $S'_n(l_{\ge})$ in general.

Proposition 3.17. *$(S'_n(l_{\ge}), \sqsupseteq)$ is not a lattice for $n, l \in \mathbb{N}$ with $n \ge 2$ and $l \ge 6$.*

Proof. Let $n \ge 2$ be arbitrary. First observe that $\{1, 1, 2, 1\}$ and $\{1, 2, 1, 2, 1, 1\}$ for $6 \le l \le 11$, and $\{1, 1, 1, 1, 1, 1, 2, 1\}$ and $\{1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 1\}$ for $12 \le l \le 14$ have no least upper bound in $S'_n(l_{\ge})$.

Let now $l \ge 15$, and choose a prime p with $\max\{3, \frac{l}{6}\} < p \le \frac{l}{3}$. (By Bertrand's postulate, for $l \ge 18$ such a prime always exists, and for $l < 18$ we may take $p = 5$.) Put $A_1 = \{1, 2\}$ and $A_2 = \{1, 1, 1, \dots, 1, 1, 1, 2\}$, where the period of A_2 is $3p$ (i.e. the first $3p - 1$ elements of the $3p$ long period of A_2 equal to 1, and the last element is 2). Moreover, let $B_1 = \{1, 2, 2, 2, 1, 2\}$ and $B_2 = \{b^{(2)}(1), \dots, b^{(2)}(2p)\}$ with $b^{(2)}(i) = 1$ if i is odd but $i \neq p$ and $b^{(2)}(i) = 2$ otherwise (that is when i is even or $i = p$). Now it is easy to check that $A_1, A_2, B_1, B_2 \in S'_n(l_\geq)$, $A_1, A_2 \sqsubseteq B_1$, $A_1, A_2 \sqsubseteq B_2$, but B_1 and B_2 cannot be compared. Moreover, in S_n we have $A_1 \vee A_2 = B_1 \wedge B_2$. However, as $A_1 \vee A_2$ is of period $6p$ (but not of shorter period), it is not in $S'_n(l_\geq)$, hence A_1 and A_2 cannot have a least upper bound in $S'_n(l_\geq)$, which completes the proof of the proposition. □

Remark 3.18. We note that $(S'_n(l_\geq), \sqsupseteq)$ is a distributive lattice if $1 \leq l \leq 5$, for every $n \in \mathbb{N}$. We omit the trivial proof of this statement.

Proposition 3.19. *$(S'_n(l), \sqsupseteq)$ is a distributive lattice for every $n, l \in \mathbb{N}$.*

Proof. As for any $A_1, A_2 \in S'_n(l)$ the sequences $A_1 \vee A_2$ and $A_1 \wedge A_2$ defined in S_n are also in $S'_n(l)$, the statement is an immediate consequence of Proposition 3.14. □

4. Neighbourhood sequences in ∞D

Throughout this chapter we denote the set of infinite integer sequences by $\mathbb{Z}^\infty$, i.e. $\mathbb{Z}^\infty = \{(z_i)_{i=1}^\infty : z_i \in \mathbb{Z}\}$. We shall refer to the elements of $\mathbb{Z}^\infty$ as points.

Our purpose is to extend the result of the previous chapter, concerning $\mathbb{Z}^n$, to this general case. Moreover, we will extend Theorem 2.6, due to Das *et al.* [2] to $\mathbb{Z}^\infty$, too. First we give some definitions that are natural generalizations of the concepts in Chapter 2.

Definition 4.1. Let p and q be two points in $\mathbb{Z}^\infty$. The ith coordinate of the point p is indicated by $\Pr_i(p)$. The points p, q in $\mathbb{Z}^\infty$ are called N-neighbours for some $N \in \mathbb{N} \cup \{\infty\}$, if

- $\forall i \in \mathbb{N} : |\Pr_i(p) - \Pr_i(q)| \leq 1$,
- $\sum_{i=1}^\infty |\Pr_i(p) - \Pr_i(q)| \leq N$.

Definition 4.2. An infinite sequence $B = \{b(i) : i \in \mathbb{N}$ and $b(i) \in \mathbb{N} \cup \{\infty\}\}$ is called an ∞D-neighbourhood sequence. If for some $l \in \mathbb{N}$, $b(i) = b(i + l)$ holds for every $i \in \mathbb{N}$, then B is called periodic, with a period l, or simply l-periodic. In this case we will use the abbreviation $B = \{b(1), \ldots, b(l)\}$.

Definition 4.3. Let p and q be two points in $\mathbb{Z}^\infty$ and let B be an ∞D-neighbourhood sequence. The point sequence $\Pi(p,q;B)$ – having the form $p=p_0,\ p_1,\ldots,\ p_m=q$, where p_{i-1} and p_i are $b(i)$-neighbours for $1\le i\le m$ – is called a path of length m from p to q determined by B. If such a path exists, then the distance of p and q (determined by B) is defined as the common length of the shortest paths from p to q determined by B. It will be denoted by $d(p,q;B)$. If there is no path from p to q determined by B, then we put $d(p,q;B)=\infty$.

Remark 4.4. Observe that the following two statements are equivalent:

- $d(p,q;B)=\infty$ for every ∞D-neighbourhood sequence B,
- the set $\{|\operatorname{Pr}_i(p)-\operatorname{Pr}_i(q)| : i\in\mathbb{N}\}$ is unbounded.

To prove our main results concerning ∞D-neighbourhood sequences, we need three lemmas. The following result shows that for any ∞D neighbourhood sequence B, the function $d(p,q;B)$ has some "symmetry" properties. We note that by Theorem 2.6, the same is also true in nD for every $n\in\mathbb{N}$.

Lemma 4.5. *Let $B=\{b(i) : i\in\mathbb{N}\}$ be an ∞D-neighbourhood sequence, and $p,q\in\mathbb{Z}^\infty$. The distance value $d(p,q;B)$ depends only on the differences of the coordinates of the points, i.e. on the numbers $|\operatorname{Pr}_i(p)-\operatorname{Pr}_i(q)|$, $i\in\mathbb{N}$. Especially, for any ∞D-neighbourhood sequence B and $p,q\in\mathbb{Z}^\infty$ we have $d(p,q;B)=d(q,p;B)$.*

Proof. First, it is clear that for every $a,b,x\in\mathbb{Z}^\infty$ and an ∞D-neighbourhood sequence B, $d(a,b;B)=d(a-x,b-x;B)$ holds. Thus we may suppose that $p=o=(0,0,0,\ldots)$. Let $q,q'\in\mathbb{Z}^\infty$ with $|\operatorname{Pr}_i(q)|=|\operatorname{Pr}_i(q')|$ for every $i\in\mathbb{N}$. To prove the first part of the lemma, it is sufficient to show that $d(o,q;B)=d(o,q';B)$. To do this, first put $d(o,q;B)=k<\infty$ and let $o=q_0,q_1,\ldots,q_k=q$ be a path from o to q. For $i=0,\ldots,k$ define the points q_i' in the following way:

$$\operatorname{Pr}_j(q_i')=\begin{cases}\operatorname{Pr}_j(q_i), & \text{if } \operatorname{Pr}_j(q)=\operatorname{Pr}_j(q'),\\ -\operatorname{Pr}_j(q_i), & \text{if } \operatorname{Pr}_j(q)=-\operatorname{Pr}_j(q').\end{cases}$$

Now we have $q_0'=o$ and $q_k'=q'$. Moreover, for $i=1,\ldots,k$ and $j\in\mathbb{N}$, $\operatorname{Pr}_j(q_{i-1}')\ne\operatorname{Pr}_j(q_i')$ implies $\operatorname{Pr}_j(q_{i-1})\ne\operatorname{Pr}_j(q_i)$. Indeed, suppose that

$\Pr_j(q_{i-1}) = \Pr_j(q_i)$ and let $\varepsilon \Pr_j(q) = \Pr_j(q')$ with $\varepsilon \in \{1, -1\}$. However, by the definition of $\Pr_j(q'_{i-1})$ and $\Pr_j(q'_i)$, in this case we would have $\Pr_j(q'_{i-1}) = \varepsilon \Pr_j(q_{i-1}) = \varepsilon \Pr_j(q_i) = \Pr_j(q'_i)$, which would be a contradiction. Hence $\sum_{i=1}^{\infty} |\Pr_j(q'_i) - \Pr_j(q'_{i-1})| \leq \sum_{i=1}^{\infty} |\Pr_j(q_i) - \Pr_j(q_{i-1})|$, which shows that q'_{i-1} and q'_i are $b(i)$-neighbours. This proves $d(o, q'; B) \leq k$. Exchanging q and q' we have $d(o, q; B) = d(o, q'; B)$ in the case $k < \infty$. However, if $d(o, q; B) = \infty$, just as above we must have $d(o, q'; B) = \infty$, as well. Indeed, $d(o, q'; B) = k < \infty$ would imply $d(o, q; B) \leq k$ which would be a contradiction.

The second statement of the lemma is an immediate consequence of the first one. □

If the points $p, q \in \mathbb{Z}^\infty$ differ only at finitely many coordinates, then their distance is certainly finite (regardless of B), and the points of a shortest path connecting them belongs to an nD subspace of $\mathbb{Z}^\infty$ for some $n \in \mathbb{N}$. By this observation, the following two lemmas are obvious. However, since these lemmas play important roles in the proof of Theorem 4.8, and for the convenience of the reader, we provide the easy proofs of these statements.

Lemma 4.6. *Let p and q be two points in $\mathbb{Z}^\infty$ such that $|\{i : \Pr_i(p) \neq \Pr_i(q)\}| = c < \infty$. Let $B = \{b(i) : i \in \mathbb{N}\}$ be an ∞D-neighbourhood sequence, and let $A = \{a(i) : i \in \mathbb{N}\}$ be an ∞D-neighbourhood sequence with $a(i) = \min\{b(i), c\}$ for $i \in \mathbb{N}$. Then $d(p, q; A) = d(p, q; B)$.*

Proof. It is clear that $d(p, q; B) \leq d(p, q; A)$, so we have only to show the opposite relation. Put $H = \{i : \Pr_i(p) \neq \Pr_i(q)\}$, and $m = d(p, q; B)$. Since $c < \infty$, we have $m < \infty$. Hence there exists a path $p = p_0$, $p_1, \ldots,\ p_m = q$ from p to q determined by B in $\mathbb{Z}^\infty$. For $1 \leq i \leq m-1$ let

$$\Pr_j(q_i) = \begin{cases} \Pr_j(p_i), & \text{for } j \in H, \\ \Pr_j(p), & \text{for } j \in \mathbb{N} \setminus H, \end{cases}$$

and set $q_0 = p$ and $q_m = q$. Now by the definitions of A and the points q_i, it is clear that $q_0, q_1, \ldots, q_m$ is a path from p to q determined by A, which yields $d(p, q; A) \leq m = d(p, q; B)$, and the lemma is proved. □

Lemma 4.7. *Let B be an ∞D-neighbourhood sequence, and p and q two points in $\mathbb{Z}^\infty$ such that the set $H = \{i : \mathrm{Pr}_i(p) \neq \mathrm{Pr}_i(q)\}$ is finite. Let $d(p, q; B) = m$. Then there exists a path $p = q_0, q_1, \dots, q_m = q$ from p to q determined by B in $\mathbb{Z}^\infty$ such that for every $i = 1, \dots, m$ and $j \in \mathbb{N} \setminus H$ we have $\mathrm{Pr}_j(q_{i-1}) = \mathrm{Pr}_j(q_i)$.*

Proof. The points $q_0, \dots, q_m$ defined in the proof of Lemma 4.6 clearly have the desired properties. □

In Theorem 4.8 we describe how $d(p, q; B)$ can be calculated. This result is of independent interest, but it will be useful in the proof of Theorem 4.10, too.

Theorem 4.8. *Let p and q be two distinct points in $\mathbb{Z}^\infty$ such that the set $\{|\mathrm{Pr}_i(p) - \mathrm{Pr}_i(q)| : i \in \mathbb{N}\}$ is bounded, and let $B = \{b(i) : i \in \mathbb{N}\}$ be an ∞D-neighbourhood sequence. For $c \geq 1$ let $H_c = \{i : |\mathrm{Pr}_i(p) - \mathrm{Pr}_i(q)| \geq c\}$, and put $k = \min\{c : |H_c| < \infty\}$ and $h = |H_k|$. For $i \in \mathbb{N}$ let $a(i) = \min\{h, b(i)\}$, and $A = \{a(i) : i \in \mathbb{N}\}$. Moreover, put $r = (\mathrm{Pr}_i(p))_{i \in H_k}$ and $s = (\mathrm{Pr}_i(q))_{i \in H_k}$. Let t be defined by the following properties:*

- $b(t) = \infty$,
- $|\{i : i \leq t \text{ and } b(i) = \infty\}| = k - 1$.

If such t does not exist, then put

$$t = \begin{cases} 0, & \text{if } k = 1, \\ \infty, & \text{otherwise.} \end{cases}$$

Now the following equality holds:

$$d(p, q; B) = \max\{d_h(r, s; A), t\},$$

where for $h \geq 1$, $d_h(r, s; A)$ is the h-dimensional distance of r and s determined by A, and $d_0(r, s; A) = 0$.

Proof. By Lemma 4.5, without loss of generality we may suppose that $\mathrm{Pr}_i(p) \leq \mathrm{Pr}_i(q)$ for every $i \in \mathbb{N}$. First, it is clear that if $h = 0$, then $d(p, q; B) = t$. Assume that $h \geq 1$, and put $d_h(r, s; A) = x$. Let $r = r_0, r_1, \dots, r_x = s$ be a path from p to q determined by A in $\mathbb{Z}^h$. First suppose that $x \geq t$, and let $1 \leq m_1 < m_2 < \dots < m_t \leq x$ be the $k - 1$

indices with $b(m_i) = \infty$, $i = 1, \dots, t$. Set $M = \{m_1, \dots, m_t\}$. Define the points q_i for $i = 0, \dots, x$ in $\mathbb{Z}^\infty$ in the following way: $q_0 = p$,

$$\mathrm{Pr}_j(q_i) = \begin{cases} \mathrm{Pr}_j(r_i), & \text{for } j \in H_k,\ i \in \{1, \dots, x\}, \\ \mathrm{Pr}_j(q_{i-1}), & \text{for } j \in \mathbb{N} \setminus H_k, \\ & i \in \{1, \dots, x\} \setminus M, \\ \min\left(\mathrm{Pr}_j(q_{i-1}) + 1, \mathrm{Pr}_j(q)\right), & \text{for } j \in \mathbb{N} \setminus H_k,\ i \in M. \end{cases}$$

By this definition, the points q_{i-1}, q_i are $b(i)$-neighbours for $i = 1, \dots, x$. Moreover, we have $q_x = q$. Indeed, for $j \in H_k$ clearly $\mathrm{Pr}_j(q_x) = \mathrm{Pr}_j(r_x) = \mathrm{Pr}_j(q)$ holds. On the other hand, as $x \geq t$, for $j \in \mathbb{N} \setminus H_k$ the jth coordinate is increased $k-1$ times (if necessary), and we obtain $\mathrm{Pr}_j(q_x) = \mathrm{Pr}_j(q)$ for these indices, too. Hence in this case we have $d(p, q; B) \leq x = \max\{d_h(r, s; A), t\}$. Now suppose that $t > x$. If $t = \infty$, then there is nothing to prove, so suppose that $t < \infty$. Let the points $r_0, \dots, r_x$ be as before, and for $0 \leq i \leq x$ define the points q_i in the same way. For $i = x + 1, \dots, t$ put

$$\mathrm{Pr}_j(q_i) = \begin{cases} \min\left(\mathrm{Pr}_j(q_{i-1}) + 1, \mathrm{Pr}_j(q)\right), & \text{if } b(i) = \infty \text{ and } j \in \mathbb{N} \setminus H_k, \\ \mathrm{Pr}_j(q_{i-1}), & \text{otherwise.} \end{cases}$$

It is clear that for $i = 1, \dots, t$ the points q_{i-1}, q_i are $b(i)$-neighbours. Moreover, we have $q_t = q$. Indeed, for $j \in H_k$ we even have $\mathrm{Pr}_j(q_x) = \mathrm{Pr}_j(q)$. On the other hand, if $j \in \mathbb{N} \setminus H_k$, then the jth coordinate is increased (at most) $k - 1$ times, which yields $\mathrm{Pr}_j(q_t) = \mathrm{Pr}_j(q)$ for such indices, too. Hence, in this case we have again

$$d(p, q; B) \leq t = \max\{d_h(r, s; A), t\}.$$

Now we prove that $d(p, q; B) \geq \max\{d_h(r, s; A), t\}$. Let $d(p, q; B) = m < \infty$. (If $m = \infty$, then we are ready.) Let $p = p_0,\ p_1, \dots,\ p_m = q$ be a path from p to q determined by B in $\mathbb{Z}^\infty$. Let q_i for $i = 0, \dots, m$ be defined in the following way:

$$\mathrm{Pr}_j(q_i) = \begin{cases} \mathrm{Pr}_j(p_i), & \text{for } j \in H_k, \\ 0, & \text{for } j \in \mathbb{N} \setminus H_k. \end{cases}$$

Clearly, for $i = 1, \dots, m$, q_{i-1} and q_i are $b(i)$-neighbours. Hence, by Lemma 4.6 and Lemma 4.7, and by the definitions of r, s and A, we have

$$d(p, q; B) \geq d(q_0, q_m; B) = d(q_0, q_m; A) \geq d_h(r, s; A).$$

On the other hand, it is clear that $d(p, q; B) \geq t$. Indeed, let u, v, and w be any three points in $\mathbb{Z}^\infty$ such that u and v are $b(i)$-neighbours for some $i \in \mathbb{N}$. For $c \in \mathbb{N}$ let $U_c = \{i : |\operatorname{Pr}_i(u) - \operatorname{Pr}_i(w)| \geq c\}$, $V_c = \{i : |\operatorname{Pr}_i(v) - \operatorname{Pr}_i(w)| \geq c\}$, and put $k_u = \min\{c : |U_c| < \infty\}$, $k_v = \min\{c : |V_c| < \infty\}$. If $|U_c| = \infty$ or $|V_c| = \infty$ for every $c \in \mathbb{N}$, then put $k_u = \infty$ or $k_v = \infty$, respectively. Now, if $b(i) < \infty$, then clearly $k_u = k_v$, and even if $b(i) = \infty$, we have $|k_u - k_v| \leq 1$ (with the agreement $|\infty - \infty| = 0$). Hence, by the definition of k and t, we obtain $d(p, q; B) \geq t$, which implies $d(p, q; B) \geq \max\{d_h(r, s; A), t\}$. The proof of the theorem is now complete. □

Remark 4.9. It is interesting to note that combining the above result with the formula provided for $d(p, q; B)$ in Theorem 2.6, it is possible to calculate explicitely the distance of two points in $\mathbb{Z}^\infty$, determined by an ∞D-neighbourhood sequence. On the other hand, if we take $p, q \in \mathbb{Z}^\infty$ such that they differ at only finitely many places, then we have $k = 1$ whence $t = 0$ in Theorem 4.8. This shows that the distance defined in $\mathbb{Z}^\infty$ is in fact a generalization of the distances introduced in the finite dimensional cases.

The following result is the extension of Theorem 3.2 to $\mathbb{Z}^\infty$.

Theorem 4.10. *Let $B_c = \{b^{(c)}(i) : i \in \mathbb{N}\}$ $(c = 1, 2)$ be two ∞D-neighbourhood sequences. For $i, k \in \mathbb{N}$, $c = 1, 2$, put $f_k^{(c)}(j) = \sum_{i=1}^{j} \min(b^{(c)}(i), k)$. Then*

$$d(p, q; B_1) \leq d(p, q; B_2) \quad \text{for all } p, q \in \mathbb{Z}^\infty$$

if and only if

$$f_k^{(1)}(i) \geq f_k^{(2)}(i) \quad \text{for all } i \in \mathbb{N},\ k \in \mathbb{N}.$$

Proof. First we derive the second property from the first one. Contrary to the second statement, suppose that for some $j, h \in \mathbb{N}$, $f_h^{(1)}(j) < f_h^{(2)}(j)$ holds. Let

$$u_i = \begin{cases} |\{b^{(2)}(t) : b^{(2)}(t) = i,\ 1 \leq t \leq j\}|, & \text{for } 1 \leq i < h, \\ |\{b^{(2)}(t) : b^{(2)}(t) \geq i,\ 1 \leq t \leq j\}|, & \text{for } i = h, \end{cases}$$

and put $p = (0, 0, 0, \dots)$, $q = (u_1, u_1 + u_2, \dots, u_1 + u_2 + \cdots + u_h, 0, 0, \dots)$. For $c = 1, 2$ and $i \in \mathbb{N}$ set $a^{(c)}(i) = \min\{b^{(c)}(i), h\}$ and sequences $A_c = \{a^{(c)}(i) : i \in \mathbb{N}\}$. By Lemma 4.6, by the constructions of p, q and A_l, $l = 1, 2$, and by $f_h^{(1)}(j) < f_h^{(2)}(j)$, we have

$$d(p, q; B_1) = d(p, q; A_1) > d(p, q; A_2) = d(p, q; B_2),$$

which contradicts the first statement, and the first part of the theorem is proved.

Now we prove that the second statement implies the first one. Fix two arbitrary points, p and q in $\mathbb{Z}^\infty$. Without loss of generality we may suppose that the set $\{|\operatorname{Pr}_i(p) - \operatorname{Pr}_i(q)| : i \in \mathbb{N}\}$ is bounded, otherwise we have $d(p, q; B_1) = d(p, q; B_2) = \infty$. Let h, r, s, t_c and A_c denote the parameters, points and sequences corresponding to p, q, and B_c, $c = 1, 2$, defined in Theorem 4.8. Using this Theorem, it is sufficient to show that $t_1 \le t_2$ and $d_h(r, s; A_1) \le d_h(r, s; A_2)$ in $\mathbb{Z}^h$.

First suppose that $t_1 > t_2$. This implies that for some $n \in \mathbb{N}$, $|\{i : i \le n \text{ and } b^{(1)}(i) = \infty\}| < |\{i : i \le n \text{ and } b^{(2)}(i) = \infty\}|$. However, in this case for some m we clearly have $f_m^{(1)}(n) < f_m^{(2)}(n)$, which contradicts the second statement. Now we prove that $d_h(r, s; A_1) \le d_h(r, s; A_2)$. To do this, observe that A_c for $c = 1, 2$ is an hD-neighbourhood sequence. Moreover, for $1 \le k \le h$ the functions $f_k^{(c)}$ corresponding to B_l are just the same as those corresponding to A_c, $c = 1, 2$ in Theorem 3.2. Hence, by Theorem 3.2 we have $d_h(r, s; A_1) \le d_h(r, s; A_2)$, which completes the proof of the theorem. □

Now we study the structure of the ∞D-neighbourhood sequences. First we define two ordering relations on them, which are just the extensions of the finite dimensional orderings to this general case.

Definition 4.11. Let B_1 and B_2 be ∞D-neighbourhood sequences. We write $B_1 \sqsupseteq^* B_2$, if for every $i, k \in \mathbb{N}$, $f_k^{(1)}(i) \ge f_k^{(2)}(i)$ holds.

Remark 4.12. Let S_∞, S'_∞, $S'_\infty(l_\ge)$ and $S'_\infty(l)$ be the sets of generalized, periodic, at most l-periodic and l-periodic ($l \in \mathbb{N}$) ∞D-neighbourhood sequences, respectively. It is clear that $\sqsupseteq^*$ is an antisymmetric, transitive relation, i.e. a partial ordering on all these sets. However, just as in the finite dimensional case, $(S_\infty, \sqsupseteq^*)$, $(S'_\infty, \sqsupseteq^*)$, $(S'_\infty(l_\ge), \sqsupseteq^*)$ and $(S'_\infty(l), \sqsupseteq^*)$ with $l \ge 2$ are not lattices. These statements are simple consequences of Propositions 3.6 through 3.11.

Definition 4.13. By the above notation, for $B_1 = \{b^{(1)}(i) : i \in \mathbb{N}\}$, $B_2 = \{b^{(2)}(i) : i \in \mathbb{N}\} \in S_\infty$ we write $B_1 \sqsupseteq B_2$ if and only if $b^{(1)}(i) \geq b^{(2)}(i)$ for every $i \in \mathbb{N}$.

It turns out that $\sqsupseteq$ has much more pleasant properties than $\sqsupseteq^*$ in case of ∞D-neighbourhood sequences, too.

Proposition 4.14. *$(S_\infty, \sqsupseteq)$, $(S'_\infty, \sqsupseteq)$ and $(S'_\infty(l), \sqsupseteq)$ for any $l \geq 1$ are distributive lattices, with greatest lower bound $\{1\}$ and least upper bound $\{\infty\}$. Moreover, the first and third lattices are complete, but the second one is not.*

Proof. The statement is a simple consequence of Propositions 3.14, 3.15, 3.16 and 3.19. We omit the details. □

Remark 4.15. $(S'_\infty(l_\geq), \sqsupseteq)$ is a complete distributive lattice for $1 \leq l \leq 5$. If $l \geq 6$, then $(S'_\infty(l_\geq), \sqsupseteq)$ is not a lattice. The proofs of these statements are trivial (see Proposition 3.17 and Remark 3.18).

The following result provides some information about the structures of several other subsets of S_∞ under $\sqsupseteq$.

Proposition 4.16. *Let $S^*_\infty = \{B : B = \{b(i) : i \in \mathbb{N}$ and $b(i) \in \mathbb{N}\}\}$, $S^+_\infty = \bigcup_{n=1}^\infty S_n$ and $S^-_\infty = \bigcup_{n=1}^\infty S'_n$, where S_n and S'_n are defined in Definition 3.3. Then $(S^*_\infty, \sqsupseteq)$, $(S^+_\infty, \sqsupseteq)$ and $(S^-_\infty, \sqsupseteq)$ are non-complete distributive lattices (sublattices of S_∞) with greatest lower bound $\{1\}$. S^*_∞, S^+_∞ and S^-_∞ have no least upper bounds.*

Proof. Let S be any of the sets S^*_∞, S^+_∞, S^-_∞. It is evident that for each $A_1, A_2 \in S$, $A_1 \vee A_2$ and $A_1 \wedge A_2$ defined in S_∞, also belong to S. Hence these three sets clearly form distributive lattices under $\sqsupseteq$. It is obvious that $\{1\}$ is the greatest lower bound of each lattice, and that they have no least upper bounds. The last observation immediately implies the non-completeness of the lattices. □

Finally, we show that the lattice $(S_\infty, \sqsupseteq)$ (and in some special cases $(S'_\infty(l_\geq), \sqsupseteq)$ or $(S'_\infty(l), \sqsupseteq)$) can be considered as the closure of the union of the finite dimensional lattices, studied in the previous chapter. We also include the lattices $(S'_\infty, \sqsupseteq)$ and $(S^*_\infty, \sqsupseteq)$ into this consideration.

Proposition 4.17. *Use the previous notation, and put* $S^-_\infty(l_\geq) = \bigcup_{n=1}^\infty S'_n(l_\geq)$ *and* $S^-_\infty(l) = \bigcup_{n=1}^\infty S'_n(l)$ *for* $l \in \mathbb{N}$, *where* $S'_n(l_\geq)$ *and* $S'_n(l)$ *are defined in Definition 3.3. In* S_∞, $(S_\infty, \sqsupseteq)$ *is the closure of the lattices* $(S'_\infty, \sqsupseteq)$, $(S^*_\infty, \sqsupseteq)$, $(S^+_\infty, \sqsupseteq)$, $(S^-_\infty, \sqsupseteq)$, $(S^-_\infty(l_\geq), \sqsupseteq)$ *and* $(S^-_\infty(l), \sqsupseteq)$. *Moreover, in* $S'_\infty(l)$ *for every* $l \in \mathbb{N}$, $(S'_\infty(l), \sqsupseteq)$ *is the closure of* $(S^-_\infty(l), \sqsupseteq)$. *Finally, in* $S'_\infty(l_\geq)$ *for* $1 \leq l \leq 5$, $(S'_\infty(l_\geq), \sqsupseteq)$ *is the closure of* $(S^-_\infty(l_\geq), \sqsupseteq)$.

Proof. The statements are trivial by noting that S_∞, $S'_\infty(l)$ $(l \in \mathbb{N})$ and $S'_\infty(l_\geq)$ $(1 \leq l \leq 5)$ are complete lattices with respect to $\sqsupseteq$, and that all the examined sets have the least upper bound $\{\infty\}$ in S_∞ (or in $S'_\infty(l)$ or $S'_\infty(l_\geq)$, respectively). $\square$

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