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# A new approach to generalized Berwald manifolds II

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**Abstract.** Generalized Berwald manifolds were introduced by V. V. Wagner and systematically investigated by M. Hashiguchi and Y. Ichijyō. They are reconsidered here in the context and with the tools of the general theory developed in the first part of our work. (However, this second part is self-contained to a reasonable extent.) Under some natural conditions we establish key relations between a horizontal endomorphism and the distinguished Barthel endomorphism on a Finsler manifold. We construct intrinsically a vector field which plays a dominant role in these and further, geometrically relevant relations. In the case of a generalized Berwald manifold $(M, E, \nabla)$ the linear connection $\nabla$ is far from unique. Our results enable us to link different generalized Berwald manifolds with common Finsler structure. Applications to Wagner manifolds and a family of examples (parallelizable manifolds endowed with one-form Finsler structure) illustrate how the general theory works in practice.

## 0. Introduction

**0.1.** In the first part ([12]) of our paper we have already introduced the generalized Berwald manifolds, providing also a preliminary characterization of this concept in terms of Ichijyō connections. – Let us mention that we called a triplet $(M, E, \nabla)$ a generalized Berwald manifold if $(M, E)$ is a Finsler manifold, $\nabla$ is a linear connection on $M$, and the horizontal endomorphism $h_\nabla$ generated by $\nabla$ is conservative, i.e., $d_{h_\nabla} E = 0$. Thus a Finsler structure is nicely related to a linear connection.

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**0.2.** A quite immediate but important consequence of the definition is that generalized Berwald manifolds are *Finsler manifolds modeled on a Finsler–Minkowski vector space* in the sense of Y. Ichijyō [9]. This means that any two tangent spaces of the Finsler manifold $(M, E)$ in question are isomorphic as Finsler–Minkowski vector spaces (see 5.1 below); the Finsler–Minkowski functional on a tangent space $T_pM$ is given by the rule $v \in T_pM \mapsto \sqrt{2E(v)} \in \mathbb{R}$.

**0.3.** It will be worth-while to present here a brief conceptual justification of our above assertion. – Let $(M, E, \nabla)$ be a generalized Berwald manifold. Choose two (different) points $p$, $q$ of $M$ and connect them with a smooth curve $c : [0,1] \to M$. Fix an arbitrary vector $v \in T_pM$ and consider the unique parallel vector field $X : [0,1] \to TM$ along $c$ satisfying the initial condition $X(0) = v$. Let $w := X(1)$. It is enough to check that $E(v) = E(w)$. – Let $\nabla_c$ be the covariant differentiation induced by $\nabla$ along $c$. Consider the connector

$$K_\nabla := \iota \circ (1_{TTM} - h_\nabla) \quad (\iota : VTM \to TM \text{ is the canonical map})$$

belonging to $\nabla$. Then

$$0 = \nabla_c X = K_\nabla \circ TX \circ \frac{d}{du} =: K_\nabla \circ \dot{X} \qquad (u := 1_{\mathbb{R}}).$$

Since $\operatorname{Ker} K_\nabla = \operatorname{Im} h_\nabla$, this means that the vector field $\dot{X} := TX \circ \frac{d}{du} : T\mathbb{R} \cong \mathbb{R} \to TTM$ is horizontal: $h_\nabla \circ \dot{X} = \dot{X}$. Using this trivial observation and (1.3c) below, for any $\tau \in [0,1]$ we obtain:

$$\begin{aligned}(E \circ X)'(\tau) &= T(E \circ X)\left(\frac{d}{du}\right)_\tau = TE \circ TX \circ \frac{d}{du}(\tau) = TE \circ \dot{X}(\tau) \\ &= dE(\dot{X}(\tau)) = dE\big[h_\nabla(\dot{X}(\tau))\big] = d_{h_\nabla} E(\dot{X}(\tau)) = 0.\end{aligned}$$

Thus $E \circ X$ is constant along $c$, which implies our claim $E(v) = E(w)$.

**0.4.** The notion of generalized Berwald manifold was originally introduced by V. V. Wagner in 1943 [19]. A modern approach to these manifolds within the framework of Matsumoto's theory, via the so-called generalized Cartan connections, was elaborated by M. Hashiguchi [8]. Our definition provides another, geometrically natural approach which is also in harmony with Matsumoto's principle of the "best" Finsler connections cited in part 1.

**0.5.** The first question that offers itself in connecetion with a generalized Berwald manifold is without any doubt the following: *to what extent is the linear connection* $\nabla$ *determined by the structure?* Concerning this problem, we are going to show that *two generalized Berwald manifolds* $(M, E, \nabla_1)$ *and* $(M, E, \nabla_2)$ *are equal if* $\nabla_1$ *and* $\nabla_2$ *have the same torsion tensor field.* More or less, this result is analogous to the well-known theorem: two linear connections on a manifold are equal if they have common geodesics and their torsion tensor fields are also the same. To derive our theorem (and for other purposes), in Section 2 we present under some – as far as possible "natural" – conditions a careful analysis of the relations between two horizontal endomorphisms given on the same manifold. "Natural conditions" in the Finslerian case certainly do exist. For example: let both horizontal endomorphisms be conservative, or let one of them be conservative and the other be the distinguished Barthel endomorphism. Nevertheless, useful relations can also be discovered in a much more general situation (see 2.1).

**0.6.** Let $(M, E)$ be a Finsler manifold. Suppose that $h$ is a horizontal endomorphism on $M$ with weak torsion $t$, and let $t^\circ := i_S t$ ($S$ is an arbitrary semispray on $M$) be the potential of $t$. We can consider the one-form $d_{t^\circ} E$, and we can construct the vector field $(d_{t^\circ} E)^\#$ which corresponds canonically to $d_{t^\circ} E$ via the fundamental two-form of $(M, E)$. In our opinion this – unfortunately, a bit complicated – vector field is at the heart of the problems (and difficulties) concerning generalized Berwald manifolds. As a justification, we refer to Propositions 2.5, 2.7; the proof of 3.8, and the key relations (4.5a) and (4.6a) below. In particular, a direct conclusion will be that a *generalized Berwald manifold* $(M, E, \nabla)$ *becomes a Berwald manifold* $(M, E)$ *if, and only if, the vector field* $(d_{\overset{\circ}{t}_{\nabla}} E)^\#$ *is quadratic.*

**0.7.** In Section 4 we deduce some useful equivalents of the property characterizing Wagner manifolds. This result has essential applications in the theory of conformal changes of a Finsler structure, see [18].

**0.8.** *Non-Berwald generalized Berwald manifolds do exist.* In Section 5 we offer a typical family of such manifolds together with their basic data. First, we build a special Finsler structure, the so-called *one-form metric*, on a parallelizable manifold. Next, we present an elegant proof of the fact, discovered originally by Y. Ichijyō, that our construction actually results in generalized Berwald manifolds. – Let us note that a systematic study of one-form metrics can be found in [11]. These metrics also occur in an interesting context in [16].

## 1. Basic setup

**1.1.** Since this paper is an immediate continuation of our previous work [12], we adopt its conceptual and notational conventions without any changes, and – in most cases – without any comment. Double numbers in italics (i.e., of form *9.99*) will refer to the first part.

Our "philosophical" attitude remains unaltered: we try to elaborate a transparent intrinsic formulation on a reasonable level of generality for the problems studied. The calculative background of our considerations is the Frölicher–Nijenhuis calculus on the velocity space, i.e., on the tangent manifold $TM$ of the given manifold $M$ (1.1). In the present second part this, a bit complicated, apparatus will be applied more explicitly and intensively than in the first part. So, for the readers' convenience, we collect here some basic facts and frequently used formulas. A more complete overview of these technical tools is available in Youssef's paper [22]; we also refer to J. Klein's stimulating survey article [10], the monograph [5], and last but not least, the original source [6].

**1.2.** Let $\Omega(M)$ be the graded algebra of the differential forms of our base manifold $M$. If $D_1$ and $D_2$ are (graded) derivations of degree $r$ and $s$ $(r, s \in \mathbb{Z})$ of $\Omega(M)$, then their bracket is

$$[D_1, D_2] := D_1 \circ D_2 - (-1)^{rs} D_2 \circ D_1; \tag{1.2a}$$

this is a graded derivation of degree $r + s$.

**1.3.** Let us denote by $\Psi^k(M)$ ($k \in \mathbb{N}$, $\Psi^\circ(M) := \mathfrak{X}(M)$) the $C^\infty(M)$-module of the *vector* $k$-forms on $M$. We recall that any vector $k$-form $K \in \Psi^k(M)$ can be interpreted as a skew-symmetric $C^\infty(M)$-multilinear map $[\mathfrak{X}(M)]^k \to \mathfrak{X}(M)$ (if $k \in \mathbb{N} \setminus \{0\}$); in particular, a vector 1-form is just a type $(1,1)$ tensor field on $M$. In the Frölicher–Nijenhuis theory to any vector $k$-form $K \in \Psi^k(M)$ two derivations of $\Omega(M)$ are associated:

(1.3a) *the derivation $i_K$ of degree $k-1$ defined by the rule*

$$i_K \restriction C^\infty(M) = 0; \quad i_K\omega := \omega \circ K, \quad \textit{if } \omega \in \Omega^1(M);$$

(1.3b) *the derivation $d_K$ of degree $k$ given by the formula*

$$d_K := [i_K, d] \overset{1.2\mathrm{a}}{=} i_K \circ d - (-1)^{k-1} d \circ i_K$$

($d$ is the operator of the "ordinary" exterior derivative).

As an immediate consequence, we obtain

(1.3c) *if* $f \in C^\infty(M)$ *and* $K \in \Psi^k(M)$, *then* $d_K f = i_K df = df \circ K$.

A characteristic property of $d_K$ is expressed by

$$[d, d_K] = 0. \tag{1.3d}$$

**1.4.** For any vector $k$-form $K \in \Psi^k(M)$ and vector $\ell$-form $L \in \Psi^\ell(M)$ there exists a unique vector $(k+\ell)$-form $[K, L] \in \Psi^{k+\ell}(M)$ such that

$$d_{[K,L]} = [d_K, d_L];$$

$[K, L]$ is said to be the *Frölicher–Nijenhuis bracket* of $K$ and $L$. This bracket is *graded anticommutative* and satisfies the *graded Jacobi identity*, i.e., for any vector forms $K^i \in \Psi^{k_i}(M)$ $(1 \le i \le 3)$ we have

$$[K_1, K_2] = -(-1)^{k_1 k_2}[K_2, K_1]; \tag{1.4a}$$

$$(-1)^{k_1 k_3}\big[K_1, [K_2, K_3]\big] + (-1)^{k_2 k_1}\big[K_2, [K_3, K_1]\big] + (-1)^{k_3 k_2}\big[K_3, [K_1, K_2]\big] = 0. \tag{1.4b}$$

In particular, let us suppose that $K$ and $L$ are vector *one-forms*. Then the following important formulas can be deduced:

(1.4c) $[K, Y]X = [KX, Y] - K[X, Y]$;

(1.4d) $[fX, K] = f[X, K] + df \wedge i_X K - d_K f \otimes X$;

(1.4e) $[K, fL] = f[K, L] + d_K f \wedge L - df \wedge (K \circ L)$;

(1.4f) $[K, \omega \otimes X] = d_K\omega \otimes X - d\omega \otimes KX + (-1)^r \omega \wedge [K, X]$;

(1.4g) $i_X \circ i_K = i_K \circ i_X + i_{KX}$;

(1.4h) $i_K \circ i_L = i_L \circ i_K + i_{L\circ K} - i_{K\circ L}$;

(1.4i) $i_X \circ d_K = -d_K \circ i_X + \mathcal{L}_{KX} + i_{[K,X]}$;

(1.4j) $i_K \circ d_X = d_X \circ i_K + i_{[K,X]}$;

(1.4k) $\quad i_K \circ d_L = d_L \circ i_K + d_{L\circ K} - i_{[K,L]}$

($X, Y \in \mathfrak{X}(M)$, $f \in C^\infty(M)$, $\omega \in \Omega^r(M)$; note that $d_X$ is just the Lie-derivative $\mathcal{L}_X$). – Observing that $d_{[K,Y]} := [d_K, d_Y] = [d_K, \mathcal{L}_Y]$, (1.4c) can be deduced immediately. (1.4d)–(1.4f) are stated in [22]. Finally, formulas (1.4g)–(1.4k) can be obtained as special cases of (5.6)a) and (5.9) of [6].

**1.5.** In our calculations some special derivations of the algebra $\Omega(TM)$ of the differential forms on the tangent manifold $TM$ will play a distinguished role. The most frequently used operators are

$$i_C,\ i_J,\ d_C = \mathcal{L}_C, \quad d_J,\ i_S,\ d_S = \mathcal{L}_S,$$

where $C \in \mathfrak{X}^{\mathrm{v}}(TM)$ is the *Liouville vector field*, $J \in \Psi^1(TM)$ is the *vertical endomorphism* and $S$ is an arbitrary semispray on $M$. Taking into account that

(1.5a) $$\operatorname{Im} J = \operatorname{Ker} J = \mathfrak{X}^{\mathrm{v}}(TM), \quad J^2 = 0;$$

(1.5b) $$[J, C] = J, \quad [J, J] = 0;$$

(1.5c) $$JS = C,$$

(1.4g), (1.4i) and (1.4j) yield the following relations:

(1.5d) $$i_C \circ i_J = i_J \circ i_C;$$

(1.5e) $$[i_C, d_J] = i_J;$$

(1.5f) $$[i_J, \mathcal{L}_C] = i_J.$$

**1.6.** A vector $k$-form $K \in \Psi^k(TM)$ ($k \in \mathbb{N} \setminus \{0\}$) is said to be *semibasic*, if $J \circ K = 0$ and, for any vector field $X \in \mathfrak{X}(TM)$, $i_{JX}K = 0$. The *potential* of a semibasic vector $k$-form $K \in \Psi^k(TM)$ is the vector $(k-1)$-form

$$K^\circ := i_S K,$$

where $S$ in an arbitrary semispray. – For more details, see [7] or [5].

**1.7.** We recall that the *complete lift* $\alpha^c$ of a function $\alpha \in C^\infty(M)$ can be introduced by

$$\alpha^c := S(\alpha \circ \pi) =: S\alpha^{\mathrm{v}}, \tag{1.7a}$$

where $S$ is again an arbitrary semispray on $M$. Then the complete lift of a vector field $X \in \mathfrak{X}(M)$ is the unique vector field $X \in \mathfrak{X}(TM)$ satisfying

$$\forall\, \alpha \in C^\infty(M) : X^c\alpha^c = (X\alpha)^c. \tag{1.7b}$$

The following useful relations can be deduced easily:

$$X^c\alpha^{\mathrm{v}} = X^{\mathrm{v}}\alpha^c = (X\alpha)^{\mathrm{v}}; \tag{1.7c}$$

$$[X,Y]^c = [X^c, Y^c], \quad [X,Y]^{\mathrm{v}} = [X^{\mathrm{v}}, Y^c]; \tag{1.7d}$$

$$[C, X^c] = 0;\ \text{i.e., } X^c \textit{ is homogeneous of degree } 1; \tag{1.7e}$$

$$JX^c = X^{\mathrm{v}}, \quad [J, X^c] = 0 \tag{1.7f}$$

$(X, Y \in \mathfrak{X}(M);\ X^{\mathrm{v}} \in \mathfrak{X}^{\mathrm{v}}(TM)$ is the vertical lift of $X$; $\alpha \in C^\infty(M))$.

**1.8. Sharp operator and gradient on a Finsler manifold.** Suppose that $(M, E)$ is a Finsler manifold with the fundamental form $\omega := dd_JE$. If $\beta$ is a 1-form on $\mathcal{T}M$, we denote by $\beta^\#$ (read: "$\beta$ sharp") the vector field corresponding to $\beta$ via $\omega$; i.e.,

$$i_{\beta^\#}\omega = \beta. \tag{1.8a}$$

In particular, the *gradient* of a function $f \in C^\infty(TM)$ is the vector field

$$\operatorname{grad} f := (df)^\#.$$

The gradient of a *vertical lift*, i.e., of a function of form $\alpha^{\mathrm{v}} = \alpha \circ \pi$, $\alpha \in C^\infty(M)$ has the following nice properties:

$$\operatorname{grad}\alpha^{\mathrm{v}} \in \mathfrak{X}^{\mathrm{v}}(\mathcal{T}M); \tag{1.8a}$$

$$[C, \operatorname{grad}\alpha^{\mathrm{v}}] = -\operatorname{grad}\alpha^{\mathrm{v}};\ \text{i.e., } \operatorname{grad}\alpha^{\mathrm{v}} \textit{ is homogeneous of degree } 0; \tag{1.8b}$$

$$(\operatorname{grad}\alpha^{\mathrm{v}})E = \alpha^c \tag{1.8c}$$

(see [14], Proposition 1).

**1.9. Theorems of M. Crampin and J. Grifone.** *The following substantial results, due to* M. Crampin [3], [4] *and* J. Grifone [7] *are among the most important theorems of the theory of connections and lie at the foundations of Finsler geometry. They will be repeatedly referred to also in our subsequent considerations.*

(A) *If $S$ is a semispray on a manifold $M$, then*

$$h := \frac{1}{2}\left(1_{\mathfrak{X}(TM)} + [J, S]\right)$$

*is a horizontal endomorphism on $M$ with vanishing weak torsion. If, in particular, $S$ is a spray, then the horizontal endomorphism $h$ is homogeneous, i.e., $H := [h, C] = 0$.*

(B) *A horizontal endomorphism arises from a semispray in the above manner if and only if its weak torsion vanishes.*

(C) *On any Finsler manifold $(M, E)$ there exists a unique conservative horizontal endomorphism with vanishing strong torsion; it is given by the formula*

$$h_0 = \frac{1}{2}\left(1_{\mathfrak{X}(TM)} + [J, S_0]\right),$$

*where $S_0$ is the canonical spray of the Finsler manifold. – $h_0$ is said to be the Barthel endomorphism of $(M, E)$.*

**1.10.** We conclude this overview with a practical convention. – The basic geometric data – such as associated semispray, tension, weak and strong torsion, almost complex structure, horizontal lifting – arising from a horizontal endomorphism $h$ or $\widetilde{h}$ will be denoted by

$$S,\ H,\ t,\ T,\ F,\ X^h \quad \text{and} \quad \widetilde{S},\ \widetilde{H},\ \widetilde{t},\ \widetilde{T},\ \widetilde{F},\ X^{\widetilde{h}} \qquad (X \in \mathfrak{X}(M)),$$

respectively. The corresponding data determined by the Barthel endomorphism $h_0$ are

$$S_0,\ H_0,\ t_0,\ T_0,\ F_0,\ X^{h_0}.$$

In particular, any linear connection $\nabla$ on $M$ gives rise to a horizontal endomorphism $h_\nabla$. Then the above data are denoted by

$$S_\nabla,\ H_\nabla,\ t_\nabla,\ T_\nabla,\ F_\nabla,\ X^{h_\nabla}$$

(here, in fact, $H_\nabla = 0$).

## 2. Horizontal endomorphisms on a Finsler manifold

**2.1. Lemma.** *Suppose that $h$ is a homogeneous horizontal endomorphism on the manifold $M$ and let $S$ be the semispray associated with $h$* ([7], Prop. I.38). *If $\widetilde{h}$ is the horizontal endomorphism determined by $S$ according to 1.9.* (A), *then $h$ and $\widetilde{h}$ are related by*

$$\widetilde{h} = h - \frac{1}{2}t^\circ,$$

*where $t$ is the weak torsion of $h$, and $t^\circ$ is its potential.*

Proof. Since $h$ is homogeneous, its associated semispray $S$ is actually a spray. In view of 1.9. (A), the weak torsion of $\widetilde{h}$ vanishes. $\widetilde{h}$ is also homogeneous, because it is generated by the *spray* $S$ ([7], Proposition I.41). Thus

$$\widetilde{h}S = S, \quad hS = S;$$

and the vector 1-form

$$K := \widetilde{h} - h$$

is obviously semibasic, therefore

$$J \circ K = K \circ J = 0.$$

Since

$$0 = \widetilde{t} := [J, \widetilde{h}] = [J, h + K] = [J, h] + [J, K] = t + [J, K],$$

it follows that $t = -[J, K]$, hence

$$t^\circ = -[J, K]^\circ.$$

The vector 1-form $[J, K]^0$ is clearly semibasic, so it is determined by its action on the complete lift vector fields. Taking into account our previous observations, for any vector field $X$ on $M$,

$$\begin{aligned}
[J, K]^\circ(X^c) = [J, K](S, X^c) &\overset{(6.4) \text{ of } [6]}{=} [JS, KX^c] + [KS, JX^c] \\
&+ J \circ K[S, X^c] + K \circ J[S, X^c] - J[S, KX^c] - J[KS, X^c] \\
&- K[S, JX^c] - K[JS, X^c] = [C, KX^c] - J[S, KX^c] - K[S, X^{\mathrm{v}}].
\end{aligned}$$

On the right hand side the first term vanishes by the homogeneity of $h$ and $\widetilde{h}$:

$$[C, KX^c] = [C, \widetilde{h}X^c] - [C, hX^c] = [C, X^{\widetilde{h}}] - [C, X^h] = 0.$$

As for the second term, we get

$$J[S, KX^c] \overset{1.2}{=} -KX^c = (h - \widetilde{h})X^c.$$

The third term can be formed as follows:

$$\begin{aligned} K[S, X^{\mathrm{v}}] &= \widetilde{h}[S, X^{\mathrm{v}}] - h[S, X^{\mathrm{v}}] = \widetilde{F} \circ J[S, X^{\mathrm{v}}] - F \circ J[S, X^{\mathrm{v}}] \\ &= -\widetilde{F}X^{\mathrm{v}} + FX^{\mathrm{v}} = -\widetilde{F} \circ JX^c + F \circ JX^c = (h - \widetilde{h})X^c. \end{aligned}$$

To sum up, it can be stated that

$$\forall\, X \in \mathfrak{X}(M) : [J, K]^\circ(X^c) = 2(\widetilde{h} - h)X^c, \quad \text{i.e.,} \quad [J, K]^\circ = 2(\widetilde{h} - h).$$

Hence $t^\circ = 2(h - \widetilde{h})$, which proves our assertion. □

**2.2. Lemma.** *If $\omega$ is the fundamental two-form of the Finsler manifold $(M, E)$ and $h$ is a conservative horizontal endomorphism on $M$, then*

$$i_h\omega = \omega + i_t dE,$$

*where $t$ is the weak torsion of $h$.*

Proof. In view of (1.4k)

$$i_h \circ d_J = d_J \circ i_h + d_{J \circ h} - i_{[h,J]} \overset{(\,1.3\mathrm{a}),\,(1.4\mathrm{b})}{=} d_J \circ i_h + d_J - i_t,$$

therefore

$$\begin{aligned} i_h\omega = i_h dd_J E &\overset{(1.3\mathrm{d})}{=} -i_h d_J dE \\ &= -d_J i_h dE - d_J dE + i_t dE \\ &= dd_J E + i_t dE = \omega + i_t dE \end{aligned}$$

(using that $i_h dE = 0$ by the conservativity of $h$). □

**2.3. Corollary.** *If $\omega$ is the fundamental two-form of the Finsler manifold $(M, E)$ and $h$ is a conservative horizontal endomorphism on $M$ with vanishing weak torsion then $i_h\omega = \omega$.*

**2.4. Lemma.** *Let $h$ be a conservative horizontal endomorphism on the Finsler manifold $(M, E)$. Then*

$$d_H E = 0,$$

*where $H$ is the tension of $h$.*

Proof. Take an arbitrary vector field $X$ on $M$. An easy calculation shows that $H(X^c) = [X^h, C]$, so

$$\begin{aligned} d_H E(X^c) &\overset{(1.3c)}{=} dE(HX^c) = dE\big([X^h, C]\big) = [X^h, C]E \\ &= X^h(CE) - C(X^h E) = X^h(2E) = 0, \end{aligned}$$

since $h$ is conservative. $\square$

**2.5. Proposition.** *Suppose that $h$ is a conservative horizontal endomorphism on the Finsler manifold $(M, E)$ with the associated semispray $S$. Then $S$ can be represented in the form*

$$S = S_0 + (d_{t^\circ} E)^{\#},$$

*where $S_0$ is the canonical spray of $(M, E)$ and $t^\circ$ is the potential of the weak torsion of $h$.*

Proof. Let us note first that the general formula (5.6)(a) of [6] yields the relation

$$i_{S_0} \circ i_t = i_t \circ i_{S_0} + i_{t^\circ},$$

while using (1.4g) we obtain

$$i_h \circ i_{S_0} = i_{S_0} \circ i_h - i_{hS_0} = i_{S_0} \circ i_h - i_S.$$

Thus, since $h$ is conservative,

$$0 = d_h E \overset{(1.3c)}{=} i_h dE = -i_h i_{S_0}\omega = i_S\omega - i_{S_0} i_h\omega.$$

Hence

$$\begin{aligned} i_S\omega &= i_{S_0} i_h \omega \overset{2.2}{=} i_{S_0}(\omega + i_t dE) = i_{S_0}\omega + i_{S_0} i_t dE \\ &= i_{S_0}\omega + i_t i_{S_0} dE + i_{t^\circ} dE = i_{S_0}\omega + d_{t^\circ} E, \end{aligned}$$

taking into account that $i_{S^0} dE = S_0 E = 0$ by the "energy conservation law". The result we have just obtained can be written in the form

$$i_{S-S_0}\omega = d_{t^\circ} E.$$

This means by (1.8a) that $S - S_0 = (d_{t^\circ} E)^{\#}$. □

**2.6. Theorem.** *Suppose $h$ and $\widetilde{h}$ are conservative horizontal endomorphism on the Finsler manifold $(M, E)$. If $h$ and $\widetilde{h}$ have common strong torsion, then $h = \widetilde{h}$.*

Proof. We are going to use systematically the conventions of 1.10. By assumption,

$$d_h E = d_{\widetilde{h}} E = 0, \qquad T = \widetilde{T}.$$

Let $S_0$ be the canonical spray of $(M, E)$. As we have seen in the preceding proof,

$$i_S\omega - i_{S_0}\omega = d_{t^\circ} E, \qquad i_{\widetilde{S}}\omega - i_{S_0}\omega = d_{\widetilde{t}^\circ} E.$$

Subtracting the second equation from the first we obtain

$$i_{S-\widetilde{S}}\omega = d_{t^\circ} E - d_{\widetilde{t}^\circ} E.$$

The strong torsion of $h$ is $T = t^\circ + H$, so

$$d_{t^\circ} E = d_{T-H} E = d_T E - d_H E \overset{2.4.}{=} d_T E.$$

In the same way,

$$d_{\widetilde{t}^\circ} E = d_{\widetilde{T}} E.$$

Combining the last three formulas we obtain

$$i_{S-\widetilde{S}}\,\omega = d_T E - d_{\widetilde{T}} E = d_T E - d_T E = 0.$$

Since $\omega$ is nondegenerate, this implies that $S = \widetilde{S}$. Thus $h$ and $\widetilde{h}$ have also the same associated semispray, therefore, by Proposition 4.9.2 of [5], $h$ and $\widetilde{h}$ coincide. □

**2.7. Proposition.** *A homogeneous, conservative horizontal endomorphism $h$ on a Finsler manifold $(M,E)$ can be expressed with the help of the Barthel endomorphism $h_0$ as follows:*

$$h = h_0 + \frac{1}{2}t^\circ + \frac{1}{2}\left[J, (d_{t^\circ}E)^{\#}\right].$$

Proof. Let $S$ be the semispray associated with $h$, and let us denote by $\widetilde{h}$ the horizontal endomorphism generated by $S$ according to 1.9 (A). Then

$$h_0 \overset{1.9\,(\mathrm{C})}{=} \frac{1}{2}\left(1_{\mathfrak{X}(TM)} + [J,S_0]\right) \overset{2.5}{=} \frac{1}{2}\left(1_{\mathfrak{X}(TM)} + [J,S] - \left[J,(d_{t^\circ}E)^{\#}\right]\right)$$
$$= \widetilde{h} - \frac{1}{2}\left[J,(d_{t^\circ}E)^{\#}\right] \overset{2.1}{=} h - \frac{1}{2}t^\circ - \frac{1}{2}\left[J,(d_{t^\circ}E)^{\#}\right],$$

which gives the desired formula. □

## 3. Applications to generalized Berwald manifolds

*3.1. Remark.* In 0.1 we have already presented a preliminary discussion of generalized Berwald manifolds, including their definition. Let us also recall from 4.1 that a generalized Berwald manifold is said to be a *Berwald manifold* if $\nabla$ is a torsion-free linear connection on $M$. Then $\nabla$ is unique and we write $(M,E)$ rather than $(M,E,\nabla)$.

**3.2. Corollary.** *If $(M,E,\nabla)$ is a generalized Berwald manifold, then (by the conventions of 1.10) we have*

$$S_\nabla = S_0 + \left(d_{t^\circ_\nabla}E\right)^{\#}; \tag{3.2a}$$

$$h_\nabla = h_0 + \frac{1}{2}t^\circ_\nabla + \frac{1}{2}\left[J, \left(d_{t^\circ_\nabla}E\right)^{\#}\right]. \tag{3.2b}$$

Proof. This is an immediate consequence of Propositions 2.5 and 2.7. □

**3.3. Theorem.** *Suppose that $(M,E,\nabla)$ and $(M,E,\overline{\nabla})$ are generalized Berwald manifolds. The linear connections $\nabla$ and $\overline{\nabla}$ are equal if and only if they have same torsion tensor field.*

Proof. Let us denote by $\mathbb{T}_\nabla$ and $\mathbb{T}_{\overline{\nabla}}$ the classical torsion tensor field of $\nabla$ and of $\overline{\nabla}$, respectively. From the Theorem of section 3 of [4] and Lemma 1 of [13] we obtain immediately that for any vector fields $X$, $Y$ on $M$

$$t_\nabla(X^c, Y^c) = [\mathbb{T}_\nabla(X,Y)]^{\mathrm{v}}. \tag{3.3a}$$

Thus in the case of $\mathbb{T}_\nabla = \mathbb{T}_{\overline{\nabla}}$ the horizontal endomorphisms $h_\nabla$ and $h_{\overline{\nabla}}$ have the same weak torsion. On the other hand, $h_\nabla$ and $h_{\overline{\nabla}}$ are homogeneous, so their strong torsions are also equal. This implies by 2.6 that $h_\nabla = h_{\overline{\nabla}}$, whence $\nabla = \overline{\nabla}$.

The necessity of the condition $\mathbb{T}_\nabla = \mathbb{T}_{\overline{\nabla}}$ is evident. □

*3.4. Remark.* For any $\tau \in \mathbb{R}$, let us denote by $\mu_\tau$ the diffeomorphism

$$TM \to TM, \qquad v \mapsto e^\tau v.$$

We recall that – in general – a vector field $X$ of class $C^k$ ($k \in \mathbb{N}$) on $TM$ or on $\mathcal{T}M$ is called *homogeneous of degree* $r$ ($r \in \mathbb{Z}$) – briefly $r$-homogeneous – if

$$\forall\, \tau \in \mathbb{R} : X \circ \mu_\tau = e^{(r-1)\tau}(T\mu_\tau) \circ X.$$

As is well-known (see e.g. [5], Proposition 4.2.5), if $X$ is of class $C^1$, then it is $r$-homogeneous if and only if

$$[C, X] = (r-1)X;$$

we have used this characterization of homogeneity up to now. In particular, a vector field of class $C^2$ on $TM$ or on $\mathcal{T}M$ is said to be *quadratic* if it is homogeneous of degree two.

**3.5. Corollary.** *A generalized Berwald manifold $(M, E, \nabla)$ reduces to a Berwald manifold $(M, E)$ if, and only if, the vector field $(d_{t^\circ_\nabla} E)^\#$ is quadratic.*

Proof. We recall (see e.g. [15], 6.6) that a Finsler manifold is a Berwald manifold if, and only if, its canonical spray $S_0$ is smooth on the whole tangent manifold; then the spray $S_0$ is necessarily quadratic. Since $\left(d_{t^\circ_\nabla} E\right)^\# \overset{(3.2a)}{=} S_\nabla - S_0$, and $S_\nabla$ is always a quadratic spray, we infer immediately that the Corollary is true. □

*3.6. Remark.* A coordinate version of 3.5 for Finsler manifolds with one-form metrics can be found in [1]. It is not needless to point out that the linear connection of the Berwald manifold $(M,E)$ in question does not coincide with the given linear connection $\nabla$ in general.

*3.7. Remark.* We recall that two sprays $S_1$ and $S_2$ on a manifold $M$ are said to be *projectively equivalent* if there exists a function $\lambda : TM \to \mathbb{R}$, smooth on $\mathcal{T}M$, $C^1$ on $TM$ such that $S_1 = S_2 + \lambda C$. Then $\lambda$ is automatically 1-homogeneous, i.e., $C\lambda = \lambda$.

**3.8. Proposition.** *Let $(M,E,\nabla)$ be a generalized Berwald manifold. If the spray $S_\nabla$ arising from $\nabla$ is projectively equivalent to the canonical spray $S_0$ then $S_\nabla = S_0$, and – consequently – $(M,E)$ is a Berwald manifold.*

Proof. In view of (3.2a), $S_\nabla$ is projectively equivalent to $S_0$ if and only if

$$\left(d_{t^\circ_\nabla} E\right)^\# = \lambda C,$$

where the function $\lambda : TM \to \mathbb{R}$ satisfies the requirements of 3.7. Then on the one hand

$$i_{S_\nabla - S_0}\omega \overset{(3.2a)}{=} i_{\left(d_{t^\circ_\nabla} E\right)^\#}\omega = i_{\lambda C}\,\omega = \lambda i_C\,\omega = \lambda d_J E,$$

on the other hand

$$i_{S_\nabla - S_0}\omega = d_{t^\circ_\nabla} E$$

(see the proof of 2.5). Comparing these two equations we obtain the formula

$$d_{t^\circ_\nabla} E = \lambda d_J E.$$

Hence, for any semispray $S$,

$$d_{t^\circ_\nabla} E(S) = \lambda d_J E(S).$$

But

$$d_{t^\circ_\nabla} E(S) \overset{(1.3c)}{=} dE\big(t^\circ_\nabla(S)\big) = dE\big(t_\nabla(S,S)\big) = dE(0) = 0,$$

while

$$\lambda d_J E(S) = \lambda dE(JS) = \lambda dE(C) = \lambda CE = 2\lambda E,$$

so it follows that $\lambda E = 0$, which implies immediately the vanishing of $\lambda$. □

## 4. Wagner–Ichijyō connections and Wagner manifolds

*4.1. Definition.* Let $\nabla$ be a linear connection on the manifold $M$. A triplet $(\overset{\nabla}{D}, h_\nabla, \alpha)$ is said to be a *Wagner–Ichijyō connection* (induced by $\nabla$) if $(\overset{\nabla}{D}, h_\nabla)$ is an Ichijyō connection (3.1, 3.2), $\alpha$ is a smooth function on $M$ and the $h$-horizontal torsion $\overset{\nabla}{\mathbb{A}}$ of $\overset{\nabla}{D}$ has the following form:

$$\overset{\nabla}{\mathbb{A}} = d\alpha^{\mathrm{v}} \wedge h_\nabla := d\alpha^{\mathrm{v}} \otimes h_\nabla - h_\nabla \otimes d\alpha^{\mathrm{v}}. \tag{4.1a}$$

**4.2. Proposition.** *Let $(\overset{\nabla}{D}, h_\nabla, \alpha)$ be a Wagner–Ichijyō connection on the manifold $M$. Then we have the following relations:*

$$\mathbb{T}_\nabla(X,Y) = d\alpha(X)Y - d\alpha(Y)X \qquad (X, Y \in \mathfrak{X}(M)); \tag{4.2a}$$

$$t_\nabla = d\alpha^{\mathrm{v}} \wedge J := d\alpha^{\mathrm{v}} \otimes J - J \otimes d\alpha^{\mathrm{v}}; \tag{4.2b}$$

$$t_\nabla^\circ = \alpha^c J - d\alpha^{\mathrm{v}} \otimes C. \tag{4.2c}$$

Proof. Let $X$ and $Y$ be arbitrary vector fields on $M$. We have already learnt in 3.5 that

$$\overset{\nabla}{\mathbb{A}}(X^c, Y^c) = \big(\mathbb{T}_\nabla(X,Y)\big)^{h_\nabla}.$$

But

$$\begin{aligned}
\overset{\nabla}{\mathbb{A}}(X^c, Y^c) &\overset{(4.1a)}{=} d\alpha^{\mathrm{v}}(X^c)h_\nabla(Y^c) - d\alpha^{\mathrm{v}}(Y^c)h_\nabla(X^c) \\
&= (X^c\alpha^{\mathrm{v}})Y^{h_\nabla} - (Y^c\alpha^{\mathrm{v}})X^{h_\nabla} \\
&\overset{(1.7c)}{=} (X\alpha)^{\mathrm{v}}Y^{h_\nabla} - (Y\alpha)^{\mathrm{v}}X^{h_\nabla} = \big[(X\alpha)Y - (Y\alpha)X\big]^{h_\nabla},
\end{aligned}$$

so we obtain the formula (4.2a).

Next we check that (4.2b) is also valid. We can write

$$\begin{aligned}
t_\nabla(X^c, Y^c) &\overset{(3.3a)}{=} \big(\mathbb{T}_\nabla(X,Y)\big)^{\mathrm{v}} \overset{(4.2a)}{=} \big[d\alpha(X)Y - d\alpha(Y)X\big]^{\mathrm{v}} \\
&= (X\alpha)^{\mathrm{v}}Y^{\mathrm{v}} - (Y\alpha)^{\mathrm{v}}X^{\mathrm{v}} \overset{(1.7c)}{=} (X^c\alpha^{\mathrm{v}})Y^{\mathrm{v}} - (Y^c\alpha^{\mathrm{v}})X^{\mathrm{v}} \\
&\overset{(1.7f)}{=} [(d\alpha^{\mathrm{v}})X^c]J(Y^c) - \big[(d\alpha^{\mathrm{v}})Y^c\big]J(X^c) \\
&= (d\alpha^{\mathrm{v}} \wedge J)(X^c, Y^c),
\end{aligned}$$

whence the desired relation.

Finally, for any semispray $S$ on $M$,

$$\begin{aligned} t^{\circ}_{\nabla}(X^c) &= t_{\nabla}(S, X^c) \overset{(4.2b)}{=} d\alpha^{\mathrm{v}}(S)JX^c - d\alpha^{\mathrm{v}}(X^c)(JS) \\ &= (S\alpha^{\mathrm{v}})JX^c - d\alpha^{\mathrm{v}}(X^c)C \overset{(1.7a)}{=} \alpha^c JX^c - d\alpha^{\mathrm{v}}(X^c)C \\ &= (\alpha^c J - d\alpha^{\mathrm{v}} \otimes C)X^c, \end{aligned}$$

which proves (4.2c). □

*4.3. Definition.* A quadruple $(M, E, \nabla, \alpha)$ is said to be a *Wagner manifold* if $(M, E, \nabla)$ is a generalized Berwald manifold, $\alpha$ is a smooth function on $M$, and the relation

$$\mathbb{T}_{\nabla}(X, Y) = d\alpha(X)Y - d\alpha(Y)X \qquad (X, Y \in \mathfrak{X}(M)) \tag{4.3a}$$

holds.

*4.4. Remark.* It can be seen immediately that for a Wagner manifold $(M, E, \nabla, \alpha)$ the Ichijyō connection induced by $\nabla$ is just a Wagner–Ichijyō connection.

**4.5. Theorem.** *Let $(M, E)$ be a Finsler manifold. Suppose that $\nabla$ is a linear connection and $\alpha$ a smooth function on $M$. Then the following assertions are equivalent:*

(i) *$(M, E, \nabla, \alpha)$ is a Wagner manifold.*

(ii) *The Wagner–Ichijyō connection $(\overset{\nabla}{D}, h_{\nabla}, \alpha)$ induced by $\nabla$ is $h_{\nabla}$-metrical, i.e.,*

$$\overset{\nabla}{D}_{h_{\nabla}} g = 0.$$

(iii) *The horizontal endomorphism $h_{\nabla}$ is of form*

$$h_{\nabla} = h_0 + \alpha^c J - E[J, \operatorname{grad} \alpha^{\mathrm{v}}] - d_J E \otimes \operatorname{grad} \alpha^{\mathrm{v}}. \tag{4.5a}$$

Proof of (i) $\iff$ (ii). This equivalence is an immediate consequence of 4.4 and 4.3/(a) $\iff$ (c).

Proof of (i) $\Longrightarrow$ (iii). Let $X$ be a vector field on $M$. Evaluating the one-form $d_{t^\circ_\nabla}$ on $X^c$, we obtain

$$\begin{aligned} d_{t^\circ_\nabla}E(X^c) &= dE\big(t^\circ_\nabla(X^c)\big) = t^\circ_\nabla(X^c)E \overset{(4.2c)}{=} \big[\alpha^c X^{\mathrm{v}} - d\alpha^{\mathrm{v}}(X^c)C\big]E \\ &= \alpha^c(X^{\mathrm{v}}E) - 2E d\alpha^{\mathrm{v}}(X^c) \overset{(*)}{=} \alpha^c i_C\,\omega(X^c) - 2E d\alpha^{\mathrm{v}}(X^c) \\ &\overset{(1.8a)}{=} \alpha^c i_C\,\omega(X^c) - 2E i_{\operatorname{grad}\alpha^{\mathrm{v}}}\omega(X^c) = i_{(\alpha^c C - 2E\operatorname{grad}\alpha^{\mathrm{v}})}\omega(X^c), \end{aligned}$$

taking into account at the step $(*)$ that

$$X^{\mathrm{v}}E = dE(JX^c) = d_J E(X^c) = i_C\omega(X^c).$$

Thus we infer immediately that

$$\big(d_{t^\circ_\nabla}E\big)^{\#} = \alpha^c C - 2E\operatorname{grad}\alpha^{\mathrm{v}}.$$

Combining this result with (3.2b), we can proceed as follows:

$$\begin{aligned} h_\nabla &= h_0 + \frac{1}{2}t^\circ_\nabla + \frac{1}{2}[J,\alpha^c C] - \frac{1}{2}[J, 2E\operatorname{grad}\alpha^{\mathrm{v}}] \\ &\overset{(4.2c),\,(1.4d)}{=} h_0 + \frac{1}{2}(\alpha^c J - d\alpha^{\mathrm{v}}\otimes C) + \frac{1}{2}\alpha^c[J,C] - \frac{1}{2}d\alpha^c\wedge i_C J \\ &+ \frac{1}{2}d_J\alpha^c\otimes C - E[J,\operatorname{grad}\alpha^{\mathrm{v}}] + dE\wedge i_{\operatorname{grad}\alpha^{\mathrm{v}}}J \\ &- d_J E\otimes\operatorname{grad}\alpha^{\mathrm{v}} \overset{(1.5b),\,(1.7c),\,(1.8a)}{=} h_0 + \frac{1}{2}\alpha^c J - \frac{1}{2}d\alpha^{\mathrm{v}}\otimes C \\ &+ \frac{1}{2}\alpha^c J + \frac{1}{2}d\alpha^{\mathrm{v}}\otimes C - E[J,\operatorname{grad}\alpha^{\mathrm{v}}] - d_J E\otimes\operatorname{grad}\alpha^{\mathrm{v}} \\ &= h_0 + \alpha^c J - E[J,\operatorname{grad}\alpha^{\mathrm{v}}] - d_J E\otimes\operatorname{grad}\alpha^{\mathrm{v}}, \end{aligned}$$

and so the implication (i) $\Longrightarrow$ (iii) is verified.

Proof of (iii) $\Longrightarrow$ (i). First we show that the horizontal endomorphism given by (4.5a) is conservative.

For any vector field $X$ on $M$ we have

$$
\begin{aligned}
X^{h_\nabla} &:= h_\nabla X^c \overset{(4.5a)}{=} X^{h_0} + \alpha^c X^{\mathrm{v}} - E[J, \operatorname{grad}\alpha^{\mathrm{v}}]X^c - d_J E \otimes \operatorname{grad}\alpha^{\mathrm{v}}(X^c) \\
&= X^{h_0} + \alpha^c X^{\mathrm{v}} - E[X^{\mathrm{v}}, \operatorname{grad}\alpha^{\mathrm{v}}] + EJ[X^c, \operatorname{grad}\alpha^{\mathrm{v}}] - (X^{\mathrm{v}}E)\operatorname{grad}\alpha^{\mathrm{v}} \\
&\overset{(1.8a)}{=} X^{h_0} + \alpha^c X^{\mathrm{v}} - E[X^{\mathrm{v}}, \operatorname{grad}\alpha^{\mathrm{v}}] - (X^{\mathrm{v}}E)\operatorname{grad}\alpha^{\mathrm{v}}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
d_{h_\nabla}E(X^c) &= dE(X^{h_\nabla}) = X^{h_\nabla}E = \alpha^c(X^{\mathrm{v}}E) - EX^{\mathrm{v}}[\operatorname{grad}\alpha^{\mathrm{v}}(E)] \\
&+ E\operatorname{grad}\alpha^{\mathrm{v}}(X^{\mathrm{v}}E) - (X^{\mathrm{v}}E)\operatorname{grad}\alpha^{\mathrm{v}}(E) \overset{(1.8c)}{=} \alpha^c(X^{\mathrm{v}}E) - E(X^{\mathrm{v}}\alpha^c) \\
&+ E\operatorname{grad}\alpha^{\mathrm{v}}(X^{\mathrm{v}}E) - (X^{\mathrm{v}}E)\alpha^c = E(\operatorname{grad}\alpha^{\mathrm{v}}(X^{\mathrm{v}}E) - X^{\mathrm{v}}\alpha^c).
\end{aligned}
$$

Since on the one hand

$$
\begin{aligned}
\omega(\operatorname{grad}\alpha^{\mathrm{v}}, X^c) &= d(d_J E)(\operatorname{grad}\alpha^{\mathrm{v}}, X^c) \\
&= \operatorname{grad}\alpha^{\mathrm{v}} d_J E(X^c) - X^c d_J E(\operatorname{grad}\alpha^{\mathrm{v}}) - d_J E\big([\operatorname{grad}\alpha^{\mathrm{v}}, X^c]\big) \\
&= \operatorname{grad}\alpha^{\mathrm{v}}(X^{\mathrm{v}}E) - X^c dE(J\operatorname{grad}\alpha^{\mathrm{v}}) - dE\big(J[\operatorname{grad}\alpha^{\mathrm{v}}, X^c]\big) \\
&\overset{(1.8.a)}{=} \operatorname{grad}\alpha^{\mathrm{v}}(X^{\mathrm{v}}E),
\end{aligned}
$$

on the other hand

$$
\omega(\operatorname{grad}\alpha^{\mathrm{v}}, X^c) = d\alpha^{\mathrm{v}}(X^c) = X^c\alpha^{\mathrm{v}} \overset{(1.7c)}{=} X^{\mathrm{v}}\alpha^c,
$$

it follows that $\operatorname{grad}\alpha^{\mathrm{v}}(X^{\mathrm{v}}E) = X^{\mathrm{v}}\alpha^c$, and therefore $d_{h_\nabla}E(X^c) = 0$ – as we claimed. Thus $(M, E, \nabla)$ is a generalized Berwald manifold.

To conclude the proof we have to check that the torsion tensor of $\nabla$ has the form (4.3a). For this let us first observe that

$$
(*) \qquad \big[J, [J, \operatorname{grad}\alpha^{\mathrm{v}}]\big] = 0,
$$

since by the graded Jacobi identity (1.4b)

$$
\begin{aligned}
0 &= \big[J, [J, \operatorname{grad}\alpha^{\mathrm{v}}]\big] - \big[J, [\operatorname{grad}\alpha^{\mathrm{v}}, J]\big] + \big[\operatorname{grad}\alpha^{\mathrm{v}}, [J, J]\big] \\
&\overset{(1.4a),\,(1.5b)}{=} 2\big[J, [J, \operatorname{grad}\alpha^{\mathrm{v}}]\big].
\end{aligned}
$$

Thus, calculating as before,

$$
\begin{aligned}
t_\nabla := [J, h_\nabla] &\overset{(4.5a)}{=} [J, h_0] + [J, \alpha^c J] - \big[J, E[J, \operatorname{grad}\alpha^{\mathrm{v}}]\big] \\
&- [J, d_J E \otimes \operatorname{grad}\alpha^{\mathrm{v}}] \overset{(1.4e,f)}{=} \alpha^c [J, J] + d_J\alpha^c \wedge J - d\alpha^c \wedge J \circ J \\
&- E\big[J, [J, \operatorname{grad}\alpha^{\mathrm{v}}]\big] - d_J E \wedge [J, \operatorname{grad}\alpha^{\mathrm{v}}] + dE \wedge J \circ [J, \operatorname{grad}\alpha^{\mathrm{v}}] \\
&- d_J d_J E \otimes \operatorname{grad}\alpha^{\mathrm{v}} + d d_J E \otimes J \operatorname{grad}\alpha^{\mathrm{v}} + d_J E \wedge [J, \operatorname{grad}\alpha^{\mathrm{v}}] \\
&\overset{(1.5a,b),\,(1.8a),\,(*)}{=} d_J\alpha^c \wedge J \overset{(1.7c)}{=} d\alpha^{\mathrm{v}} \wedge J = d\alpha^{\mathrm{v}} \otimes J - J \otimes d\alpha^{\mathrm{v}}.
\end{aligned}
$$

This implies the relation (4.3a), as we have already seen in the proof of (4.2b). □

**4.6. Corollary.** *If $(M, E, \nabla, \alpha)$ is a Wagner manifold then the spray $S_\nabla$ generated by $h_\nabla$ and the canonical spray $S_0$ are related by*

$$S_\nabla = S_0 + \alpha^c C - 2E \operatorname{grad}\alpha^{\mathrm{v}}. \tag{4.6a}$$

Proof. It turned out in the proof of (i) $\Longrightarrow$ (iii) that $(d_{t^\circ_\nabla} E)^\# = \alpha^c C - 2E \operatorname{grad}\alpha^{\mathrm{v}}$. In view of (3.2a) this implies (4.6a). □

*4.7. Remark.* The relations (4.5a) and (4.6a) have already been obtained by Cs. Vincze in [17], but his reasoning follows a very different, less direct path.

## 5. Examples: Finsler manifolds with "one-form metric"

**5.1. Finsler–Minkowski functionals.** To avoid any confusion, we lay down here the following definition.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be a *Finsler–Minkowski functional* and the pair $(\mathbb{R}^n, f)$ a *Finsler–Minkowski vector space* if

(5.1a) $\forall\, v \in \mathbb{R}^n : f(v) \geq 0; \quad f(v) = 0 \iff v = 0$ (*positivity*);

(5.1b) $\forall\, \tau \in [0, \infty[\,, \quad \forall\, v \in \mathbb{R}^n : f(\tau v) = \tau f(v)$
(*positive homogeneity*);

(5.1c) $f \in C^3(\mathbb{R}^n \setminus \{0\})$ (*differentiability*);

(5.1d) the second Fréchet derivative of the function $F := \dfrac{1}{2}f^2$ is a positive definite symmetric bilinear function from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ at any point of $\mathbb{R}^n \setminus \{0\}$ (*strong convexity*).

Then the mapping

$$\langle\ ,\ \rangle : p \in \mathbb{R}^n \setminus \{0\} \mapsto \langle\ ,\ \rangle_p, \tag{5.1e}$$

$$\forall\, v, w \in T_p\mathbb{R}^n \cong \mathbb{R}^n : \langle\ ,\ \rangle_p(v, w) =: \langle v, w\rangle_p := F''(p)(v, w)$$

is a Riemannian metric on $\mathbb{R}^n \setminus \{0\}$.

**5.2.** Let $\beta$ be a one-form on a manifold $M$. In the sequel we are going to denote by $\widetilde{\beta}$ the function

$$TM \to \mathbb{R}, \quad v \mapsto \widetilde{\beta}(v) := \beta_{\pi(v)}(v).$$

We shall also utilize the following fact.

*If $\nabla$ is a linear connection on $M$, then for any vector field $X \in \mathfrak{X}(M)$ and one-form $\beta \in \Omega(M)$*

$$X^{h_\nabla}\widetilde{\beta} = \widetilde{\nabla_X \beta} \tag{5.2 a}$$

(see [21], Lemma 2).

**5.3.** Let us suppose for our subsequent considerations that $M$ is a *parallelizable* manifold with a *parallelization* $(X_i)_{i=1}^n$; $X_i \in \mathfrak{X}(M)$, $1 \leq i \leq n$ ([2], p. 117). Let $(\lambda^i)_{i=1}^n$ be the coframe dual to $(X_i)_{i=1}^n$. Using the convention fixed in 5.2, consider the mapping

$$\widetilde{\lambda} := (\widetilde{\lambda^1}, \dots, \widetilde{\lambda^n}) : TM \to \mathbb{R}^n, \quad v \mapsto \widetilde{\lambda}(v) = \big(\widetilde{\lambda}^1(v), \dots, \widetilde{\lambda}^n(v)\big).$$

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a Finsler–Minkowski functional, and let us introduce the functions

$$\mathcal{L} := f \circ \widetilde{\lambda}, \qquad E := \frac{1}{2}\mathcal{L}^2.$$

Then $(M, E)$ is a Finsler manifold, the Finsler structure constructed in this way is said to be a *one-form Finsler structure*. Following (at least partly) the traditions, in the sequel we shall mention $(M,E)$ as a "*Finsler manifold with one-form metric*" (cf. [11]).

Concerning some basic analytic data of $(M,E)$, we have the following results.

(5.3a) The fundamental two-form of $(M,E)$ is

$$\omega = \overset{\sim}{\lambda}^{*} df \wedge i_J \overset{\sim}{\lambda}^{*} df - (f \circ \widetilde{\lambda}) d_J \overset{\sim}{\lambda}^{*} df$$

(the $*$ denotes pull-back).

(5.3b) The vertical metric $\overline{g}$ (1.10) of $(M,E)$ is the pull-back of the Riemannian metric $\langle\, ,\, \rangle$ via $\widetilde{\lambda}$, i.e.

$$\overline{g} = \overset{\sim}{\lambda}^{*} \langle\, ,\, \rangle;$$

hence the mapping $\widetilde{\lambda}$ preserves the Finslerian norms.

(5.3c) The (lowered) first Cartan tensor of $(M,E)$ can be represented in the form

$$\mathcal{C}_\flat = \frac{1}{2}\Big(\eta \odot d_J(f \circ \widetilde{\lambda}) + (f \circ \widetilde{\lambda}) \overset{\circ}{D}_J \eta\Big),$$

where $(\overset{\circ}{D}, h)$is a Finsler connection of Berwald-type ([13]), $\eta$ is a type $(0,2)$ tensor field given by

$$(X,Y) \in \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) \\ \mapsto \eta(X,Y) := dd_J(f \circ \widetilde{\lambda})(JX,Y) \in C^\infty(TM),$$

and $\odot$ is the symbol of the symmetric product.

**5.4. Proposition.** *Let $(M,E)$ be the Finsler manifold with the one-form metric constructed in 5.3. Consider the linear connection $\nabla$ determined by the parallelization $(X_i)_{i=1}^n$ ([2], 9.1.2), and let $h_\nabla$ be the horizontal endomorphism arising from $\nabla$. Then $(M,E,\nabla)$ is a generalized Berwald manifold. If, in addition, $\nabla$ is torsion-free, then $(M,E)$ is a locally Minkowski manifold.*

Proof. First we show that $(M,E,\nabla)$ is a generalized Berwald manifold, i.e. $d_{h_\nabla} E = 0$. The members of the dual coframe $(\lambda^i)_{i=1}^n$ are clearly parallel with respect to $\nabla$, so for any vector fields $X$, $Y$ on $M$ we have

$$0 = [(\nabla\lambda^i)(Y,X)]^{\mathrm{v}} = \left[\left(\nabla_X \lambda^i\right)(Y)\right]^{\mathrm{v}} \overset{\text{I.2.8 of [20]}}{=} Y^{\mathrm{v}} \widetilde{\nabla_X \lambda^i} \qquad (1 \le i \le n);$$

hence for any vector field $X$ on $M$,

$$\widetilde{\nabla_X \lambda^i} = 0 \qquad (1 \leq i \leq n).$$

On the other hand

$$X^{h_\nabla} \widetilde{\lambda}^i = \widetilde{\nabla_X \lambda^i} \qquad (1 \leq i \leq n)$$

by (5.2a). So we conclude that

$$\forall\, X \in \mathfrak{X}(M) : X^{h_\nabla} \widetilde{\lambda}^i = d_{h_\nabla} \widetilde{\lambda}^i (X^c) = 0 \qquad (1 \leq i \leq n),$$

therefore

$$d_{h_\nabla} \widetilde{\lambda}^i = 0, \qquad 1 \leq i \leq n.$$

From this we obtain the desired relation $d_{h_\nabla} E = 0$ by the chain rule.

To prove the second assertion, let us consider the Ichijyō connection $(\overset{\nabla}{D}, h_\nabla)$. Since – as we have just seen – $(M, E, \nabla)$ is a generalized Berwald manifold, 4.3 assures that $\overset{\nabla}{D}$ is $h$-metrical. Furthermore $\nabla$ is clearly a flat connection and – by assumption – it is torsion-free. These three properties imply by 4.8 that $(M, E)$ is indeed a locally Minkowski manifold. □

**5.5. Curvature and torsion data.** Suppose that $(M, E)$ is a Finsler manifold with one-form metric, according to 5.3. Let $\nabla$ be the linear connection determined by the parallelization of $M$, and let us consider the Ichijyō connection $(\overset{\nabla}{D}, h_\nabla)$. (Using local coordinates, this connection has already been constructed in [11] under the name "one-form connection".) Concerning the partial curvatures and torsion of $(\overset{\nabla}{D}, h_\nabla)$, the following tables can be obtained as easy consequences of our preceding considerations and 3.5.

| **Curvature** | $(X, YZ \in \mathfrak{X}(\mathcal{T}M))$ |
|---|---|
| *horizontal* | $\overset{\nabla}{\mathbb{R}} = 0$ |
| *mixed* | $\overset{\nabla}{\mathbb{P}} = 0$ |
| *vertical* | $\overset{\nabla}{\mathbb{Q}}(X, Y)Z = \mathcal{C}(F(\mathcal{C}(X, Z), Y) - \mathcal{C}(X, F\mathcal{C}(Y, Z))$ |

| **Torsion** | $(X, Y \in \mathfrak{X}(M))$ |
|---|---|
| *h-horizontal* | $\overset{\nabla}{\mathbb{A}}(X^{h_\nabla}, Y^{h_\nabla}) = (\mathbb{T}_\nabla(X,Y))^{h_\nabla}$ |
| *h-mixed* | $\overset{\nabla}{\mathbb{B}}(X^{h_\nabla}, Y^{\mathrm{v}}) = -F_\nabla \mathcal{C}(X^{h_\nabla}, Y^{h_\nabla})$ |
| *v-horizontal* | $\overset{\nabla}{\mathbb{R}}_1 = 0$ |
| *v-mixed* | $\overset{\nabla}{\mathbb{P}}_1 = 0$ |
| *v-vertical* | $\overset{\nabla}{\mathbb{S}}_1 = 0$ |

($F$ is an arbitrary almost complex structure on $TM$, $\mathcal{C}$ is determined by (5.3c).)

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