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Isometric or harmonic mappings of complete Riemannian manifolds

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Abstract. We investigate (a) the isometric and (b) the harmonic mappings φ of a complete Riemannian manifold M^n whose sectional curvature is bounded from below, into euclidean space E^{n+m} and in case of (a) also into the unit sphere $S^{n+m-1} \subset E^{n+m}$. In case of (a) we obtain conditions in terms of the euclidean norm $\|H(\varphi(x))\|$ $x \in M^n$ of the mean curvature vector of $\varphi(M^n)$ on the radius r of the euclidean ball $B(r)$ in order that $\varphi(M^n)$ cannot be pinched in any such $B(r)$ $(\varphi(M^n) \not\subset B(r))$. In case of (b) we show that under a mild condition on the Ricci curvature the positivity of the energy density $e(\varphi)$ is necessary in order that $\varphi(M^n)$ spreads out to infinity.

In [1] the present authors investigated isometric mappings φ of complete Riemannian manifolds $V^n = (M, g)$ into euclidean space E^{n+m} (respectively, into the unit sphere $S^{n+m-1} \subset E^{n+m}$), where $\varphi(V^n)$ cannot be pinched by certain geodesic balls $B(r)$ of radius $r : \varphi(V^n) \not\subset B(r) \subset E^{n+m}$ (respectively $\varphi(V^n) \not\subset B(r) \subset S^{n+m-1}$).

After an introduction, in Section 1 of this note we make some remarks on our previous paper [1]. We weaken a condition of the results obtained in it and we point out that some of them are sharp. In Section 2, we replace in our problem the isometric mappings by harmonic ones, and obtain a necessary condition in order that $\varphi(V^n)$ cannot be pinched by any ball $B(r)$ of E^{n+m} or, what is the same, $\varphi(V^n)$ stretches out to infinity in E^{n+m}.

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1. Isometric mappings

Isometric mappings of compact Riemannian manifolds $V^n = (M, g)$ have a long history, and they were investigated under different conditions by a number of mathematicians. We mention here only a few them.

S. S. Chern and C. C. Hsiung [3] showed in 1963 that a compact V^n has no isometric and minimal immersion φ into euclidean space E^{n+m}. This means also that a euclidean space has no compact minimal submanifold. If V^n is compact, and $\varphi : V^n \to E^{n+m}$ is not necessarily minimal, but an isometric immersion only, also then $\varphi(V^n)$ is compact, thus bounded, and so contained in a ball $B(R) \subset E^{n+m}$ of a sufficiently big radius R. But how big this radius must be? Under which value of the radius r is $\varphi(V^n) \subset B(r) \subset E^{n+m}$ impossible? Such a bound was found recently by S. S. Yang [11]. He proved in 1998 that if V^n is compact and

$$\|H(\varphi(x))\| < \frac{1}{\sqrt{m}r} \quad \forall x \in V^n,\ m \geq 1, \tag{1}$$

where $\|H\|$ is the euclidean length of the mean curvature vector H of $\varphi(V^n)$, then no ball $B(r) \subset E^{n+m}$ can contain $\varphi(V^n) : \varphi(V^n) \not\subset B(r)$. This result is sharp in the case of $m = 1$. Indeed, let φ be the identity, and $V^n = S^n(R)$ a sphere of radius R in E^{n+1}. Thus $\|H(\varphi(x))\| = 1/R$. Then (1) yields $r < R$, and in fact $B(r)$, $r < R$ are those balls only which do not contain $S^n(R)$. It is easy to see that (1) is not sharp for every V^n and φ. E.g. if $\varphi = \mathrm{id}$, $m = n = 1$, V^1 is an ellipse $\mathcal{E}$ in E^2 with main axes a and b $(a > b)$, then $\|H\| = \kappa$ is the curvature of the ellipse, which is the biggest namely a/b^2, at the endpoint of the big axis. So we obtain from Yang's theorem that $\mathcal{E} \not\subset B(r)$ if $r < b^2/a$, however, clearly $\mathcal{E} \not\subset B(r)$ also if $r < a$, while $a > b^2/a$. This means that there are balls which are bigger than those deduced from Yang's theorem and yet they cannot contain $\mathcal{E}$. Thus in this example Yang's result is not sharp.

Chern and Hsiung's theorem is a consequence of Yang's theorem. Suppose that V^n is compact and φ is isometric and minimal as in Chern and Hsiung's theorem. Then $H(\varphi(x))$ vanishes, and $\varphi(V^n) \subset B(R) \subset E^{n+m}$ with a sufficiently big R, since $\varphi(V^n)$ is compact and hence bounded. However these H and R satisfy (1), yet $\varphi(V^n) \subset B(R)$. This contradicts Yang's theorem. So, if V^n is compact and φ is an isometric immersion, then $\varphi(V^n)$ cannot be minimal.

This problem (i.e. for which r is $\varphi(V^n) \not\subset B(r) \subset E^{n+m}$) was investigated under the weaker condition of completeness (instead of compactness) of V^n in our paper [1]. Our result says:

Theorem A [1]. *If V^n is complete, its sectional curvature is bounded from below: $K > K_0$, and $\varphi : V^n \to E^{n+m}$ is an isometric immersion, then $\varphi(V^n) \not\subset B(r) \subset E^{n+m}$, provided that the radius r satisfies*

$$\|H(\varphi(x))\| < \frac{1}{r} \quad \forall x \in V^n. \tag{2}$$

This allows a slightly bigger balls $B(r)$ not containing $\varphi(V^n)$, for in contrust to (1), the factor $1/\sqrt{m}$ does not appear on the right-hand side of (2) in contrary to (1). The above discused example of $\varphi = \mathrm{id}$, $V^n = S^n(R) \subset E^{n+1}$ shows that this result is sharp for any m in E^{n+m}. Applying an old result of H. W. E. Jung [6], also a lower bound for the diameter d of $\varphi(V^n)$ can be obtained ([1], Corollary 1), namely

$$d(\varphi(V^n)) \geq \sqrt{\frac{2(s+1)}{s}} r_0, \quad s = n+m,$$

where r_0 is the $\limsup$ of the r satisfying (2). We guess that this is not the best estimate for the lower bound of the diameter of $\varphi(V^n)$.

A result similar to the one of Theorem A concerns the isometric immersion $\varphi : V^n \to S^{n+m-1}$ into the euclidean unit sphere S^{n+m-1}.

Theorem B [1]. *If V^n is complete, its sectional curvature $K > K_0$, and φ is an isometric immersion of V^n into S^{n+m-1}, then $\varphi(V^n)$ cannot be contained in any geodesic ball $B(r)$ of S^{n+m-1} of radius r: $\varphi(V^n) \not\subset B(r) \subset S^{n+m-1}$, provided*

$$\|H(\varphi(x))\| < \frac{\cos r}{2 \sin \frac{r}{2}} \quad \forall x \in V^n,\ m \geq 1. \tag{3}$$

We remark that the condition in Theorems A and B saying that the sectional curvature K of V^n is bounded from below: $K > K_0$, can be weakened and replaced by

$$\mathrm{Ric}(X)(x) > c(1 + \rho^2 \log^2(\rho + 2)) \quad \forall x \in V^n \text{ and } X \in T_x V^n, \tag{4}$$

where Ric means the Ricci curvature of V^n for an arbitrary unit vector X, c is a negative constant, and ρ is the distance function on V^n from a fixed point $x_0 \in V^n$ to $x \in V^n$. If $K(X,Y)(x) > K_0$ $(K_0 < 0)$, $\forall X, Y \in T_x V^n$, $x \in V^n$, then (4) holds with $c = K_0$. The converse is however not true in general, (4) allows K to tend to $-\infty$. Thus $K > K_0$ is stronger than (4).

The proof of Theorems A and B with the condition (4) in place of $K > K_0$ runs exactly the same way as in [1], only we must use in place of Omori's theorem ([8], Theorem 1) the following

Theorem (Q. Chen and Y. L. Xin [2], Theorem 2.2)**.** *For any smooth and bounded function ϕ on a complete connected Riemannian manifold Q which satisfies* (4), *and for any $\varepsilon > 0$ there exists a sequence of points $\{x_k \in Q\}$ such that $\lim_{k\to\infty} \phi(x_k) = \sup_M \phi$, and for sufficiently large k $\|(\operatorname{grad}\phi)(x_k)\| < E$, and at these x_k the Hessian $\mathcal{H}_\phi$ of ϕ is smaller than $\varepsilon : \mathcal{H}_\phi(X,X) < \varepsilon$ for any unit vector $X \in T_{x_k}Q$.*

We still note an immediate consequence of Theorem B (either in its original ([1], Theorem 2) or in its present form). If $\varphi(V^n)$ is minimal, then $H = 0$. Thus (3) is satisfied by any $r \in (0, \frac{\pi}{2})$. This means that no complete minimal submanifold of a sphere can be contained in a hemisphere. In case of a compact V^n and $m = 1$ this is a result of S. B. Meyers ([7], Theorem 4).

Finally we mention that there are investigations which use conditions on the sectional curvature K and not on $\|H\|$ in order to conclude that $\varphi(V^n)$ is not pinched into certain balls $B(r)$. H. Jacobowitz [5] proved in 1973 that if V^n is a compact Riemannian manifold, $\varphi : V^n \to E^{2n-1}$ is an isometric immersion, and everywhere $K < \frac{1}{r^2}$, then there exists no ball $B(r)$ containing $\varphi(V^n)$. This is a generalization of a result of S. S. Chern and N. H. Kuiper [4] from 1952 saying that a compact Riemannian manifold V^n with everywhere nonpositive K cannot be isometrically embedded into E^{2n-1}. Also this result contains as a corollary the old theorem of C. Tompkins [9] from 1939, according to which the n-dimensional flat torus cannot be embedded isometrically in E^{2n-1}.

2. Harmonic mappings

We want to replace the isometric immersions of the previous section by harmonic mappings $\varphi : V^n \to E^{n+m}$ and we investigate conditions under which the image $\varphi(V^n)$ of a complete Riemannian manifold $V^n = (M, g)$ cannot be pinched by a ball $B(r)$ of E^{n+m}.

Let $e_1, \dots, e_n \in T_pV^n$ be an orthonormal system, and $E_1, \dots, E_n$ their parallel translateds according to the Levi Civita connection ∇ of V^n along geodesics emanating from $p = x_0 \in M$ in a domain within the range of a geodesic polar coordinate system centered at p. $\varphi(x) = y$, $x \in M$ is a point of E^r $r = n + m$ with origin 0. We denote the parallel translated of $\overrightarrow{0y}$ in E^r to y by $\vec{\varphi}(x)$, which is an element of T_yE^r, and we denote the canonical scalar product in E^r by $\langle , \rangle$. Then

$$F(x) := \langle \vec{\varphi}(x), \vec{\varphi}(x) \rangle$$

is a smooth function on M. The Hessian $\mathcal{H}_F(X,X)(p)$ of F at p for a unit vector $X \in \mathfrak{X}(M)$ is

$$\mathcal{H}_F(X,X)(p) = XX(F)|_p - (\nabla_X X)F|_p.$$

We want to calculate $XX(F)$. The pullback of the tangent bundle $\tau_E = (TE^r, \pi, E^r)$ of E^r by φ is the bundle

$$\varphi^*(\tau_E) = M \times_{E^r} TE^r = \{(x,z) \mid z \in \pi^{-1}(\varphi(x))\}.$$

We denote the pullback of $\vec{\varphi}(y)$ by $\varphi^*(\vec{\varphi}(y)) \equiv \vec{\varphi}(x)$, and the pullback (or associated) connection on $\varphi^*(\tau_E)$ by $\hat{\nabla}$ (see e.g. [10], p. 10). Then

$$\hat{\nabla}\tilde{\varphi} = \overset{E}{\nabla}_{\varphi_* X}\vec{\varphi} \overset{(*)}{=} \varphi_* X,$$

where $\varphi_* : T_xM \to T_{\varphi(x)}E^r$, and $\overset{E}{\nabla}$ denotes the Levi Civita connection of E^r with respect to $\langle , \rangle$. (For $(*)$ compare [1] formula (4).) Also

$$\langle \vec{\varphi}, \vec{\varphi} \rangle(\varphi(x)) = \langle \tilde{\varphi}, \tilde{\varphi} \rangle^*(x),$$

where $\langle\ ,\ \rangle^*$ denotes the metric on $\varphi^*(\tau_E)$ induced by $\langle\ ,\ \rangle$ ([10], p. 10). Using these relations we obtain

$$X(F) = \langle \tilde{\varphi}, \tilde{\varphi} \rangle^* = 2\langle \hat{\nabla}_X \tilde{\varphi}, \tilde{\varphi} \rangle^* = 2\langle \varphi_* X, \tilde{\varphi} \rangle^*.$$

Then

$$XX(F) = 2X\langle \varphi_* X, \tilde{\varphi} \rangle^* = 2[\langle \hat{\nabla}_X \varphi_* X, \tilde{\varphi} \rangle^* + \langle \varphi_* X, \varphi_* X \rangle^*].$$

Applying this for the vector fields E_i, and taking into account that $\nabla_{E_i} E_i = 0$, since E_i are parallel vector fields, we obtain at p

$$\sum_{i=1}^{n} \mathcal{H}_F(E_i, E_i) = \sum_{i=1}^{n} E_i E_i(F) \tag{5}$$

$$= \sum_{i=1}^{n} 2\left[\langle \hat{\nabla}_{E_i} \varphi_* E_i - \varphi_*(\nabla_{E_i} E_i), \tilde{\varphi} \rangle^* + \langle \varphi_* E_i, \varphi_* E_i \rangle^*\right].$$

There are well known the following notions and facts ([10], pp. 10–13). The energy density $e(\varphi)$ of a map $\varphi : (M, g) \to (N, h)$ between two Riemannian manifolds is defined by

$$e(\varphi) := \frac{1}{2}\langle g, \varphi^* h\rangle_2 = \frac{1}{2}\sum_{i=1}^{n}\langle \varphi_* e_i, \varphi_* e_i\rangle_h,$$

where $\langle\ ,\ \rangle_2$ is the inner product of two $(0,2)$ tensors on (M, g). $2e(\varphi)$ is often denoted also by $|d\varphi|^2$. The energy of φ is

$$E(\varphi) = \int_M e(\varphi)\, dM.$$

If φ is a critical point of an energy functional $E(\varphi)$, then φ is said to be a harmonic map, and the cross-section

$$\tau(\varphi) = \sum_{i=1}^{n}(\nabla_{e_i} d\varphi)e_i$$

of the bundle $\varphi^{-1}(TN)$ is called its tension. It is also known that φ is a harmonic map iff $\tau(\varphi) = 0$.

Now (5) can be written in the form

$$\sum_{i=1}^{n}\mathcal{H}_F(E_i, E_i) = 2\left(\langle \tau(\varphi), \tilde{\varphi}\rangle^* + \sum_{i=1}^{n}\langle \varphi_* E_i, \varphi_* E_i\rangle^*\right).$$

If we suppose that $\varphi(V^n) \subset B(r)$, then $F = \langle \vec{\varphi}, \vec{\varphi}\rangle < r^2$ is bounded, and we can apply the theorem of Q. Chen and Y. L. Xin ([2], Theorem 2.2) cited in Section 1 (with $\phi = F$, $Q = M$). Thus, for sufficiently large k

$$\sum_{i=1}^{n}\mathcal{H}_F(E_i, E_i)(x_k) = 2\langle \tau(\varphi), \tilde{\varphi}\rangle^*(x_k) + e(\varphi)(x_k) < \varepsilon.$$

We know that in case of a harmonic φ $\tau(\varphi)$ vanishes ([10], p. 11). Thus we obtain the following

Proposition. *If V^n is a complete Riemannian manifold, $\varphi : V^n \to E^{n+m}$ is harmonic, $\varphi(V^n) \subset B(r)$, and the Ricci curvature of V^n satisfies (4), i.e.*

$$\mathrm{Ric}(X)(x) > c(1 + \rho^2 \log^2(\rho + 2)) \quad \forall x \in V^n \text{ and } X \in T_x V^n,$$

then

$$\inf_M e(\varphi) = 0.$$

This also yields the following

Theorem. *If V^n is a complete Riemannian manifold, $\varphi : V^n \to E^{n+m}$ is harmonic, and the Ricci curvature satisfies (4), then $\inf_M e(\varphi) > 0$ is necessary in order that $\varphi(V^n) \not\subset B(r)$ for any r.*

This means that under the above conditions $\varphi(V^n)$ spreads out to infinity in E^{n+m}.

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