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Some Tauberian theorems for Schwartz distributions

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Abstract. We give a condition on a space of test functions $\mathcal{A}$ for the inclusion $\mathcal{A} \cap \mathcal{A}' \subset \mathcal{S}$ to hold. These kinds of results are Tauberian theorems that guarantee that a generalized function of rapid distributional decay at infinity is a rapidly decreasing smooth function, that is, an element of $\mathcal{S}$.

1. Introduction

The purpose of this article is to study some class of Tauberian theorems for generalized functions. Our main concern is to study some supplementary conditions on a smooth generalized function that decays distributionally at infinity which guarantee that the function is a rapidly decreasing smooth function, that is, an element of the space of test functions $\mathcal{S}$.

More specifically, we identify a condition on the space of test functions $\mathcal{A}$ that guarantees that

$$\mathcal{A} \cap \mathcal{A}' \subset \mathcal{S}. \tag{1.1}$$

Results of this kind are very useful in mathematical physics [4, Section 4]. We mention the results $\mathcal{O}_M \cap \mathcal{O}'_M \subset \mathcal{S}$ and $\mathcal{O}_C \cap \mathcal{O}'_C \subset \mathcal{S}$ given by Ortner and Wagner [12] and later in [4]. Observe the Tauberian character of such a result: the elements of $\mathcal{O}'_M$ are generalized functions that in some sense (made precise in Section 2) decay at infinity; the elements of $\mathcal{O}_M$ are smooth functions that do not increase too fast at infinity, but which are usually not of rapid decay. The fact that $\mathcal{O}_M \cap \mathcal{O}'_M \subset \mathcal{S}$ is thus a rather interesting result.

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The main tool that we use in our analysis is the concept of the Cesàro behavior of a distribution [3]. The notion of the Cesàro behavior of a distribution generalizes the classical methods for the summability of series and integrals [8]. It is also closely related to the parametric or distributional behavior [5], [6], [13], [15]. We give a short account of these ideas in Section 2. We also explain the notation for the spaces of generalized functions needed in this article in that section.

The third section gives the main results of this study. We give a condition on a space of test functions $\mathcal{A}$ that yields the inclusion $\mathcal{A}\cap\mathcal{A}'\subset\mathcal{S}$. The elements of $\mathcal{A}'$ are usually generalized functions of rapid decay in the Cesàro sense while being in $\mathcal{A}$ gives the smoothness, and something else, to obtain elements of $\mathcal{S}$. The results are useful because there are many smooth functions, like $f(x)=\cos x$ for instance, which show rapid distributional decay at infinity but which do not belong to $\mathcal{S}$. In this and the next sections we present several examples and counterexamples to illustrate the results.

2. Preliminaries

In this section we explain the spaces of test functions and distributions needed in this paper. We also give a summary of the notion of Cesàro behavior of a distribution. All of our functions and spaces are over the space $\mathbb{R}$.

The spaces of test functions $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{S}$ and the corresponding spaces of distributions $\mathcal{D}'$, $\mathcal{E}'$, and $\mathcal{S}'$ are well-known [6], [9], [10], [14]. In general [16] we call a topological vector space $\mathcal{A}$ as a space of test functions if $\mathcal{D}\subset\mathcal{A}\subset\mathcal{E}$, the inclusions being continuous, and if the derivative d/dx is a continuous operator of $\mathcal{A}$.

We shall also need the spaces of test functions $\mathcal{O}_M$, $\mathcal{O}_C$, $\mathcal{G}_\alpha$ and $\mathcal{K}$ and their duals [6], [9], [10], [14]. A smooth function ϕ belongs to $\mathcal{O}_M$ if there are constants γ_k such that $\phi^{(k)}(x)=O(|x|^{\gamma_k})$ as $|x|\to\infty$ for $k=0,1,2,\ldots$. If $\gamma_k=\gamma$ for all k then $\phi\in\mathcal{O}_C$. When $\gamma_k=\gamma-\alpha k$ then $\phi\in\mathcal{G}_\alpha$ [1]. Notice that $\mathcal{G}_0=\mathcal{O}_C$. When $\alpha=1$ we obtain the space $\mathcal{K}=\mathcal{G}_1$ of so-called GLS symbols [7]. The topologies of these spaces are given by the canonical seminorms.

The space $\mathcal{K}'$ plays a fundamental role in the theory of asymptotic expansions of generalized functions [5], [6] and in the theory of summability of distributional evaluations [3]. The elements of $\mathcal{K}'$ are exactly the generalized functions that decay very rapidly at infinity in the distributional sense or, equivalently, in the Cesàro sense.

The Cesàro behavior of a distribution at infinity is studied by using the order symbols $O(x^\alpha)$ and $o(x^\alpha)$ in the Cesàro sense. If $f \in \mathcal{D}'(\mathbb{R})$ and $\alpha \in \mathbb{R} \setminus \{-1, -2, -3, \dots\}$, we say that $f(x) = O(x^\alpha)$ as $x \to \infty$ in the Cesàro sense and write

$$f(x) = O(x^\alpha) \quad \text{(C) as } x \to \infty, \tag{2.1}$$

if there exists $N \in \mathbb{N}$ such that every primitive of order N of f, i.e., $F^{(N)} = f$, is an ordinary function for large arguments and satisfies the ordinary order relation

$$F(x) = p(x) + O(x^{\alpha+N}) \quad \text{as } x \to \infty, \tag{2.2}$$

for a suitable polynomial p of degree $N-1$ at the most. A similar definition applies to the little o symbol. The definitions when $x \to -\infty$ are clear.

The equivalent notations $f(x) = O(x^{-\infty})$ and $f(x) = o(x^{-\infty})$ mean that $f(x) = O(x^{-\beta})$ for each $\beta > 0$. It is shown in [3] that a distribution $f \in \mathcal{D}'$ is of rapid decay at $\pm\infty$ in the (C) sense,

$$f(x) = O(|x|^{-\infty}) \quad \text{(C) as } |x| \to \infty, \tag{2.3}$$

if and only if $f \in \mathcal{K}'$.

Functions like $\sin x$, $J_0(x)$, or $x^2 e^{ix}$ belong to $\mathcal{K}'$ and thus are "distributionally small".

Since $\mathcal{A}' \subset \mathcal{K}'$ for $\mathcal{A} = \mathcal{O}_M$, $\mathcal{O}_C$, and $\mathcal{G}_\alpha$ for $\alpha \leq 1$, it follows that the elements of $\mathcal{O}'_M$, $\mathcal{O}'_C$ and $\mathcal{G}'_\alpha$ for $\alpha \leq 1$ are distributionally small. But $\mathcal{G}'_\alpha$ is not a subset of $\mathcal{K}'$ for $\alpha > 1$.

The space $\mathcal{K}'$ is a distributional analogue of the space $\mathcal{S}$ of rapidly decreasing smooth functions [3].

We say that a distribution $f \in \mathcal{D}'$ has the limit L in the (C) sense as $x \to \infty$ and write $\lim_{x\to\infty} f(x) = L$ (C), if $f(x) = L + o(1)$ (C) as $x \to \infty$.

Distributional evaluations are treated as follows. Let $f \in \mathcal{D}'$ and let $\phi \in \mathcal{E}$. Then in general the evaluation $\langle f(x), \phi(x) \rangle$ does not make sense. Of course, if we can find a space $\mathcal{B} \subset \mathcal{E}$ such that $f \in \mathcal{B}'$ and $\phi \in \mathcal{B}$ then there is no problem. A different approach is to use the summability ideas as we now explain. Suppose first that $\operatorname{supp} f$ is bounded on the left. Let

$g(x)$ be the first order primitive of $\phi(x)f(x)$ with support bounded on the left. We say that the evaluation $\langle f(x), \phi(x)\rangle$ exists in the Cesàro sense and equals L, and write

$$\langle f(x), \phi(x)\rangle = L \quad \text{(C)}, \tag{2.4}$$

if $\lim_{x\to\infty} g(x) = L$ (C). The case when $\operatorname{supp} f$ is bounded on the right is similar and the general case is handled by writing $f = f_1 + f_2$ with $\operatorname{supp} f_1$ bounded on the left and $\operatorname{supp} f_2$ bounded on the right. It is easy to see that the Cesàro limit of the evaluation is independent of the decomposition.

3. The main result

In order to give our Tauberian theorem we need several preliminary lemmas. The first one is proved in [3].

Lemma 1. *Let $f \in \mathcal{K}'$ and let $\phi \in \mathcal{K}$. Then the evaluation $\langle f(x), \phi(x)\rangle$ exists in the* (C) *sense.*

Proof. We may suppose that $\operatorname{supp} f$ is bounded on the left, since the analysis when the support is bounded on the right is similar, and we can always decompose f as the sum of a distribution with support bounded on the left and another with support bounded on the right.

Let g be the first order primitive of ϕf with support bounded on the left. Since $\phi \in \mathcal{K}$, $f \in \mathcal{K}'$, then $\phi f \in \mathcal{K}'$, and then $\phi(x)f(x) = o(x^{-\infty})$ (C), as $x \to \infty$, that is, $\phi(x)f(x) = o(x^{-\beta})$ (C), as $x \to \infty$, $\forall\beta$. Take $\beta = -3/2$; then we can find $N \in \mathbb{N}$ and a primitive of order N of ϕf, $G^{(N)} = \phi f$, and a polynomial p of order $N-1$ such that the ordinary relation

$$G(x) = p(x) + o\big(x^{N-3/2}\big), \quad x \to \infty, \tag{3.1}$$

holds. But G is a primitive of order $N-1$ of g, so that if we write $p(x) = Ax^{N-1} + p_1(x)$, where p_1 is of order $N-2$, we obtain

$$G(x) - Ax^{N-1} = p_1(x) + o\big(x^{N-3/2}\big), \quad x \to \infty. \tag{3.2}$$

The definition of the order relations in the Cesàro sense therefore yields

$$g(x) - (N-1)!A = o\big(x^{-1/2}\big) \quad \text{(C)}\ x \to \infty, \tag{3.3}$$

and consequently,

$$\lim_{x\to\infty} g(x) = (N-1)!A \quad \text{(C)}, \tag{3.4}$$

which means that

$$\langle f(x), \phi(x)\rangle = (N-1)!A \quad \text{(C)}, \tag{3.5}$$

as required. □

Observe that the unit function, $\phi(x) = 1$, belongs to $\mathcal{K}$. It follows that if $f \in \mathcal{K}'$ then $\langle f(x), 1\rangle$ (C) exists. In particular, if $f \in \mathcal{K}'$ is locally integrable then the integral $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ is Cesàro summable to the value $\langle f(x), 1\rangle$. Similarly, if $f(x) = \sum_{n=1}^{\infty} a_n \delta(x - b_n)$ belongs to $\mathcal{K}'$, then the series $\sum_{n=1}^{\infty} a_n$ is Cesàro summable to $\langle f(x), 1\rangle$.

In the classical theory of summability [8] it is shown that if the series $\sum_{n=1}^{\infty} a_n$ diverges to infinity in the ordinary sense then $\sum_{n=1}^{\infty} a_n = \infty$ (C). The same idea can be used in the summability of distributional evaluations to obtain the following Tauberian result.

Lemma 2. *Let g be a locally integrable function. Suppose $g(x) \geq 0$ for $x \in \mathbb{R}$. If the integral $\int_{-\infty}^{\infty} g(x)\,\mathrm{d}x = \langle g(x), 1\rangle$ is* (C) *summable then the integral converges.*

Proof. It is enough to prove the result if $\operatorname{supp} g \subset [0, \infty)$. If the integral $\int_0^{\infty} g(x)\,\mathrm{d}x$ diverges, then it diverges to infinity. Let $A > 0$. Then we can find $y > 0$ such that $\int_0^y g(t)\,\mathrm{d}t > A$. Define the function $h(x)$ by $h(x) = g(x)$ for $x < y$ and $h(x) = 0$ for $x \geq y$. Then

$$\limsup_{x\to\infty}\left(\frac{1}{x^n}\int_0^x \frac{(x-t)^n}{n!}\,g(t)\,\mathrm{d}t\right) \geq \lim_{x\to\infty}\left(\frac{1}{x^n}\int_0^x \frac{(x-t)^n}{n!}\,h(t)\,\mathrm{d}t\right) > A,$$

and it follows that $\int_0^{\infty} g(t)\,\mathrm{d}t = \infty$ (C). □

We remark that this lemma has an obvious generalization, which will not be needed presently, namely, if μ is a positive Radon measure for which the integral $\int_{-\infty}^{\infty} \mathrm{d}\mu = \langle \mu, 1\rangle$ is (C) summable then the integral is convergent.

We shall also need the following simple variation of a well-known result.

Lemma 3. *Let ψ be a smooth function. Suppose the integrals*

$$\int_{-\infty}^{\infty} |\psi^{(j)}(x)|^2 x^{2n}\,\mathrm{d}x, \tag{3.6}$$

converge for all $j, n = 0, 1, 2, \ldots$. Then $\psi \in \mathcal{S}$.

Proof. If $n > 0$, $x \neq 0$, and $j \in \mathbb{N}$, then the Cauchy–Schwartz inequality yields

$$\begin{aligned}
\left|\psi^{(j)}(x)\right| &= \left|\int_0^x \left(t^{n+1/2}\psi^{(j)}(t)\right)'\mathrm{d}t\right||x|^{-n-1/2} \\
&= \left|\int_0^x \left\{(n+1/2)t^{n-1/2}\psi^{(j)}(t) + t^{n+1/2}\psi^{(j+1)}(t)\right\}\mathrm{d}t\right|\,|x|^{-n-1/2} \\
&\leq \left\{(n+1/2)\left(\int_0^x \left|t^{n-1/2}\psi^{(j)}(t)\right|^2\mathrm{d}t\right)^{1/2}\right. \\
&\qquad \left. + \left(\int_0^x \left|t^{n+1/2}\psi^{ft(j+1)}(t)\right|^2\mathrm{d}t\right)^{1/2}\right\}|x|^{-n} \\
&\leq \left\{(n+1/2)\left(\int_0^\infty \left|t^{n-1/2}\psi^{(j)}(t)\right|^2\mathrm{d}t\right)^{1/2}\right. \\
&\qquad \left. + \left(\int_0^\infty \left|t^{n+1/2}\psi^{(j+1)}(t)\right|^2\mathrm{d}t\right)^{1/2}\right\}|x|^{-n}.
\end{aligned}$$

Thus

$$\psi^{(j)}(x) = O(|x|^{-n}), \quad x \to \infty, \tag{3.7}$$

for each $j \in \mathbb{N}$, for each $n > 0$. Therefore, $\psi \in \mathcal{S}$. □

We are going to use the following notation. If $\mathcal{A}$ and $\mathcal{B}$ are spaces of functions then $\mathcal{AB} = \{\phi\psi : \phi \in \mathcal{A},\ \psi \in \mathcal{B}\}$.

We can now give our result.

Theorem 1. *Let $\mathcal{A}$ be a space of test functions closed under complex conjugation. Suppose $\mathcal{AK} \subset \mathcal{A}$. Then*

$$\mathcal{A} \cap \mathcal{A}' \subset \mathcal{S}. \tag{3.8}$$

Proof. Observe first that the condition $\mathcal{AK} \subset \mathcal{A}$ yields the inclusion $\mathcal{AA}' \subset \mathcal{K}'$. Indeed, if $f \in \mathcal{A}'$, $\phi \in \mathcal{A}$ and $\psi \in \mathcal{K}$, then the identity

$\langle \phi f, \psi \rangle = \langle f, \phi\psi \rangle$ shows that ϕf is a well-defined element of $\mathcal{K}'$ since the right hand side is defined because $\phi\psi \in \mathcal{A}$.

Let now $f \in \mathcal{A} \cap \mathcal{A}'$. Then, for each $j \in \mathbb{N}$, the function $|f^{(j)}|^2 = f^{(j)} \bar{f}^{(j)}$ belongs to $\mathcal{A}\mathcal{A}'$ and, consequently, belongs to $\mathcal{K}'$. Hence if $n \in \mathbb{N}$ then $\langle |f^{(j)}(x)|^2, x^{2n} \rangle$ is a well-defined evaluation in $\mathcal{K}' \times \mathcal{K}$ since x^{2n} belongs to $\mathcal{K}$, and therefore the integral $\int_{-\infty}^{\infty} |f^{(j)}(x)|^2 x^{2n}\, \mathrm{d}x$, namely this evaluation, is Cesàro summable. The Lemma 2 then yields the convergence of these integrals and by the Lemma 3 we conclude that $f \in \mathcal{S}$. $\square$

Notice that if the conditions of the theorem are satisfied then $\mathcal{A} \cap \mathcal{A}'$ might be a proper subset of $\mathcal{S}$. However, when $\mathcal{S} \subset \mathcal{A} \subset \mathcal{S}'$ then $\mathcal{S} \subset \mathcal{A} \cap \mathcal{A}'$ and therefore in this case the theorem would give $\mathcal{A} \cap \mathcal{A}' = \mathcal{S}$.

The following examples illustrate the use of the theorem.

Example 1. If $\mathcal{A} = \mathcal{K}$ then clearly $\mathcal{K}\mathcal{K} \subset \mathcal{K}$ and so

$$\mathcal{K} \cap \mathcal{K}' = \mathcal{S}. \tag{3.9}$$

Observe that there are functions like $\sin x$, $\cos x$ or $J_0(x)$ which are of rapid distributional decay at infinity, i.e., belong to $\mathcal{K}'$, but which do not belong to $\mathcal{S}$: such functions do not belong to $\mathcal{K}$.

Example 2. If $\mathcal{A} = \mathcal{O}_M$ we have $\mathcal{O}_M \mathcal{K} \subset \mathcal{O}_M$. Thus

$$\mathcal{O}_M \cap \mathcal{O}_M' = \mathcal{S}. \tag{3.10}$$

Example 3. If $\mathcal{A} = \mathcal{O}_C$ we have $\mathcal{O}_C \mathcal{K} \subset \mathcal{O}_C$. Thus

$$\mathcal{O}_C \cap \mathcal{O}_C' = \mathcal{S}. \tag{3.11}$$

This can also be obtained from (3.10) by taking the Fourier transform.

Example 4. Let us take $\mathcal{A} = \mathcal{P}$, the space of test functions ϕ that satisfy $\phi^{(j)}(x) = o(e^{\alpha|x|})$ for each $j \in \mathbb{N}$ and for each $\alpha > 0$. This space was introduced in [11]. Then $\mathcal{P}\mathcal{K} \subset \mathcal{P}$ and so

$$\mathcal{P} \cap \mathcal{P}' = \mathcal{S}. \tag{3.12}$$

Example 5. If $\mathcal{A} = \mathcal{E}$ then we clearly have that $\mathcal{E} \cap \mathcal{E}' = \mathcal{D}$, a subspace of $\mathcal{S}$. Observe however that in this case $\mathcal{E} \cap \mathcal{E}'$ is a *proper* subspace of $\mathcal{S}$.

Example 6. The spaces $\mathcal{G}_\alpha$ for $\alpha \le 1$ satisfy $\mathcal{G}_\alpha \mathcal{K} \subset \mathcal{G}_\alpha$. Hence

$$\mathcal{G}_\alpha \cap \mathcal{G}'_\alpha = \mathcal{S}, \quad \text{for } \alpha \le 1. \tag{3.13}$$

This does not hold if $\alpha > 1$.

The theorem also holds in $\mathbb{R}^d$. Actually the proof is the same, except that in the multi-dimensional case one needs to consider the Cesàro summability of distributional evaluations by spherical means [2].

4. Counterexamples

We now give two counterexamples to show how the conclusion of the theorem might not hold if the assumptions are not satisfied.

Let us first consider the space $\mathcal{G}_\alpha$ for $\alpha > 1$. It is clear that $\mathcal{K}$ is not a subset of $\mathcal{G}_\alpha$ and since $1 \in \mathcal{G}_\alpha$, then $\mathcal{G}_\alpha \mathcal{K} \not\subset \mathcal{G}_\alpha$. In this case we also have $\mathcal{G}_\alpha \cap \mathcal{G}'_\alpha \not\subset \mathcal{S}$ since any polynomial belongs to $\mathcal{G}_\alpha \cap \mathcal{G}'_\alpha$. Indeed, if p is a polynomial and $\phi \in \mathcal{G}_\alpha$ then $\langle p, \phi \rangle = \int_{-\infty}^{\infty} P(x)\phi^{(N)}(x)\,\mathrm{d}x$ where $P^{(N)} = p$ and N is large enough for the integral to converge. Observe that in general if $f \in \mathcal{G}'_\alpha$ and $\phi \in \mathcal{G}_\alpha$ then the evaluation $\langle f, \phi \rangle$ is not (C) summable.

As a second example, consider the space $\mathcal{C} = \varinjlim \mathcal{C}_\gamma$, where $\mathcal{C}_\gamma$ is the space of continuous functions g that satisfy $g(x) = O(|x|^\gamma)$ as $x \to \infty$, with the canonical norm,

$$\|g\|_\gamma = \max\bigl\{ \sup\{|g(x)| : |x| \le 1\},\ \sup\{|x|^{-\gamma}|g(x)| : |x| \ge 1\}\bigr\}. \tag{4.1}$$

Observe that $\mathcal{C}\mathcal{K} \subset \mathcal{C}$. However, $\mathcal{C} \cap \mathcal{C}' \not\subset \mathcal{S}$. Indeed, let $g_0 \in \mathcal{C}$ be a positive function with $\operatorname{supp} g_0 \subset [0,1]$. Let $\varrho > 1$. Define the new function $g \in \mathcal{C}$ by

$$g(x) = \sum_{n=1}^{\infty} g(\varrho^n(x-n)). \tag{4.2}$$

Observe that $g(x) = O(1)$ as $|x| \to \infty$ and so g indeed belongs to $\mathcal{C}$. But also

$$\int_{-\infty}^{\infty} g(x)|x|^n\,\mathrm{d}x < \infty, \quad \forall n \in \mathbb{N}, \tag{4.3}$$

and so

$$\int_{-\infty}^{\infty} g(x)\phi(x)\,\mathrm{d}x \tag{4.4}$$

converges for any $\phi \in \mathcal{C}$. Therefore $g \in \mathcal{C}'$. Clearly, however, $g \notin \mathcal{S}$.

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