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The structure of two-sided networks for completely simple semigroups

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Abstract. As a byproduct of the kernel-trace approach to congruences on regular semigroups S we have the operators $\Gamma = \{T_l, K, T_r, t_l, k, t_r\}$ induced by the classes of the left trace, kernel and right trace relations via their upper and lower ends. The semigroup generated by these operators forms the two-sided network of S.

For S a Rees matrix semigroup with a normalized sandwich matrix, the two-sided network Ω was characterized in a previous paper by the author in terms of generators and relations. The theme of this paper is the structure of Ω. After isolating the right zeros of Ω, consisting of constant operators, we find the structure of the rest by means of certain triples. These triples resemble those of a Rees matrix semigroup and indeed a part of Ω can be embedded into such a semigroup. Other structural features of Ω are studied together with a special case.

1. Introduction and summary

The kernel-trace approach to congruences on regular semigroups has brought unexpected results; maybe the most interesting one is the appearance of operators T_l, K, T_r, t_l, k, t_r defined as follows. Let S be a regular semigroup with congruence lattice $\mathcal{C}(S)$. For each $\rho \in \mathcal{C}(S)$, $\ker\rho$, the kernel of ρ, is the union of idempotent ρ-classes, $\operatorname{tr}\rho$, the trace of ρ, is the restriction of ρ to the set $E(S)$ of idempotents of S. Further,

$$\operatorname{ltr}\rho = \operatorname{tr}\,(\rho \vee \mathcal{L})^\circ, \quad \operatorname{rtr}\rho = \operatorname{tr}\,(\rho \vee \mathcal{R})^\circ,$$

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where ( )° means the greatest congruence contained in ( ), are the left and right traces of ρ, respectively. The relations $\mathcal{T}_l$, $\mathcal{K}$, $\mathcal{T}_r$ defined on $\mathcal{C}(S)$ by

$$\begin{aligned} \lambda\ \mathcal{T}_l\ \rho &\text{ if } \operatorname{ltr}\lambda = \operatorname{ltr}\rho, \\ \lambda\ \mathcal{K}\ \rho &\text{ if } \ker\lambda = \ker\rho, \\ \lambda\ \mathcal{T}_r\ \rho &\text{ if } \operatorname{rtr}\lambda = \operatorname{rtr}\rho \end{aligned}$$

are the left trace, kernel and right trace relations on $\mathcal{C}(S)$, respectively. The first and third of these are complete congruences and the second one is a complete $\wedge$-congruence on $\mathcal{C}(S)$. Each of their classes is an interval, so for $\rho \in \mathcal{C}(S)$, denoting by $\rho\mathcal{P}$ the $\mathcal{P}$-class of ρ, we may write

$$\rho\mathcal{T}_l = [\rho t_l, \rho T_l], \quad \rho\mathcal{K} = [\rho k, \rho K], \quad \rho\mathcal{T}_r = [\rho t_r, \rho T_r]$$

which introduces the above mentioned operators on $\mathcal{C}(S)$. For a treatment of this subject, consult [2].

For a fixed $\rho \in \mathcal{C}(S)$, the set

$$\rho,\ \rho T_l,\ \rho K,\ \rho T_r,\ \rho t_l,\ \rho k,\ \rho t_r,\ \rho T_l K,\ \rho T_l T_r,\ \ldots,$$

partially ordered by inclusion, is the two-sided network for ρ. This network was studied in [6] on a free completely regular semigroup of infinite rank with a view of application to the semigroups generated by corresponding operators on the lattice of varieties of completely regular semigroups. The semigroup generated by the set Γ of the above six operators we call the *two-sided network* for S.

In [5], we have characterized the two-sided network Ω for a Rees matrix semigroup by means of generators Γ and relations Σ and found an isomorphic copy of Ω within the free semigroup Γ^+ with the product of representatives of the congruence μ on Γ^+ induced by Σ. The network so obtained turns out to be infinite (in general) and appears quite complicated. It is the purpose of the present paper to determine the semigroup structure of this network.

Defining the trace relation $\mathcal{T}$ on $\mathcal{C}(S)$ by

$$\lambda\ \mathcal{T}\ \rho \quad \text{if } \operatorname{tr}\lambda = \operatorname{tr}\rho$$

we are led to operators T and t defined for each $\rho \in \mathcal{C}(S)$ by $\rho\mathcal{T} = [\rho t, \rho T]$. In [4], we performed a similar analysis for the operators K, T, k, t. In this case, the corresponding network is finite and of quite simple structure. We may order these (two-sided) networks by letting $u \leq v$ if $\rho u \subseteq \rho v$ for all $\rho \in \mathcal{C}(S)$. In the same paper, we characterized the lattice generated by the network of congruences in terms of generators and relations. In the case of the two-sided networks this seems a formidable task and is not even attempted.

Section 2 contains all the necessary notation as well as the main result in [5]. We treat in Section 3 the semigroup Δ of right zeros and prove some related lemmas to be used later. Section 4 contains the main result of the paper namely a faithful representation of Ω by Δ and certain triples; its proof requires seven additional lemmas. A part of the quotient Ω/Δ is embedded into a Rees matrix semigroup in Section 5. The regularity of elements of Ω and its $\mathcal{D}$-structure are determined in Section 6. The case of rectangular groups is treated in Section 7 which represents the simplest case that can occur; the section ends with an example showing what might happen in the rectangular group case.

2. Notation

We state here first the minimum notation needed from [5] and for a complete discussion refer to that paper. Let

$$\Gamma = \{T_l, K, T_r, t_l, k, t_r\}$$

where these are operators on the congruence lattice $\mathcal{C}(S)$ of a fixed Rees matrix semigroup S. We consider the free semigroup Γ^+ generated by Γ as the set of all (nonempty) words over Γ. We refer to elements of Γ as letters. For any $w \in \Gamma^+$,

$h(w)$, the *head* of w, is the first letter occurring in w,

$l(w)$, the *length* of w, is the length of w,

$t(w)$, the *tail* of w, is the last letter occurring in w.

Let Σ be the set of relations:

(i) $T_lT_r = T_rT_l = T_lK = T_rK$,

(ii) $T_lt_rk = T_rt_lk$,

(iii) $t_lt_r = t_rt_l = kt_l = kt_r = t_lt_rk$,

(iv) $(Kp)^2 = Kp$, $(pK)^2 = pK$ for $p \in \{t_l, t_r\}$,

(v) $P^2 = pP$, $p^2 = Pp = p$ for $P \in \{T_l, K, T_r\}$, $p \in \{t_l, k, t_r\}$ and P corresponding to p

and μ be the congruence on Γ^+ induced by Σ.

Let Φ' be the set of all subwords of the words of the form $Kt_lKt_rKt_l \dots Kt_r$, and

$$\begin{aligned}
\Phi_0 &= \{w \in \Phi' \mid l(w) \geq 2\}, \\
\Phi_l &= \{wKT_l \mid w \in \Phi',\, t(w) = t_r\}, \\
\Phi_k &= \{wk \mid w \in \Phi',\, l(w) \geq 2,\ t(w) \in \{t_l, t_r\}\}, \\
\Phi_r &= \{wKT_r \mid w \in \Phi',\, t(w) = t_l\}, \\
\Phi_e &= \{t_lKt_l, t_lKT_l, t_rKt_r, t_rKT_r\}, \\
\Phi &= \Phi_0 \cup \Phi_l \cup \Phi_k \cup \Phi_r \cup \Phi_e, \\
J &= \{t_lk, t_rk, KT_l, KT_r\}.
\end{aligned} \tag{1}$$

We use the letters ω and ϵ for the universal and equality relations on any set. The letters σ and τ denote the least group and greatest idempotent pure congruences, respectively, and $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ Green's relations on any semigroup. Beside these meanings, we use these letters, with some meets and joins, for certain words over Γ as follows. Let Δ be the following set of words:

$$\begin{array}{lll}
\omega = T_lT_r, & \mathcal{L} = t_rT_l, & \mathcal{R} = t_lT_r, \\
\mathcal{H} = T_lk, & \sigma = T_lt_rK, & \tau = t_lt_rk, \\
\mathcal{L} \wedge \sigma = T_lt_r, & \mathcal{R} \wedge \sigma = T_rt_l, & \mathcal{H} \wedge \sigma = T_lt_rk, \\
\mathcal{L} \vee \tau = t_lt_rKT_l, & \mathcal{R} \vee \tau = t_lt_rKT_r, & \mathcal{L} \wedge \tau = t_lt_rKt_r, \\
 & \mathcal{R} \wedge \tau = t_lt_rKt_l. &
\end{array}$$

Finally let

$$\Omega = \Gamma \cup J \cup \Phi \cup \Delta. \tag{2}$$

It is proved in [5] that Ω is a set of representatives for the μ-classes and the products of representatives are given. As a consequence, any word can be reduced to its normal form by using the relations in Σ. When writing elements of Ω in the form $w = x_1 x_2 \dots x_n$ or $u = u_1 u_2 \dots u_n$ we shall always mean that $x_i, u_i \in \Gamma$ for $i = 1, 2, \dots, n$. The main result of paper [5] follows.

Theorem 2.1. *For any completely simple semigroup S, the semigroup generated by Γ on $\mathcal{C}(S)$ is a homomorphic image of Ω ($\cong \langle \Gamma, \Sigma \rangle$). No proper homomorphic image of Ω has this property.*

In order to avoid cumbersome notation, we shall denote by juxtaposition both the product in Γ^+ and in Ω and will make their distinction explicit only when this is needed for clarity. Note that the union in (1) is disjoint. We shall need in this paper a somewhat different decomposition of Ω. So we introduce some more notation. Let

$$\Psi = \Omega \setminus \Delta,$$

$$\Pi = \{w \in \Psi \mid \text{either } t_l, t_r \text{ or } t_l, T_r \text{ or } T_l, t_r \text{ occur in } w\},$$

$$\Theta = \{w \in \Psi \mid \text{not both } t_l, t_r \text{ or } t_l, T_r \text{ or } T_l, t_r \text{ occur in } w\}.$$

We thus get $\Omega = \Theta \cup \Pi \cup \Delta$, again a disjoint union. Let

$$I = \{t_l K, t_r K, K t_l, K t_r\}, \quad \Psi_e = \{K t_l K, K t_r K, K t_l k, K t_r k\}.$$

Hence $\Theta = \Gamma \cup I \cup J \cup \Phi_e \cup \Psi_e$, also a disjoint union. For $n = 1, 2, \dots,$ let

$$\Psi_n = \{w \in \Psi \mid l(w) = n\}$$

so that $\Psi = \bigcup_{n=1}^{\infty} \Psi_n$, a disjoint union. Note that $\Psi_1 = \Gamma$, $\Psi_2 = I \cup J$,

$$\Psi_3 = \{t_l K t_r, t_r K t_l, t_r K T_l, t_l K T_r\} \cup \Phi_e \cup \Psi_e,$$

$$\Psi_4 = \{t_l K t_r K, t_r K t_l K, t_l K t_r k, t_r K t_l k, K t_l K t_r, K t_r K t_l, K t_l K T_r, K t_r K T_l\},$$

and so on with each Ψ_n having eight elements for any $n \geq 4$. Also let

$$\mathcal{T}_l = \{T_l, t_l\}, \quad \mathcal{K} = \{K, k\}, \quad \mathcal{T}_r = \{T_r, t_r\},$$

and for any $u \in \Gamma$,

$$\overline{u} = \begin{cases} t_l & \text{if } u \in \mathcal{T}_r \\ K & \text{if } u \in \mathcal{K} \\ t_r & \text{if } u \in \mathcal{T}_l \end{cases}.$$

Elements of Ω are ordered as indicated in Section 1. However, we can write congruences on a Rees matrix semigroup by means of admissible triples, see ([1], Section III.4) and compare them componentwise. This is particularly suitable for elements of Δ, which are constant operators. The partially ordered set of Δ is represented in Diagram 1. With this ordering, we introduce the following product. For $u \in \Delta$ and $v \in \{t_l, K, t_r\}$, let

$$u \circ v = \begin{cases} \sigma & \text{if } u \geq \mathcal{R} \wedge \sigma,\ v = t_l \text{ or } u \not\leq \tau,\ v = K \text{ or } u \geq \mathcal{L} \wedge \sigma,\ v = t_r \\ \tau & \text{otherwise.} \end{cases}$$

We generally follow the standard notation and terminology which can be found in books [1] and [3].

3. Constant operators

According to ([5], Lemma 4.1), elements of Δ act on $\mathcal{C}(S)$ as constant functions and thus we refer to them as constant operators. These elements play an important role in our deliberations. We paraphrase this reference as follows.

Lemma 3.1. *Elements of Δ act as right zeros of Ω.*

Lemma 3.2. *The products $(\Gamma \cup \Delta)\Gamma$ in Ω are given in Table 1.*

Proof. This follows by a simple calculation from the relations in Σ. As a sample, we compute

$$T_r k = T_r(Kk) = (T_r K)k = (T_l K)k = T_l(Kk) = T_l k = \mathcal{H},$$

$$kT_l = k(t_l T_l) = (kt_l)T_l = (t_r t_l)T_l = t_r(t_l T_l) = t_r T_l = \mathcal{L},$$

$$\omega T_l = (T_l T_r)T_l = (T_r T_l)T_l = T_r(T_l T_l) = T_r T_l = T_l T_r = \omega,$$

$$\omega k = (T_l T_r)k = (T_l K)k = T_l(Kk) = T_l k = \mathcal{H}.$$

The remainning cases follow just as easily. □

The products $\Delta(\Psi \setminus \Gamma)$ can be expressed by means of the multiplication $u \circ v$ defined in Section 2 and have the following form.

	T_l	K	T_r	t_l	k	t_r
T_l	T_l	ω	ω	t_l	$\mathcal{H}$	$\mathcal{L} \wedge \sigma$
K	KT_l	K	KT_r	Kt_l	k	Kt_r
T_r	ω	ω	T_r	$\mathcal{R} \wedge \sigma$	$\mathcal{H}$	t_r
t_l	T_l	t_lK	$\mathcal{R}$	t_l	t_lk	ϵ
k	$\mathcal{L}$	K	$\mathcal{R}$	ϵ	k	ϵ
t_r	$\mathcal{L}$	t_rK	T_r	ϵ	t_rk	t_r
ω	ω	ω	ω	$\mathcal{R} \wedge \sigma$	$\mathcal{H}$	$\mathcal{L} \wedge \sigma$
$\mathcal{L} \vee \tau$	$\mathcal{L} \vee \tau$	ω	ω	$\mathcal{R} \wedge \tau$	$\mathcal{H}$	$\mathcal{L} \wedge \sigma$
$\mathcal{R} \vee \tau$	ω	ω	$\mathcal{R} \vee \tau$	$\mathcal{R} \wedge \sigma$	$\mathcal{H}$	$\mathcal{L} \wedge \tau$
$\mathcal{L}$	$\mathcal{L}$	ω	ω	ϵ	$\mathcal{H}$	$\mathcal{L} \wedge \tau$
$\mathcal{R}$	ω	ω	$\mathcal{R}$	$\mathcal{R} \wedge \sigma$	$\mathcal{H}$	ϵ
$\mathcal{H}$	$\mathcal{L}$	ω	$\mathcal{R}$	ϵ	$\mathcal{H}$	ϵ
σ	ω	σ	ω	$\mathcal{R} \wedge \sigma$	$\mathcal{H} \wedge \sigma$	$\mathcal{L} \wedge \sigma$
$\mathcal{L} \wedge \sigma$	$\mathcal{L}$	σ	ω	ϵ	$\mathcal{H} \wedge \sigma$	$\mathcal{L} \wedge \sigma$
$\mathcal{R} \wedge \sigma$	ω	σ	$\mathcal{R}$	$\mathcal{R} \wedge \sigma$	$\mathcal{H} \wedge \sigma$	ϵ
$\mathcal{H} \wedge \sigma$	$\mathcal{L}$	σ	$\mathcal{R}$	ϵ	$\mathcal{H} \wedge \sigma$	ϵ
τ	$\mathcal{L} \vee \tau$	τ	$\mathcal{R} \vee \tau$	$\mathcal{R} \wedge \tau$	ϵ	$\mathcal{L} \wedge \tau$
$\mathcal{L} \wedge \tau$	$\mathcal{L}$	τ	$\mathcal{R} \vee \tau$	ϵ	ϵ	$\mathcal{L} \wedge \tau$
$\mathcal{R} \wedge \tau$	$\mathcal{L} \vee \tau$	τ	$\mathcal{R}$	$\mathcal{R} \wedge \tau$	ϵ	ϵ
ϵ	$\mathcal{L}$	τ	$\mathcal{R}$	ϵ	ϵ	ϵ

Table 1

Lemma 3.3. *For $u \in \Delta$ and $v \in \Psi \setminus \Gamma$, we have in Ω,* $uv = (u \circ h(v))t(v)$.

Proof. We consider several cases in which we make repeated use of Diagram 1 and Table 1.

Case: $u \geq \mathcal{R} \wedge \sigma$, $h(v) = t_l$. Then $ut_l \;\mu\; \mathcal{R} \wedge \sigma$ and the elements of the sequence

$$ut_lK,\ ut_lKt_r,\ ut_lKt_rK,\ ut_lKt_rKt_l,\ ut_lKt_rKt_lK,\ \dots$$

are μ-related to

$$\sigma,\ \mathcal{L} \wedge \sigma,\ ,\sigma,\ \mathcal{R} \wedge \sigma,\ \sigma,\ \dots$$

and we see that these are μ-related to

$$\sigma K,\ \sigma t_r,\ \sigma K,\ \sigma t_l,\ \sigma K,\ \dots$$

Diagram 1

which evidently implies the assertion of the lemma for the cases $t(v) \in \{t_l, K, t_r\}$. Using this, we obtain

$$\begin{aligned}
&ut_lKt_r \dots KT_l \ \mu \ \sigma T_l, \\
&ut_lKt_r \dots t_lk \ \mu \ (\mathcal{R} \wedge \sigma)k \ \mu \ \mathcal{H} \wedge \sigma \ \mu \ \sigma k, \\
&ut_lKt_r \dots t_rk \ \mu \ (\mathcal{L} \wedge \sigma)k \ \mu \ \mathcal{H} \wedge \sigma \ \mu \ \sigma k, \\
&ut_lKt_r \dots KT_r \ \mu \ \sigma T_r.
\end{aligned}$$

Case: $u \not\geq \mathcal{R} \wedge \sigma, \ h(v) = t_l$. Then

$$ut_l \ \mu \begin{cases} \mathcal{R} \wedge \tau & \text{if } \mathcal{R} \wedge \tau \leq u \leq \mathcal{L} \vee \tau \\ \epsilon & \text{if } u \leq \mathcal{L} \end{cases}$$

and thus $ut_lK \ \mu \ \tau$. From now on, the argument is the same as in the preceding case if we substitute each σ by τ.

Case: $u \not\leq \tau, \ h(v) = K$. Then

$$uK \ \mu \begin{cases} \sigma & \text{if } \mathcal{H} \wedge \sigma \leq u \leq \sigma \\ \omega & \text{if } u \geq \mathcal{H} \end{cases}$$

and thus the elements of the sequence

$$uKt_l,\ uKt_lK,\ uKt_lKt_r,\ uKt_lKt_rK,\ \dots$$

are μ-related to

$$\mathcal{R}\wedge\sigma,\ \sigma,\ \mathcal{L}\wedge\sigma,\ \sigma,\ \dots$$

and we see that these are μ-related to

$$\sigma t_l,\ \sigma K,\ \sigma t_r,\ \sigma K,\ \dots$$

which evidently implies the assertion of the lemma for the cases $t(v) \in \{t_l, K, t_r\}$. Using this, we obtain the assertion of the lemma for the cases $t(v) \in \{T_l, k, T_r\}$ exactly as in the first case above.

Case: $u \geq \tau$, $h(v) = K$. Then $uK\ \mu\ \tau$ and the rest of the argument is the same as in the preceding case if we subtitute each σ by τ.

The two cases concerning the instance $h(v) = t_r$ are dual to the first two cases above. □

The next lemma gives the first inkling into the structure of Ω.

Lemma 3.4. *Both* Δ *and* $\Pi \cup \Delta$ *are ideals of* Ω*.*

Proof. By Lemma 3.1, elements of Δ are right zeros of Ω and hence Δ is a left ideal of Ω. By Lemmas 3.2 and 3.3, or by repeated application of Lemma 3.2 alone, we deduce that Δ is also a right ideal of Ω.

Next let $u \in \Omega$ and $v \in \Pi$ be such that $uv \notin \Delta$. Since $v \in \Pi$, either both t_l and t_r or both t_l and T_r or both T_l and t_r occur as letters in v. Table 2 and its dual obtained by interchanging l and r show that after the reductions made due to taking the product uv at least one of the listed pairs occurs in uv. Therefore $uv \in \Pi$ showing that $\Pi \cup \Delta$ is a left ideal of Ω. The same type of argument shows that $\Pi \cup \Delta$ is also a right ideal of Ω. □

The next lemma identifies elements of Δ by the behavior of pairs of their adjacent letters.

Lemma 3.5. *Let* $w = x_1x_2\dots x_n \in \Omega$, $n \geq 2$*. Then* $x_ix_{i+1} \notin \Delta$ *for* $i = 1, 2, \dots, n-1$ *if and only if* $w \notin \Delta$*.*

Proof. *Necessity.* The argument is by induction on n. If $n = 2$, there is nothing to prove. Assume the statement valid for $n - 1$ where

	t_lK	t_lk	t_lKt_l	t_lKT_l	Kt_l	KT_l	Kt_lK
t_lK	t_lK	t_lk	t_lKt_l	t_lKT_l	t_lKt_l	t_lKT_l	t_lK
Kt_lK	Kt_lK	Kt_lk	Kt_l	KT_l	Kt_l	KT_l	Kt_lK
t_rK	t_rKt_lK	t_rKt_lk	t_rKt_l	t_rKT_l	t_rKt_l	t_rKT_l	t_rKt_lK
Kt_l	Kt_lK	Kt_lk	Kt_l	KT_l	Kt_l	KT_l	Kt_lK
t_lKt_l	t_lK	t_lk	t_lKt_l	t_lKT_l	t_lKt_l	t_lKT_l	t_lK
t_lk					t_lKt_l	t_lKT_l	t_lK
Kt_lk					Kt_l	KT_l	Kt_lK
t_rk					t_rKt_l	t_rKT_l	t_rKt_lK
Kt_rk					Kt_rKt_l	Kt_rKT_l	Kt_rKt_lK

Table 2

$n > 2$ and consider w with $l(w) = n$. Then $w = ux$ for some $u \in \Omega$ with $l(w) = n-1$ and $x \in \Gamma$. By the induction hypothesis, we have that $u \notin \Delta$. The problem reduces to showing that

$$u \in \Psi \setminus \Gamma, \quad x \in \Gamma, \quad t(u)x \in \Psi \setminus \Gamma \implies ux \notin \Delta.$$

Assume the antecedent of this implication. Hence $t(u)x \in I \cup J$. We distinguish several cases.

Case: $t(u) = t_l$. Then

$$u \in \{Kt_l, t_lKt_l, t_rKt_l\} \cup \{vt_rKt_l \in \Psi \mid v \in \Phi'\}$$

and hence for $x \in \{K, k\}$, we get

$$ux \in \{Kt_lx, t_lx, t_rKt_lx\} \cup \{vt_rKt_lk \mid v \in \Phi',\ t(v) = K\}$$

which shows that $ux \notin \Delta$.

Case: $t(u) = K$. Then

$$u \in \{t_lK, t_rK\} \cup \{vt_lK \in \Psi \mid v \in \Phi'\} \cup \{vt_rK \in \Psi \mid v \in \Phi'\}$$

and hence for $x = t_l$ we obtain

$$ux \in \{t_lKt_l, t_rKt_l\} \cup \{vt_l \in \Psi \mid v \in \Phi'\} \cup \{vt_rKt_l \in \Psi \mid v \in \Phi'\}$$

and analogously for $x = t_r$; for $x = T_r$, we get

$$ux \in \{t_l K T_l, t_r K T_l\} \cup \{vT_l \in \Psi \mid v \in \Phi'\} \cup \{vt_r K T_l \in \Psi \mid v \in \Phi'\}$$

and analogously for $x = T_l$. Thus again $ux \notin \Delta$.

The case when $t(u) = t_r$ is dual to the case $t(u) = t_l$. This exhausts all the choices for $t(u)x$ in $I \cup J$.

Sufficiency. By contrapositive, assume that $x_i x_{i+1} \in \Delta$ for some i. By Lemma 3.1, we have $x_1 x_2 \dots x_i x_{i+1} = x_i x_{i+1}$ and by Lemmas 3.2 and 3.3, we conclude that $x_i x_{i+1} \dots x_n \in \Delta$ and thus $w = x_1 x_2 \dots x_n \in \Delta$ where all these products are taken in Ω. □

Lemma 3.5 will now be used to establish a lemma which will find many applications in the next two sections.

Lemma 3.6. *Let $u, v \in \Psi$. Then $uv \notin \Delta$ if and only if $t(u)h(v) \notin \Delta$.*

PROOF. *Necessity.* If no contraction occurs in the forming of the product uv, then $t(u)h(v) \notin \Delta$ follows directly from Lemma 3.5. Hence assume that some contraction takes place in uv.

Case: $l(u) = 1$. Then $u \in \Gamma$ and the possible contractions are:

$$\begin{aligned} t_p(Kt_pK) &= t_pK, & t_p(Kt_pk) &= t_pk, \\ K(t_pKt_p) &= Kt_p, & K(t_pKT_p) &= KT_p \end{aligned}$$

for $p \in \{l, r\}$ and $uh(v) = h(v)$. In all these cases $t(u)h(v) \notin \Delta$.

Case: $l(v) = 1$. Then $v \in \Gamma$ and the only possible contractions are

$$\begin{aligned} (t_pKt_p)K &= t_pK, & (t_pKt_p)k &= t_pk, \\ (Kt_pK)t_p &= Kt_p, & (Kt_pK)T_p &= KT_p \end{aligned}$$

for $p \in \{l, r\}$ and $t(u)v = v$. In all these cases $t(u)h(v) \notin \Delta$.

Case: $l(u), l(v) > 1$. Table 2 and its dual obtained by interchanging l and r show all the contractions that may occur when forming the product uv. Suppose that the end of u appears in the first column of Table 2 and denote it by u'. Similarly, assume that the beginning of v appears in the first row of Table 2 and denote it by v'. From Table 2 we see that $u'v' \notin \Delta$ if and only if the conjunction of $t(u) = k$ and $h(v) = t_l$ fails. It follows from Lemma 3.5 that the hypothesis implies that $u'v' \notin \Delta$ and hence from Table 2 that $t(u)h(v) \notin \Delta$.

Sufficiency. In view of Lemma 3.5, the cases $l(u) = 1$ or $l(v) = 1$ follow similarly as above. Hence assume that $l(u), l(v) \geq 2$. Again we use Table 2: the argument for its dual is obtained by interchanging the roles of l and r. By Table 2, we conclude that the hypothesis implies that the case $t(u) = k$ and $h(v) = t_l$ does not occur. The other instances yield the product $u'v'$, defined as above, where the product of adjacent letters does not fall into Δ. From the hypothesis that $u, v \notin \Delta$, this also holds for the remaining pairs of adjacent factors in uv. Now Lemma 3.5 implies that $uv \notin \Delta$. □

We shall use Lemma 3.6 without express reference.

4. A structure theorem

By Lemma 3.4, Δ is an ideal of Ω and by Lemma 3.1, its elements are right zeros of Ω. Hence Δ is the kernel of Ω. In Lemmas 3.2 and 3.3, we have the products $\Delta\Psi$. Therefore we have all the products $\Omega\Delta$ and $\Delta\Omega$ in a sufficiently explicit form. It remains to find the structure of the semigroup Ω/Δ and to find the product of any two elements of Ψ which falls into Δ.

Our structure theorem is based on the observation that any element of $\Psi \setminus \Gamma$, considered as a word over Γ, is uniquely determined by (the product of) its first two letters, its length and (the product of) its last two letters. This can be seen easily from the disjoint union

$$\Psi = \Gamma \cup J \cup \Phi_0 \cup \Phi_l \cup \Phi_k \cup \Phi_r \cup \Phi_e,$$

see (1) and (2). Hence to each element of $\Psi \setminus \Gamma$ we can uniquely assign a triple consisting of a word in I, an integer greater than 1 and a word in $I \cup J$. We can extend this representation to Γ by assigning to each element x of Γ the triple $(x, 1, x)$. This establishes a one-to-one correspondence of the set Ψ with the set of triples whose form carries the obvious resemblance with the triples appearing in the construction of a Rees matrix semigroup! There remains the problem of characterizing these triples, describing their multiplication and determining their product with elements of Δ.

Recall the notation from Section 2. Below $p, q \in \{l, r\}$, $\tau_p \in \mathcal{T}_p$, $\tau_q \in \mathcal{T}_q$, $\kappa \in \mathcal{K}$ for all choices. Let

$$\begin{aligned} M_1 &= \{(u, 1, u) \mid u \in \Gamma\}, \\ M_2 &= \{(uv, 2, uv) \mid uv \in I \cup J\}, \\ M_3 &= \{(Kt_p, 3, t_p\kappa), (t_pK, 3, K\tau_q)\}, \end{aligned}$$

and for $n \geq 1$,

$$\begin{aligned} M_{4n} &= \{(t_pK, 4n, t_q\kappa), (Kt_p, 4n, K\tau_q) \mid p \neq q\}, \\ M_{4n+1} &= \{(Kt_p, 4n+1, t_q\kappa), (t_pK, 4n+1, K\tau_q) \mid p \neq q\}, \\ M_{4n+2} &= \{(t_pK, 4n+2, t_p\kappa), (Kt_p, 4n+2, K\tau_q)\}, \\ M_{4n+3} &= \{(Kt_p, 4n+3, t_p\kappa), (t_pK, 4n+3, K\tau_q) \mid p \neq q\}, \end{aligned}$$

finally set

$$M = \left(\bigcup_{n=1}^{\infty} M_n \right) \cup \Delta.$$

We now proceed to define a product in M. For this we need the following symbolism. Generally, we denote the product in Ω by juxtaposition to avoid cumbersome notation. As an exception to this rule, we let $w \in \Gamma^+$ and denote by $\overline{w}$ its representative in Ω, that is w reduced according to the relations Σ. With this setting, define

$$[w] = l(\overline{w}) - l(w).$$

Hence $[w]$ represents the loss of length due to the reduction according to Σ. For any $w \in \Psi$,

$i(w)$, the *initial part* of w, is equal to w if $l(w) = 1$, and is equal to the product of the first two letters of w otherwise,

$f(w)$, the *final part* of w, is equal to w if $l(w) = 1$ and it is equal to the product of the last two letters of w otherwise.

Now let $U_m \in M_m$ with the notation

$$U_m = \begin{cases} (u_1, 1, u_1) & \text{if } m = 1 \\ (u_1u_2, 2, u_1u_2) & \text{if } m = 2 \\ (u_1u_2, 3, u_2u_3) & \text{if } m = 3 \\ (u_1u_2, m, u_{m-1}u_m) & \text{if } m > 3 \end{cases}$$

and $\hat{u}_m = u_1 \ldots u_m$ if $m \leq 3$. Also let $V_n \in M_n$ with the analogous notation v_i and $\hat{v}_n$.

Define

$$U_m V_n = (a, m+b+n, c) \quad \text{if } u_m v_1 \notin \Delta \tag{3}$$

where

$$a = \begin{cases} i(\hat{u}_m \hat{v}_n) & \text{if } m, n \leq 3 \\ i(\hat{u}_m v_1 v_2) & \text{if } m \leq 3,\ n > 3 \\ u_1 u_2 & \text{if } m > 3 \end{cases},$$

$$b = \begin{cases} [\hat{u}_m \hat{v}_n] & \text{if } m, n \leq 3 \\ [\hat{u}_m v_1 v_2 \bar{v}_1] & \text{if } m \leq 3,\ n > 3 \\ [\bar{u}_m u_{m-1} u_m \hat{v}_n] & \text{if } m > 3,\ n \leq 3 \\ [u_{m-1} u_m v_1 v_2 \bar{v}_1] & \text{if } m, n > 3 \end{cases},$$

$$c = \begin{cases} u_1 v_1 & \text{if } m = n = 1 \\ f(u_1 u_2 v_1) & \text{if } m = 2,\ n = 1 \\ f(u_2 u_3 v_1) & \text{if } m = 3,\ n = 1 \\ f(u_{m-1} u_m v_1) & \text{if } m > 3,\ n = 1 \\ v_{n-1} v_n & \text{if } n > 1 \end{cases},$$

where the notation $\bar{v}_1$ was defined at the end of Section 2, and if $u_m v_1 \in \Delta$, then

$$U_m V_n = \begin{cases} u_m v_1 v_2 v_1 v_n & \text{if } v_1 v_2 \in \{K t_l, K t_r\} \\ u_m v_1 v_2 v_n & \text{if } v_1 v_2 \in \{t_l K, t_r K\} \\ u_m v_1 v_2 & \text{if } v_1 v_2 \in J \\ u_m v_1 & \text{if } n = 1 \end{cases}.$$

For $\theta, \delta \in \Delta$, we define

$$U_m \theta = \theta, \quad \theta U_m = \begin{cases} \theta \hat{u}_m & \text{if } m \leq 3 \\ (\theta \circ u_1) u_m & \text{if } m > 3 \end{cases}, \quad \theta \delta = \delta,$$

where $\theta \circ u_1$ was defined in Section 2.

This makes M into a groupoid. Note that the arguments of the functions i and f are in Ψ, that is, they are reduced, while the argument of the function $[\ ]$ is an element of Γ^+. In the last instance above, $\theta\hat{u}_m$ can be obtained by (repeated if $m > 1$) application of Table 1.

We are now ready for the statement of the main theorem of the paper.

Theorem 4.1. *The mapping*

$$\varphi : w \longmapsto \begin{cases} (i(w), l(w), f(w)) & \textit{if } w \in \Psi \\ w & \textit{if } w \in \Delta \end{cases}$$

is an isomorphism of Ω *onto* M.

The proof is preceded by seven lemmas. We start with the bijective property of φ.

Lemma 4.2. *The mapping* φ *in Theorem 4.1 is a bijection.*

Proof. Recall the notation Ψ_n from Section 2. It suffices to show that $\varphi_n = \varphi|_{\Psi_n}$ is a bijection of Ψ_n onto M_n for $n = 1, 2, \ldots$.

Now $\Psi_1 = \Gamma$ and $\Psi_2 = I \cup J$ so that the statement holds trivially for $n = 1, 2$. For $n = 3$, we list the elements of Ψ_3 and their images as follows:

$$\begin{aligned} &Kt_lK \longmapsto (Kt_l, 3, t_lK), && t_lKT_l \longmapsto (t_lK, 3, KT_l), \\ &Kt_lk \longmapsto (Kt_l, 3, t_lk), && t_lKt_r \longmapsto (t_lK, 3, Kt_r), \\ &t_lKt_l \longmapsto (t_lK, 3, Kt_l), && t_lKT_r \longmapsto (t_lK, 3, KT_r), \end{aligned}$$

and those obtained by interchanging l and r.

For the remaining Ψ_n, we first establish a concrete representation of their members. We illustrate the genesis of their form by starting with Ψ_4 and then affixing letters in front of its elements thereby obtaining $\Psi_5, \Psi_6, \Psi_7, \Psi_8, \Psi_9$ successively, see Table 3.

It will be convenient to have the following notation. For $u \in \Gamma$, let

$$\tilde{u} = \begin{cases} t_l & \text{if } u \in \mathcal{T}_l \\ K & \text{if } u \in \mathcal{K} \\ t_r & \text{if } u \in \mathcal{T}_r \end{cases},$$

Ψ_9	Ψ_8	Ψ_7	Ψ_6	Ψ_5	Ψ_4
K	t_l	K	t_r	K	t_lKt_rK
K	t_r	K	t_l	K	t_rKt_lK
K	t_l	K	t_r	K	t_lKt_rk
K	t_r	K	t_l	K	t_rKt_lk
t_r	K	t_l	K	t_r	Kt_lKt_r
t_l	K	t_r	K	t_l	Kt_rKt_l
t_r	K	t_l	K	t_r	Kt_lKT_r
t_l	K	t_r	K	t_l	Kt_rKT_l

Table 3

u_1	u_2	u_3	v_1	v_2	$u_3v_1v_2$	$u_2u_3v_1v_2$	$u_1u_2u_3v_1v_2$	v_3
K	t_l	K	t_l	K	Kt_lK	t_lK	Kt_lK	t_r
K	t_r	K	t_l	K		t_rKt_lK	Kt_rKt_lK	t_r
K	t_l	K	t_r	K	Kt_rK	t_lKt_rK	Kt_lKt_rK	t_l
K	t_r	K	t_r	K		t_rK	Kt_rK	t_l
t_l	K	t_l	K	t_l	t_lKt_l	Kt_l	t_lKt_l	K
t_r	K	t_l	K	t_l			t_rKt_l	K
t_l	K	t_l	K	t_r	t_lKt_r	Kt_lKt_r	t_lKt_r	K
t_r	K	t_l	K	t_r			$t_rKt_lKt_r$	K
t_l	K	t_r	K	t_l	t_rKt_l	Kt_rKt_l	$t_lKt_rKt_l$	K
t_r	K	t_r	K	t_l			t_rKt_l	K
t_l	K	t_r	K	t_r	t_rKt_r	Kt_r	t_lKt_r	K
t_r	K	t_r	K	t_r			t_rKt_r	K

Table 4

and for $u = u_1u_2u_3u_4 \in \Psi_4$, write $\tilde{u} = u_1u_2u_3\tilde{u}_4$ and let u^0 be the empty word. From Table 3, we easily deduce the form of elements of Ψ_n for $n > 3$. For $n \geq 1$, we get, with $u = u_1u_2u_3u_4$,

$$\begin{aligned}
\Psi_{4n} &= \{\tilde{u}^{n-1}u \mid u \in \Psi_4\},\\
\Psi_{4n+1} &= \{\tilde{u}_4\tilde{u}^{n-1}u \mid u \in \Psi_4\},\\
\Psi_{4n+2} &= \{u_3\tilde{u}_4\tilde{u}^{n-1}u \mid u \in \Psi_4\},\\
\Psi_{4n+3} &= \{u_2u_3\tilde{u}_4\tilde{u}^{n-1}u \mid u \in \Psi_4\},
\end{aligned}$$

and thus, by the definition of the mapping φ, we have

$$\begin{aligned}
\varphi_{4n} \quad &: \tilde{u}^{n-1}u \longmapsto (u_1u_2, 4n, u_3u_4),\\
\varphi_{4n+1} &: \tilde{u}_4\tilde{u}^{n-1}u \longmapsto (\tilde{u}_4u_1, 4n+1, u_3u_4),\\
\varphi_{4n+2} &: u_3\tilde{u}_4\tilde{u}^{n-1}u \longmapsto (u_3\tilde{u}_4, 4n+2, u_3u_4),\\
\varphi_{4n+3} &: u_2u_3\tilde{u}_4\tilde{u}^{n-1}u \longmapsto (u_2u_3, 4n+3, u_3u_4),
\end{aligned}$$

where $u = u_1u_2u_3u_4$ ranges over all elements of Ψ_4. Now a simple comparison of the list of elements in Ψ_4 in the last column of Table 3 and the definition of M_{4n+i} shows that φ_{4n+i} is a bijection of Ψ_{4n+i} onto M_{4n+i} for $n = 1, 2, \dots$ and $i = 0, 1, 2, 3$. Therefore φ is a bijection of Ω onto M. □

The next two lemmas treat the initial part of uv.

Lemma 4.3. *Let* $u = u_1u_2 \dots u_m$, $v = v_1v_2 \dots v_n \in \Psi$, $uv \notin \Delta$, $m \leq 3$, $n > 3$. *Then* $i(uv) = i(uv_1v_2)$.

Proof. We distinguish three cases.

Case: $m = 1$. If $u_1v_1 = v_1$, then

$$i(uv) = i(v) = v_1v_2 = i(v_1v_2) = i(u_1v_1v_2).$$

So suppose that $u_1v_1 \neq v_1$. Now writing u_3 for v_1, the third and fourth columns of Table 4 give all choices for the product u_3v_1. The fifth column of Table 4 provides all choices for v_2. The sixth column of Table 4 gives the product $u_3v_1v_2$. The last column of Table 4 contains all the choices for v_3. Comparing these two columns we see that affixing letters to the back of words in the sixth column does not change the initial part.

Case: $m = 2$. If $u_2v_1 = v_1$, then by the first case, we get

$$i(uv) = i(u_1v) = i(u_1v_1v_2) = i(uv_1v_2).$$

So suppose that $u_2v_1 \neq v_1$. Writing u_3 for u_2, we see that the choices for u_3, v_1, v_2 and v_3 are the same as above. We write u_2 for u_1 and observe that the second column of Table 4 lists all the corresponding choices of u_2. The seventh column of Table 4 gives the products $u_2u_3v_1v_2$. Comparing this column with the last column which gives all choices of v_3, we conclude, as above, that affixing letters to the back of the words in the seventh column does not change their initial part.

Case: $m = 3$. If $u_3v_1 = v_1$, then by the second case, we get

$$i(uv) = i(u_1u_2v) = i(u_1u_2v_1v_2) = i(uv_1v_2).$$

So suppose that $u_3v_1 \neq v_1$. The choices for u_2, u_3, v_1, v_2, v_3 are the same as above. The first column of Table 4 lists all the corresponding choices of u_1. The eighth column of Table 4 gives the products $u_1u_2u_3v_1v_2$. Comparing this column with the last column which gives all choices of v_3, we again conclude that affixing to the back of the words in the eighth column does not change their initial part. $\square$

Lemma 4.4. *Let* $u, v \in \Psi$, $uv \notin \Delta$, $l(u) > 3$. *Then* $i(uv) = i(u)$.

Proof. Since $l(u) > 3$, we have that u contains as letters at least one of the pairs t_l, t_r or t_l, T_r or T_l, t_r. In forming the product uv the possible contractions that may occur will take place after $h(u)$ for they can not involve the first occurrence of t_l or t_r because of the second occurrence of one of the letters in the above pairs. Whether $i(u)$ is Kt_l or Kt_r or t_lK or t_rK, it follows that the reduction $i(u)$ will not change. $\square$

The next lemma takes care of $[uv]$. Recall the notation $\bar{v}$ from Section 2.

Lemma 4.5. *Let* $u = u_1u_2\dots u_m$, $v = v_1v_2\dots v_n \in \Psi$, $uv \notin \Delta$. *Then*

$$[uv] = \begin{cases} [uv_1v_2\bar{v}_1] & \text{if } m \leq 3,\ n > 3 \\ [\bar{u}_m u_{m-1}u_m v] & \text{if } m > 3,\ n \leq 3\ . \\ [\bar{u}_m u_{m-1}u_m v_1v_2] = [u_{m-1}u_m v_1v_2\bar{v}_1] & \text{if } m, n > 3 \end{cases}$$

Proof. We consider the three cases separately.

Case: $m \leq 3$, $n > 3$. Note that $v_3 = \bar{v}_1$ so that $v = v_1v_2\bar{v}_1v_4$ if $n = 4$ and $v = v_1v_2\bar{v}_1v_4v'$ for a suitable v' if $n > 4$. Forming the product uv, we see that v_4 acts as a barrier to possible reductions since either v_1 or v_2 is equal to t_l or t_r; if v_1 or v_2 equals t_l, then $v_4 \in \mathcal{T}_l$ and if v_1 or v_2 equals t_r, then $v_4 \in \mathcal{T}_r$. Hence the possible loss of length already occurs up to $\bar{v}_1$.

Case: $m > 3$, $n \leq 3$. First note that $\bar{u}_m = u_{m-2}$. Now the argument here is essentially the same as in the preceding case only in the reverse order and may be omitted.

Case: $m, n > 3$. The product of the last three letters of u, namely $\bar{u}_m u_{m-1} u_m$, with the first two letters of v, namely $v_1 v_2$, as above, are flanked by barriers to any possible reduction when forming the product uv. Hence the loss of length is already given in $[\bar{u}_m u_{m-1} u_m v_1 v_2]$. Essentially the same argument will show that this loss is also given by $[u_{m-1} u_m v_1 v_2 \bar{v}_1]$. □

The next two lemmas handle the final part of uv.

Lemma 4.6. *Let* $u = u_1 u_2 \dots u_m \in \Psi$, $v_1 \in \Gamma$, $uv_1 \notin \Delta$, $m > 2$. *Then* $f(uv_1) = f(u_{m-1} u_m v_1)$.

Proof. If $u_m = v_1$, then

$$f(uv_1) = f(u) = u_{m-1} u_m = u_{m-1} u_m v_1 = f(u_{m-1} u_m v_1).$$

So suppose that $u_m \neq v_1$. The third and fourth columns of Table 5 show all the choices for $u_m v_1$. For these choices, the second column of Table 5 gives all the choices for u_{m-1} and then the final column all the choices for u_{m-2}. The fifth and sixth columns of Table 5 give the products $u_{m-1} u_m v_1$ and $u_{m-2} u_{m-1} u_m v_1$, respectively. By inspection of the table, we see that $f(u_{m-2} u_{m-1} u_m v_1) = f(u_{m-1} u_m v_1)$. From the form of the words $u_{m-2} u_{m-1} u_m v_1$ in the table, we see that affixing letters to the front of these words does not influence their final part. □

Lemma 4.7. *Let* $u, v \in \Psi$, $uv \notin \Delta$, $l(v) \geq 2$. *Then* $f(uv) = f(v)$.

Proof. We consider first the case $v \in I \cup J \cup \Phi_e \cup \Psi_e$. Table 6 provides the products uv when $u \in \Gamma$. These products show that $f(uv) = f(v)$ for the case $l(u) = 1$. For the case $l(u) = 2$, in the first column of Table 6, we would have the following substitutions:

$$T_l \to KT_l,\ K \to t_l K \text{ or } K \to t_r K,\ T_r \to KT_r,$$

$$t_l \to Kt_l,\ k \to t_l k \text{ or } k \to t_r k,\ t_r \to Kt_r.$$

By simple inspection, the effect of these substitutions on the corresponding entries of Table 6 does not change their final part. This takes care of the case $l(u) = 2$.

u_{m-2}	u_{m-1}	u_m	v_1	$u_{m-1}u_mv_1$	$u_{m-2}u_{m-1}u_mv_1$
t_l	K	T_l	t_l	Kt_l	t_lKt_l
t_r	K	T_l	t_l	Kt_l	t_rKt_l
K	t_l	K	T_l	t_lKT_l	KT_l
K	t_r	K	T_l	t_rKT_l	Kt_rKT_l
K	t_l	K	T_r	t_lKT_r	Kt_lKT_r
K	t_r	K	T_r	t_rKT_r	KT_r
K	t_l	K	t_l	t_lKt_l	Kt_l
K	t_r	K	t_l	t_rKt_l	Kt_rKt_l
K	t_l	K	k	t_lk	Kt_lk
K	t_r	K	k	t_rk	Kt_rk
K	t_l	K	t_r	t_lKt_r	Kt_lKt_r
K	t_r	K	t_r	t_rKt_r	Kt_r
t_l	K	T_r	t_r	Kt_r	t_lKt_r
t_r	K	T_r	t_r	Kt_r	t_rKt_r
t_l	K	t_l	T_l	KT_l	t_lKT_l
t_r	K	t_l	T_l	KT_l	t_rKT_l
t_l	K	t_l	K	Kt_lK	t_lK
t_r	K	t_l	K	Kt_lK	t_rKt_lK
t_l	K	t_l	k	Kt_lk	t_lk
t_r	K	t_l	k	Kt_lk	t_rKt_lk
K	t_l	k	K	t_lK	Kt_lK
K	t_r	k	K	t_rK	Kt_rK
t_l	K	t_r	T_r	KT_r	t_lKT_r
t_r	K	t_r	T_r	KT_r	t_rKT_r
t_l	K	t_r	K	Kt_rK	t_lKt_rK
t_r	K	t_r	K	Kt_rK	t_rK
t_l	K	t_r	k	Kt_rk	t_lKt_rk
t_r	K	t_r	k	Kt_rk	t_rk

Table 5

For the case $l(u) = 3$, we perform further substitutions

$$KT_l \to t_lKT_l \text{ or } KT_l \to t_rKT_l,\ t_lK \to Kt_lK,\ t_rK \to Kt_rK,$$

$$KT_r \to t_lKT_r \text{ or } KT_r \to t_rKT_r,\ Kt_l \to t_lKt_l \text{ or } Kt_l \to t_rKt_l,$$

$$t_lk \to Kt_lk,\ t_rk \to Kt_rk,\ Kt_r \to t_lKt_r \text{ or } Kt_r \to t_rKt_r$$

	t_lK	t_rK	Kt_l	Kt_r	t_lk	t_rk	KT_l	KT_r
T_l	t_lK				t_lk			
K	Kt_lK	Kt_rK	Kt_l	Kt_r	Kt_lk	Kt_rk	KT_l	KT_r
T_r		t_rK				t_rk		
t_l	t_lK		t_lKt_l	t_lKt_r	t_lk		t_lKT_l	t_lKT_r
k			Kt_l	Kt_r			KT_l	KT_r
t_r		t_rK	t_rKt_l	t_rKt_r		t_rk	t_rKT_l	t_rKT_r

	t_lKt_l	t_lKT_l	t_rKt_r	t_rKT_r	Kt_lK	Kt_lk	Kt_rK	Kt_rk
T_l	t_lKt_l	t_lKT_l						
K	Kt_l	KT_l	Kt_r	KT_r	Kt_lK	Kt_lk	Kt_rK	Kt_rk
T_r			t_rKt_r	t_rKT_r				
t_l	t_lKt_l	t_lKT_l			t_lK	t_lk	t_rK	t_lKt_rk
k					Kt_lK	Kt_lk	Kt_rK	Kt_rk
t_r			t_rKt_r	t_rKT_r	t_rKt_lK	t_rKt_lk	t_rK	t_rk

Table 6

and again none of these substitutions changes the final part of the corresponding entries. In the next step, for $l(u) = 4$, we reach the words u in which occur pairs of letters of the form: either t_l, t_r or t_l, T_r or T_l, t_r with the final parts still unchanged.

From now on, there is no change in the final part since the last pair of t_l, t_r or t_l, T_r or T_l, t_r blocks the possible reductions if we affix an element of Γ to the front of the word u.

It remains to consider the case $v \in \Pi$. Now v contains a pair of letters of the form: t_l, t_r or t_l, T_r or T_l, t_r. When forming the product uv, any reduction that may occur must precede $f(v)$ since the occurrence of first t_l or t_r preceding $f(v)$ represents the last occasion for a possible reduction. Hence the final part of v remains unchanged. □

The final lemma deals with the case $uv \in \Delta$.

Lemma 4.8. *Let* $u = u_1u_2 \dots u_m$, $v = v_1v_2 \dots v_n \in \Psi$, $uv \in \Delta$. *Then*

$$uv = \begin{cases} u_m v_1 v_2 v_1 v_n & \text{if } v_1v_2 \in \{Kt_l, Kt_r\} \\ u_m v_1 v_2 v_n & \text{if } v_1v_2 \in \{t_lK, t_rK\} \\ u_m v & \text{if } v \in \Gamma \cup J \end{cases} .$$

Proof. In view of Lemma 3.1, we deduce that $uv = u_m v$. It suffices to consider the case $v_1 v_2 \in I$. To this end, we consider several cases.

Case: $v_1 v_2 = K t_l$. Then $u_m K \in \Delta$ and by Table 1, we get that $u_m \in \{T_l, T_r\}$, say $u_m = T_p$. Hence

$$u_m v_1 v_2 v_1 = T_p K t_l K = \omega t_l K = (\mathcal{R} \wedge \sigma) K = \sigma. \tag{4}$$

By Lemma 3.3, we have

$$u_m v = ((u_m v_1) \circ v_2) v_n = ((T_p K) \circ t_l) v_n = (\omega \circ t_l) v_n = \sigma v_n$$

which together with (4) yields that $uv = u_m v = v_1 v_2 v_1 v_n$.

The case $v_1 v_2 = K t_r$ is dual. This proves the first formula.

Case: $v_1 v_2 = t_l K$. By Table 1, $u_m t_l \in \Delta$ gives

$$u_m v_1 v_2 = \begin{cases} (\mathcal{R} \wedge \sigma) K & \text{if } u_m = T_r \\ \epsilon K & \text{if } u_m \in \{k, t_r\} \end{cases} = \begin{cases} \sigma & \text{if } u_m = T_r \\ \tau & \text{if } u_m \in \{k, t_r\} \end{cases}. \tag{5}$$

On the other hand, by Lemma 3.3, we get

$$\begin{aligned} u_m v = ((u_m v_1) \circ v_2) v_n &= \begin{cases} ((\mathcal{R} \wedge \sigma) \circ K) v_n & \text{if } u_m = T_r \\ (\epsilon \circ K) v_n & \text{if } u_m \in \{k, t_r\} \end{cases} \\ &= \begin{cases} \sigma v_n & \text{if } u_m = T_r \\ \tau v_n & \text{if } u_m \in \{k, t_r\} \end{cases} \end{aligned}$$

which together with (5) yields that $uv = u_m v_1 = u_m v_1 v_2 v_n$.

The case $v_1 v_2 = t_r K$ is dual. This proves the second formula. □

Proof of Theorem 4.1. By Lemma 4.2, φ is a bijection of Ω onto M. For the homomorphism property, we consider several cases.

Let $u = u_1 u_2 \dots u_m,\ v = v_1 v_2 \dots v_n \in \Psi$.

Case: $uv \notin \Delta$. By definition of φ we have

$$(uv)\varphi = (i(uv), l(uv), f(uv)) \tag{6}$$

and by the definition of the product in M,

$$(u\varphi)(v\varphi) = (a, m + b + n, c) \tag{7}$$

as in (3) and the notation that follows it.

If $m, n \leq 3$, then $a = i(\hat{u}_m \hat{v}_n) = i(uv)$ trivially. If $m \leq 3$ and $n > 3$, then by Lemma 4.3, we have $a = i(\hat{u}_m v_1 v_2) = i(uv_1 v_2) = i(uv)$. If $m > 3$, then Lemma 4.4 implies that $a = u_1 u_2 = i(u) = i(uv)$.

If $m, n \leq 3$, then $b = [\hat{u}_m \hat{v}_n] = [uv]$. If $m \leq 3$ and $n > 3$, then by Lemma 4.5, we get

$$b = [\hat{u}_m v_1 v_2 \bar{v}_1] = [uv_1 v_2 \bar{v}_1] = [uv].$$

If $m > 3$ and $n \leq 3$, then again by Lemma 4.5, we obtain

$$b = [\bar{u}_m u_{m-1} u_m \hat{v}_n] = [\bar{u}_m u_{m-1} u_m v] = [uv].$$

If $m, n > 3$, then the same Lemma 4.5 yields that

$$b = [\bar{u}_m u_{m-1} u_m v_1 v_2] = [uv].$$

It follows that

$$m + b + n = l(u) + [uv] + l(v) = l(uv).$$

If $m = n = 1$, then trivially $c = v_1 v_n = f(uv)$. If $m > 1$ and $n = 1$, then $c = f(uv)$ in the next three cases for c by Lemma 4.6. If $n > 1$, then Lemma 4.7 implies that $c = v_{n-1} v_n = f(uv)$.

Now comparing (6) and (7), we see that $(uv)\varphi = (u\varphi)(v\varphi)$ in this case.

Case: $uv \in \Delta$. The required formula $(u\varphi)(v\varphi) = (uv)\varphi$ follows directly from Lemma 4.8.

We now let $\theta \in \Delta$ and consider its product with u. By Lemma 3.1, we immediately obtain that $u\theta = \theta$ so that

$$(u\varphi)(\theta\varphi) = (u\varphi)\theta = \theta = \theta\varphi = (u\theta)\varphi.$$

Moreover, $\theta u = \theta \hat{u}_m$ and thus, for $m \leq 3$, we get

$$(\theta\varphi)(u\varphi) = \theta(u\varphi) = \theta\hat{u}_m = \theta u = (u\theta)\varphi.$$

If $m > 3$, then by Lemma 3.3, we have $\theta u = (\theta \circ u_1)u_m$ whence

$$(\theta\varphi)(u\varphi) = \theta(u\varphi) = (\theta \circ u_1)u_m = \theta u = (\theta u)\varphi.$$

If $\theta, \delta \in \Delta$, then $\theta\delta = \delta$ whence $(\theta\varphi)(\delta\varphi) = \delta\varphi = (\theta\delta)\varphi$.

Therefore φ is a homomorphism and thus an isomorphism of Ω onto M. $\square$

We have seen in the formulas for $i(uv)$, $[uv]$, and $f(uv)$ that these can be obtained by taking the function of the product of a few u_i and v_i. One might ask if one could do the same task with a fewer number of u_i or v_i. The following examples show that, except in two trivial cases, this cannot be done.

For the function i we have the following examples:

$$\begin{aligned} t_l(t_lk) &= t_l(Kt_lk) = (t_lK)k = (t_lK)(t_lk) = (t_lK)(Kt_lk) \\ &= (t_lKt_l)k = (t_lKt_l)(t_lk) = t_lk, \end{aligned}$$

$$(Kt_lK)(t_lKT_l) = KT_l,$$

where, for example, $t_l(t_lk) = t_lk$ shows that $i(u_1v_1v_2)$ cannot be expressed as $i(u_1v_1)$, etc.

For the function $[\ ]$, we actually have $[u_1v_1v_2] = [u_1v_1]$ and $[u_1u_2v_1] = [u_2v_1]$, the exceptions mentioned above. Otherwise, we have the following examples:

$$K(t_lKT_l) = (Kt_l)(KT_l) = KT_l$$

$$(t_lK)(t_lKt_r) = t_lKt_r \text{ we cannot omit } u_1,$$

$$(Kt_l)(t_lKt_l) = Kt_l \text{ we cannot omit } v_2 \text{ or } v_1v_2,$$

$$(Kt_lK)t_l = (Kt_lK)(Kt_l) = (Kt_lK)(t_lKt_l) = Kt_l.$$

For $n > 3$, we further have

$$t_l(Kt_lK) = (t_lK)(Kt_lK) = (t_lKt_l)(Kt_lK) = t_lK.$$

The case $m > 3$ is symmetric.

For the function f, it suffices to mention $(t_lK)k = t_lk$.

5. An embedding

We can use the triple representation of a part of Ψ to embed it into a Rees matrix semigroup as follows. As we shall see, it is the part Π

of Ψ, with a zero adjoined, that is suitable for this embedding. This depends on the products of the relevant triples since they have to conform to the Rees multiplication. The essential ingredients for this are already in Theorem 4.1. We now provide the necessary details.

Our index sets will be I and $I \cup J$ with ordering of their elements as follows:

$$t_lK,\ t_rK,\ Kt_l,\ Kt_r,\ t_lk,\ t_rk,\ KT_l,\ KT_r.$$

We shall see that the following sandwich matrix faithfully reflects the loss of length when forming the corresponding products. Our group will be the additive group of integers $\mathbb{Z}$ with a zero z adjoined. Let

$$P = \begin{bmatrix} -2 & 0 & -3 & -1 \\ 0 & -2 & -1 & -3 \\ -1 & z & -2 & 0 \\ z & -1 & 0 & -2 \\ z & z & -3 & -1 \\ z & z & -1 & -1 \\ -1 & z & z & z \\ z & -1 & z & z \end{bmatrix}.$$

Recall from Lemma 3.4 that both Δ and $\Pi \cup \Delta$ are ideals of Ω.

Theorem 5.1. *Let $\Lambda = (\Pi \cup \Delta)/\Delta$. Then the mapping*

$$\varphi_0 : w \longmapsto \begin{cases} (i(w), l(w), f(w)) & \text{if } w \in \Pi \\ 0 & \text{otherwise} \end{cases}$$

is an embedding of Λ into $S = \mathcal{M}^0(I, \mathbb{Z}, I \cup J; P)$.

Proof. Let

$$\Psi_0 = \{t_lKt_r, t_rKt_l, t_lKT_r, t_rKT_l\}$$

and recall from Section 2 that

$$\Pi = \Psi_0 \cup \{w \in \Psi \mid l(w) > 3\}.$$

Hence letting

$$M_0 = \{(t_lK, 3, Kt_r),\ (t_rK, 3, Kt_l),\ (t_lK, 3, KT_r),\ (t_rK, 3, KT_l)\}$$

	t_lKt_r	t_rKt_l	Kt_lK	Kt_rK
t_lK	t_lKt_r	$t_lKt_rKt_l$	t_lK	t_lKt_rK
t_rK	$t_rKt_lKt_r$	t_rKt_l	t_rKt_lK	t_rK
Kt_l	Kt_lKt_r	$\mathcal{R} \wedge \tau$	Kt_lK	Kt_lKt_rK
Kt_r	ϵ	Kt_rKt_l	Kt_rKt_lK	Kt_rK
t_lk	ϵ	$\mathcal{R} \wedge \tau$	t_lK	t_lKt_rK
t_rk	ϵ	$\mathcal{R} \wedge \tau$	t_rKt_lK	t_rKt_rK
KT_l	Kt_lKt_r	$\mathcal{R} \wedge \sigma$	σ	σ
KT_r	$\mathcal{L} \wedge \sigma$	Kt_rKt_l	σ	σ

Table 7

by the proof of Lemma 4.2, we obtain

$$\Pi\varphi = M_0 \cup \left(\bigcup_{n=3}^{\infty} M_n \right).$$

Since φ_0 and φ in Theorem 4.1 agree on Π, in order to prove the theorem, it suffices to show that the products in $\big(M_0 \cup \big(\bigcup_{n=3}^{\infty} M_n\big) \cup \Delta\big)/\Delta$ and in S coincide.

We first treat the part $N = \bigcup_{n=3}^{\infty} M_n$. In the notation of the preceding section, for $U_m, V_n \in N$ and $u_m v_1 \notin \Delta$, we have

$$U_m V_n = (u_1 u_2,\ m + [u_{m-1} u_m v_1 v_2 \bar{v}_1] + n,\ v_{n-1} v_n).$$

So we must show that

$$p_{u_{m-1}u_m, v_1 v_2} = [u_{m-1} u_m v_1 v_2 \bar{v}_1]$$

which we present in the form of Table 7. For example, in it, $(t_lK)(t_lKt_r) = t_lKt_r$ in the $(1,1)$-position, so the loss in length is 2 and thus in the same position in the matrix P we must have -2. The remaining entries are checked just as easily. Therefore the two products agree on N.

It remains to check the products $M_0 M_0$, $N_0 N$ and $N M_0$. With the same notation as above, in view of Theorem 4.1, for $U_m, V_n \in M_0 \cup N$ and $u_m v_1 \notin \Delta$, we have

$$U_m V_n = (a, m + b + n, v_{n-1} v_n)$$

where

$$a = \begin{cases} i(u_1u_2u_3v_1v_2v_3) & \text{if } m = n = 3 \\ i(u_1u_2u_3v_1v_2) & \text{if } m = 3,\ n > 3 \\ u_1u_2 & \text{if } m > 3,\ n = 3 \end{cases},$$

$$b = \begin{cases} [u_1u_2u_3v_1v_2v_3] & \text{if } m = n = 3 \\ [\hat{u}_1u_2u_3v_1v_2\bar{v}_1] & \text{if } m = 3,\ n > 3 \\ [\bar{u}_mu_{m-1}u_mv_1v_2v_3] & \text{if } m > 3,\ n = 3 \end{cases}.$$

The product in S of the elements U_m and V_n has the form

$$(u_1u_2, m + p_{u_{m-1}u_m, v_1v_2} + n, v_{n-1}v_n)$$

where

$$p_{u_{m-1}u_m, v_1v_2} = [u_{m-1}u_mv_1v_2\bar{v}_1] = \begin{cases} [u_2u_3v_1v_2v_3] & \text{if } m = n = 3 \\ [u_2u_3v_1v_2\bar{v}_1] & \text{if } m = 3,\ n > 3 \\ [u_{m-1}u_mv_1v_2\bar{v}_1] & \text{if } m > 3,\ n = 3 \end{cases}$$

as we have seen above.

That $a = u_1u_2$ in all cases follows from the form of the elements u of Ψ_0. Indeed, each contains a pair of letters of the form t_l, t_r or t_l, T_r or T_l, t_r. When multiplying uv the reductions can not influence the first two letters of u because of the occurrence of either t_p or T_p as the last letter of u and t_q with $p \neq q$ as the first letter of u.

The same type of argument is valid for b. Indeed, if $m = 3$, then for the reasons just expounded above, we get $b = [u_2u_3v_1v_2v_3]$ if $n = 3$ and $b = [u_2u_3v_1v_2\bar{v}_1]$ if $n > 3$. Assume that $m > 3$ and $n = 3$. Then $v_1v_2v_3 = v_1v_2\bar{v}_1v_3$ so that

$$b = [\bar{u}_mu_{m-1}u_mv_1v_2\bar{v}_1v_3] = [\bar{u}_mu_{m-1}u_mv_1v_2\bar{v}_1].$$

Here $v_1v_2 \in \{t_lK, t_rK\}$ so that the possible reductions in the product $\bar{u}_mu_{m-1}u_mv_1v_2\bar{v}_1$ are independent of $\bar{u}_m$ and thus $b = [u_{m-1}u_mv_1v_2\bar{v}_1]$, as required.

We see from the matrix P that $p_{u_1u_2, v_1v_2} = z$ if and only if $u_2v_1 \in \Delta$. This completes the proof that the product in $M_0 \cup N$ coincides with that in S. $\square$

6. General Properties

We consider first the regularity and idempotency of elements of Ω. This is followed by a complete description of the $\mathcal{D}$-structure of Ω. The quotient $\Omega/(\Pi \cup \Delta)$ is finite and regular so completely semisimple. Beside the $\mathcal{D}$-classes contained in Γ, all three of which are 2-element right zero semigroups, there are two nonzero classes in this quotient for which we construct the Rees matrix representation.

Proposition 6.1.

(i) Δ *coincides with right zeros of* Ω.

(ii) $\Theta \setminus J$ *consists of idempotents.*

(iii) J *coincides with regular nonidempotent elements of* Ω.

(iv) Π *coincides with nonregular elements of* Ω.

(v) *If* $u \in \Pi$, *then either* $u^2 \in \Delta$ *or* u *is of infinite order.*

(vi) *If* $u \in J$, *then* $u^2 \in \Delta$.

Proof. (i) By Lemma 3.1, every element of Δ is a right zero of Ω. Since Δ is a right zero semigroup and an ideal of Ω, it must be the kernel of Ω. But any right zero lies in the kernel and is thus in Δ.

(ii) This may be checked by direct multiplication.

(iii) For $p \in \{l, r\}$, we have

$$(KT_p)t_p(KT_p) = KT_p, \quad (t_pk)K(t_pk) = t_p$$

and thus all elements of J are regular. Since $(KT_p)^2 = K(T_pK)T_p = K\omega T_p$ and $(t_pk)^2 = t_p(kt_p)k = t_p\epsilon k$, none of them is idempotent. That there are no other regular nonidempotent elements will follow from parts (i), (ii) and the remaining part of the proof of part (iv).

(iv) It remains to show that all elements of Π are nonregular. For this let $u \in \Pi$, $v \in \Psi$, consider the product uvu and assume that $uvu \notin \Delta$. Since u contains at least one pair of letters t_l, t_r or t_l, T_r or T_l, t_r, in any reduction arising from forming the product uvu, some of these letters create barriers so that uvu can never reduce to u. Therefore u is not regular.

(v) Let $u \in \Pi$ be such that $u^2 \notin \Delta$. Since then $t(u)h(u) \notin \Delta$, we get from Table 1 that

$$t(u)h(u) \in \{KK, t_lK, t_rK, T_lt_l, Kt_l, t_lt_l, Kt_r, T_rt_r, t_rt_r\}$$

which evidently implies that $u, u^2, u^3, \dots$ are all distinct.

(vi) This can be checked directly. □

For the description of the $\mathcal{D}$-structure of Ω, we shall need the following notation. Let

$$Q_l = t_lKt_rK, \quad Q_r = t_rKt_lK,$$

and let Q_p^0 stand for the empty word, $p \in \{l, r\}$. For $n = 1, 2, \dots$ and $P \in \{L, R\}$, $\{p, q\} = \{l, r\}$, we denote by P_n the array

$$\begin{array}{cccc} Q_p^n & Q_p^{n-1}t_pKt_qk & Q_p^{n-1}t_pKt_q & Q_p^{n-1}t_pKT_q \\ KQ_p^n & KQ_p^{n-1}t_pKt_qk & KQ_p^{n-1}t_pKt_q & KQ_p^{n-1}t_pKT_q \end{array}$$

and by P'_n the array

$$\begin{array}{cccc} Q_p^nt_pK & Q_p^nt_pk & Q_p^nt_p & Q_p^nT_p \\ KQ_p^nt_pK & KQ_p^nt_pk & KQ_p^nt_p & KQ_p^nT_p. \end{array}$$

Theorem 6.2. *Diagram 2 represents the $\mathcal{D}$-structure of Ω.*

Proof. Simple verification shows that the elements in the rows of the egg-box pictures in the diagram and in the arrays P_n and P'_n are $\mathcal{R}$-related and the elements in the columns are $\mathcal{L}$-related.

That the vertical, lower-left to upper-right and lower-right to upper-left lines indicate the inclusion of $\mathcal{J}$-clases is seen by observing that some element of the lower array contains some element of the upper array as a (possibly interior) factor. For note that $t_rKQ_l = Q_rt_rK$.

Again comparing two arrays, say A and B, such that A is immediately above B, it is easy to see that no element of A can be a factor of any element of B even if the representation of elements of B is expanded using the relations in Σ. This implies that the ideal generated by B is strictly contained in the ideal generated by A. Assume next that A and B are not comparable in the order indicated in the diagram. If they are in the first two rows of the diagram, it is clear that no element of A can be $\mathcal{J}$-related to an element of B. Assume that either $\{A, B\} = \{L_n, R_n\}$ or

Diagram 2

$\{A, B\} = \{L'_n, R'_n\}$ for some n. In the first case, it is easy to see that $Q_p^n \notin J(Q_q^n)$ for $\{p, q\} = \{l, r\}$ so the ideals generated by A and B are incomparable. The second case is treated similarly.

Therefore the arrays in the diagram represent $\mathcal{J}$-classes of Ω so by the first part of the proof also its $\mathcal{D}$-classes. $\square$

From Lemma 3.4, we know that $\Omega/(\Pi \cup \Delta)$ is a finite regular semigroup with five nonzero $\mathcal{D}$-classes, namely

$$\mathcal{T}_l,\ \mathcal{K},\ \mathcal{T}_r,\ \mathcal{M}_l,\ \mathcal{M}_r$$

where for $p \in \{l, r\}$,

$$\mathcal{M}_p = \{t_pK, t_pk, t_pKt_p, t_pKT_p, Kt_pK, Kt_pk, Kt_p, KT_p\}.$$

The first three of these $\mathcal{D}$-classes are right zero semigroups of order two. The last two $\mathcal{D}$-classes, together with a zero and the undeclared products equal to that zero, must be completely 0-simple. We now find a Rees matrix representation for them.

Lemma 6.3. *Let* $p \in \{l, r\}$, $\Theta_p = (\mathcal{M}_p \cup \Pi \cup \Delta)/(\Pi \cup \Delta)$ *and*

$$M_p = \mathcal{M}^0(\{K, t_p\}, \{e\}, \{K, k, T_p, t_p\}; Q),$$

where

$$Q = \begin{bmatrix} e & e \\ e & 0 \\ 0 & e \\ e & e \end{bmatrix}.$$

Then the mapping

$$\chi : \begin{cases} w \mapsto (h(w), e, t(w)) & (w \in \mathcal{M}_p) \\ 0 \mapsto 0 \end{cases}$$

is an isomorphism of Θ_p *onto* M_p.

Proof. Direct verification shows that

$$q_{x,y} = e \iff xy \notin \Delta \iff xy \notin \Pi \cup \Delta. \tag{8}$$

For any $u, v \in \mathcal{M}_p$ we get

$$\begin{aligned} (u\chi)(v\chi) &= (h(u), e, t(u))(h(v), e, t(v)) \\ &= \begin{cases} (h(u), e, t(v)) & \text{if } q_{t(u),h(v)} = e \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

$$(uv)\chi = \begin{cases} (h(uv), e, t(uv)) & \text{if } uv \notin \Delta \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 3.6 and (8), we obtain

$$uv \notin \Delta \iff t(u)h(v) \notin \Delta \iff q_{t(u),h(v)} = e$$

and if $uv \notin \Delta$, then $h(u) = h(uv)$, $t(v) = t(uv)$. Therefore χ is a homomorphism. Clearly χ is injective and since both Θ_p and M_p have nine elements, it is also surjective. □

7. Rectangular groups

By definition, they are semigroups which are isomorphic to the direct product of a rectangular band and a group. Equivalently, they are completely simple semigroups in which idempotents form a subsemigroup. For $S = \mathcal{M}(I, G, \Lambda; P)$, with P normalized, S is a rectangular group if and only if all entries of P are equal to e, the identity of G. For proofs of these staments and an extensive discussion, we refer to ([3], Section IV.3). We add that S is a rectangular group if and only if $\sigma = \tau$.

Congruences on a Rees matrix semigroup are best represented in terms of admissible triples. This development is explained in ([5], Section 2). For a complete treatment of this subject, we refer to ([1], Section III.4). When $S = \mathcal{M}(I, G, \Lambda; P)$ with all entries of P equal to e, the identity of G, the admissible triples for S are all $\theta = (r, N, \pi)$ where r and π are partitions of I and Λ, respectively, and N is a normal subgroup of G. In this simple situation, we immediately obtain

$$\theta T_l = (\omega, G, \pi), \quad \theta K = (\omega, N, \omega), \quad \theta T_r = (r, G, \omega),$$

$$\theta t_l = (\epsilon, e, \pi), \quad \theta k = (\epsilon, N, \epsilon), \quad \theta t_r = (r, e, \epsilon).$$

For rectangular groups we have the following result.

Theorem 7.1. *The following conditions on a completely simple semigroup S are equivalent.*

(i) *S is a rectangular group.*

(ii) *For S we have $t_l t_r = t_r k$.*

(iii) *Every element of $\Omega \setminus \Gamma$ is μ-related to some element of Δ.*

(iv) $\Omega = \Gamma \cup \Delta$.

(v) *Ω is a band.*

Proof. (i) $\Longrightarrow$ (ii). In view of the above comments, for an admissible triple $\theta = (r, N, \pi)$, we have

$$\theta t_r K = (r, e, \epsilon)k = (\epsilon, e, \epsilon)$$

and we always have $t_l t_r = \epsilon$ so that $t_l t_r = t_r k$.

(ii) $\Longrightarrow$ (iii). Using the relations in Σ and the hypothesis we get

$$\begin{aligned}\sigma &= T_l t_r K = T_l t_r (kK) = T_l (t_r k) K = T_l (t_l t_r) K \\ &= (T_l t_l) t_r K = t_l t_r K = \tau.\end{aligned}$$

This means that S is E-unitary and thus orthodox so a rectangular group. As above, we can get that also $t_l t_r = t_l K$ holds for S, or by duality. Hence $t_l k \ \mu \ t_r k \ \mu \ \epsilon$ which in view of Theorem 6.2 (see Diagram 2) implies the assertion.

(iii) $\Longrightarrow$ (iv). This is trivial.

(iv) $\Longrightarrow$ (v). This is obvious since all elements of Γ and Δ are idempotent.

(v) $\Longrightarrow$ (i). In particular KT_l is idempotent which implies that $KT_l = K(T_l K)T_l \in \Delta$. We may let $S = \mathcal{M}(I, G, \Lambda; P)$ with P normalized. By ([5], Lemma 2.3), for an admissible triple $\theta = (r, N, \pi)$, we get

$$\theta K T_l = (N\alpha, N, N\gamma) T_l = (\omega, G, N\gamma)$$

and thus $N\gamma$ must be a constant. But then also $N\gamma\delta$ is a constant. By ([5], Lemma 2.4), we obtain $e\gamma\delta = e$ and $G\gamma\delta = \omega\delta = \bar{\omega}$. Therefore $\bar{\omega} = \{e\}$ which evidently implies that all entries of G are equal to e and S is a rectangular group. $\square$

It follows from Theorem 7.1 that for a rectangular group S, Ω is a band and hence a right regular band since all its $\mathcal{D}$-classes are right zero semigroups. From Table 1, we see that, for example, in the natural partial ordering, T_l is greater than ω, $\mathcal{L} \vee \tau$ and $\mathcal{L}$ so that Ω is not a normal band. Its multiplication table is given by Table 1 together with the observation that the elements of Δ are the right zeros of Ω. The partially ordered set Ω for a rectangular group is depicted in Diagram 3 where each vertex is provided with the corresponding admissible triple.

According to Theorem 7.1, $\Sigma \cup \{t_l t_r = t_r k\}$ provides a system of relations for rectangular groups. We now give a somewhat simpler set of relations for them. Let Σ_{rg} denote the set

$\Sigma(i),\ \Sigma(ii),\ \Sigma(v)$ from Section 2,

$(vi)\ t_l t_r = t_l k = t_r k,$

$(vii)\ t_l K t_l = K t_l,\ t_r K t_r = K t_r.$

Diagram 3

Lemma 7.2. *The relations* $\Sigma \cup \{t_l t_r = t_l k = t_r k\}$ *and* Σ_{rg} *induce the same congruence on* Γ^+.

Proof. Assuming the former, we obtain

$$\begin{aligned} t_l K t_l &= t_l (kK) t_l = (t_l k) K t_l = (k t_l) K t_l = (Kk) t_l K t_l \\ &= K(k t_l) K t_l = K(t_l k) K t_l = K t_l (kK) t_l = K t_l K t_l = K t_l \end{aligned}$$

and dually for $t_r K t_r = K t_r$.

Conversely, assume Σ_{rg}. Then

$$\begin{aligned} T_l t_r k &= T_l (t_l k) = (T_l t_l) k = t_l k = t_r k = (T_r t_r) k = T_r (t_r k) \\ &= T_r (t_l k) = T_r t_l k, \\ (K t_l)^2 &= K t_l K t_l = K K t_l = K t_l, \end{aligned}$$

	T_l	K	T_r	t_l	k	t_r	t_lK	t_lk	t_lKt_l	t_lKT_l	Kt_l	Kt_lK	Kt_lk	KT_l	t_lKt_r	Q_l
ω	ω	ω	ω	$\mathcal{R}$	$\mathcal{H}$	$\mathcal{L}$	ω	$\mathcal{H}$	$\mathcal{R}$	ω	$\mathcal{R}$	ω	$\mathcal{H}$	ω	$\mathcal{L}$	ω
$\mathcal{L}$	$\mathcal{L}$	ω	ω	ϵ	$\mathcal{H}$	$\mathcal{L}$	ϵ	ϵ	ϵ	$\mathcal{L}$	$\mathcal{R}$	ω	$\mathcal{H}$	ω	ϵ	ϵ
$\mathcal{R}$	ω	ω	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{H}$	ϵ	ω	$\mathcal{H}$	$\mathcal{R}$	ω	$\mathcal{R}$	ω	$\mathcal{H}$	ω	$\mathcal{L}$	ω
$\mathcal{H}$	$\mathcal{L}$	ω	$\mathcal{R}$	ϵ	$\mathcal{H}$	ϵ	ϵ	ϵ	ϵ	$\mathcal{L}$	$\mathcal{R}$	ω	$\mathcal{H}$	ω	ϵ	ϵ
ϵ	$\mathcal{L}$	ϵ	$\mathcal{R}$	ϵ	ϵ	ϵ	ϵ	ϵ	ϵ	$\mathcal{L}$	ϵ	ϵ	ϵ	$\mathcal{L}$	ϵ	ϵ

Table 8

$$\begin{aligned}(t_lK)^2 &= t_lKt_lK = t_lKt_l(kK) = t_lK(t_lk)K = t_lK(kt_l)K \\ &= t_l(Kk)t_lK = t_lkt_lK = t_lkK = t_lK\end{aligned}$$

and dually for t_r. □

We illustrate the situation for non rectangular groups on the smallest possible example.

Example 7.3. Let $S = \mathcal{M}^0(\{1,2\}, \mathbb{Z}_2, \{1,2\}; P)$ where

$$P = \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix}.$$

It follows easily that the congruences on S are:

$$\omega \sim (\omega, \mathbb{Z}_2, \omega), \quad \mathcal{L} \sim (\omega, \mathbb{Z}_2, \epsilon), \quad \mathcal{R} \sim (\epsilon, \mathbb{Z}_2, \omega),$$
$$\mathcal{H} \sim (\epsilon, \mathbb{Z}_2, \epsilon), \quad \epsilon \sim (\epsilon, \{\bar{0}\}, \epsilon),$$

with $\sigma = \omega$ and $\tau = \epsilon$. From Table 8, we see that $t_lK = Q_l$ and thus dually also $t_rK = Q_r$. It follows from Theorem 6.2 (see Diagram 2) that $\Omega = \Delta \cup \Theta$ where

$$\Delta = \{\omega, \mathcal{L}, \mathcal{R}, \mathcal{H}, \omega\}$$

and the $\mathcal{D}$-structure of Θ is as in the general case.

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