The Chern–Connes character formula
for families of Dirac operators

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Abstract. A bivariant Chern–Connes character is used, by incorporating the JLO formula and Bismut's superconnection formalism, to compute the local cyclic cycle formula for families of Dirac operator D acting on a fibre bundle M over B. The fundamental techniques used are, the rescaling of Bismut's superconnection and the canonical order calculus.

Introduction

Consider a family of Dirac operators $D = \{D_y \mid y \in B\}$ parameterized by a smooth compact manifold B, then index(D) is a well defined element in $K^0(B)$. Atiyah–Singer index theorem for families of Dirac operator, gives a cohomological formula for the Chern character of the index bundle. Over the past few years many proofs have appeared concerning the computation of the index theorem for families of Dirac operators. Some of these have utilized the heat equation method, and each proof has its own advantages.

Motivated by the problem of generalizing the heat equation proof of the index theorem to prove a local index theorem for families, by finding a suitable representative for the Chern character as a differential form, QUILLEN [Q1] introduced the concept of superconnection. To avoid analytical technicalities, he treated the finite dimensional case only. BISMUT [Bi] extended Quillen's formalism to infinite dimension. He introduced a Levi–Civita superconnection $\mathcal{A}$ associated to families of Dirac operator D, and

used it to give a heat kernel representation for the Chern character for families.

CONNES [Co1], [Co2] defined a new cohomology theory, cyclic cohomology, whose cycles play for a noncommutative algebra the role that de Rham currents play for smooth manifolds. Cyclic homology was also introduced by TSYGAN [T] and LODAY–QUILLEN [L–Q]. KAROUBI [Ka] constructed Chern character from Quillen's algebraic K-theory to cyclic homology. Moreover, he defined noncommutative de Rham homology associated to an algebra A and showed its relation to cyclic homology. CONNES [Co3] defined Chern character of θ-summable Fredholm module $(\mathcal{H}, D)$ over a unital C^*-algebra A with values in the entire cyclic cohomology of A.

Over the past few years, explicit computation of the cyclic cocycle associated to the Dirac operator have appeared, using different approaches. Some have utilized asymptotic pseudodifferential operators and Getzler's rescaling technique [Bl–F], while others have used heat equation, graph projection method together with some rescaling techniques (c.f. [Co–M], [L2]).

The main result of this paper, is the computation of the local cyclic cocycle formula associated to family of Dirac operators. The fundamental techniques used are the rescaling of Bismut's superconnection and the canonical order calculus.

In the case of families of Dirac operator, we deal with a special class of θ-summable modules. The Hilbert space $\mathcal{H}$ is replaced by a bimodule $\mathcal{M}$ and the Dirac operator by Bismut's superconnection. Thus, the natural setting requires Kasparov's bivariant theory and bivariant Chern Connes character. There are several different approaches to bivariant character, some are algebraic while others are topological and analytical. To motivate our framework, Let us first recall some of these approaches.

QUILLEN [Q2] presented a new approach to the algebraic formalism of cyclic cohomology. He defined a Hom complex from differential graded algebra to another algebra which played the rule of cyclic cochains. Then using the JLO formula he established a bivariant character. KASSEL [K] introduced a bivariant algebraic k-theory for unital algebras. The bivariant group $K(A, B)$ is defined as the Grothendieck group of an exact category of suitable $(A - B)$ bimodules. He established a bivariant Chern character from the bivariant cyclic group $K(A, B)$ with values in the bivariant cyclic

theory. NISTOR [N1] introduced a bivariant Chern–Connes character for finitely summable Kasparov kk-bimodules and established the compatibility with the periodicity operator in cyclic (co)homology and with Kasparov's product. Furthermore, in [N2], he made a connection between the superconnection formalism of Quillen and cyclic cohomology. Moreover he extended Quillen's superconnection formalism to arbitrary operators and established the coincidence of the Connes–Karoubi character of the graph projection with the Chern character of superconnection whenever some sort of pseudodifferential calculus exists.

Giving up the compatibility with the periodicity operator and Kasparov's product, WU [W] constructed a bivariant Chern–Connes character for (a special class) of θ summable modules, by incorporating the JLO formula and the superconnectiom formalism of Quillen. His construction is influenced by the work of CONNES (c.f. [Co2], [Co4], see also [Q2]). In fact when $(\mathcal{H}, D)$ is the dual Dirac on a locally symmetric space, then the Chern character is essentially the bivariant character of CONNES (c.f. [Co4] page 336–337).

Wu's bivariant character takes values in the bivariant cyclic theory described by LOTT [L1], who constructed it as a combination of entire cyclic (co)homology and non commutative de Rham homology of graded differential algebra (Ω, d). Hence by adopting Wu's method and employing Bismut's superconnection together with the canonical order calculus, we express the local formula for families in terms of differential forms on the base and the Chern roots of the fibration M/B.

In our forth coming paper [Az2], we construct an equivariant bivariant Chern Connes character, in the presence of a compact lie group which acts by automorphism on algebras. In the future, we hope to compute a local equivariant formula for families of invariant Dirac operators.

Let us summarize the contents of this article:

Consider a fibre bundle $\pi : M \to B$, where M and B are compact connected smooth manifolds, and a smooth family of Dirac operators $D = \{D_y\}$, one for each fibre $M_y = \pi^{-1}(y)$. π defines the fibration M/B of M where $M/B = \{M_y \ni y \in B\}$.

Denote by $\mathcal{E}$ the spinor bundle over M. This bundle gives rise to an infinite dimensional superbundle $\mathcal{W}$ over B with a $C^\infty(B)$ valued inner product. Let $\mathcal{H} = L^2(\Gamma(B, \mathcal{W}))$ be Hilbert space, then $(\mathcal{H}, D)$ is a θ-summable Fredholm module over the algebra $C(M)$. Let $\mathcal{M} = \wedge(B) \otimes \mathcal{H}$,

then $\mathcal{M}$ is a $\mathcal{V} - C^\infty(B)$ bimodule, where $\mathcal{V} = C^\infty(B) \otimes C^1(M)$. Bismut's superconnection $\mathcal{A}$ is an operator on $\mathcal{M}$ whose term of exterior degree zero is D, i.e. $\mathcal{A} = D + \tilde{\nabla} + E$, where $\tilde{\nabla}$ is a connection on $\wedge(B) \otimes \Gamma(B, \mathcal{W})$ defined in a certain way and $E \in \bigwedge(B) \otimes \mathrm{End}(\mathcal{W})$. Moreover $\mathcal{A}$ satisfies the following:

$$\mathcal{A}(ws) = (d_B w)s + (-1)^{|w|} w \mathcal{A}s, \ \text{where} \ w \in \wedge(B), \ s \in \wedge(B) \otimes \Gamma(B, \mathcal{W}).$$

The square of $\mathcal{A}$ is the curvature operator $\mathcal{F}$. The bivariant Chern–Connes character $\mathrm{Ch}^n(\mathcal{M}, \mathcal{A})$ over $\mathcal{V}$ is defined by

(1)
$$\mathrm{Ch}^n(\mathcal{M}, \mathcal{A})\big(v_0, v_1, \ldots, v_n\big)$$

$$= \int_{\Delta_{2k}} \mathrm{Tr}_s \Big(v_0 e^{-t_1 \mathcal{F}} [\mathcal{A}, v_1] e^{-(t_2 - t_1)\mathcal{F}} \ldots [\mathcal{A}, v_n] e^{-(1 - t_n)\mathcal{F}} \Big) dt_1 \ldots dt_n$$

where the supertrace Tr_s is the trivial extension of the operator supertrace on $\mathcal{H}$ to a $\bigwedge(B)$-valued map. The total bivariant character

$$\mathrm{Ch}(\mathcal{M}, \mathcal{A}) = \{\mathrm{Ch}^0(\mathcal{M}, \mathcal{A}), \mathrm{Ch}^1(\mathcal{M}, \mathcal{A}), \ldots, \mathrm{Ch}^n(\mathcal{M}, \mathcal{A}), \ldots\}$$

sends the entire cyclic cycle of $\mathcal{V}$ to de Rham cocycle in $\wedge(B)$, i.e. the bivariant character is a homomorphism from the entire cyclic homology $HE_*(\mathcal{V})$ of $\mathcal{V}$ to the de Rham cohomology $H^*_{dR}(B)$ of B.

Consider the asymptotic expansion of the heat kernel. The key observation is the relation between the number of Clifford variables and powers of t. In fact each Clifford variable corresponds to $1/2$ power of t. This relation plays an important role in our computation.

On one hand, the supertrace takes care of the singular part in the asymptotic expansion of the heat kernels in (1), since it vanishes on elements in which the number of Clifford variables is less than the dimension of the manifold. On the other hand, to keep track of powers of t in the asymptotic expansion we use the canonical order calculus which is due to SIMON ([C–F–K–S], Chapter 12). This calculus provides information on the integral kernel of an operator, in the sense of how fast it blows up or decays as t tends to zero.

The use of canonical order calculus and its relation to the Clifford variables is the main technique of this article. In [Az1] we have used similar techniques to compute the equivariant cyclic cocycle associated with the Dirac operator.

The explicit expression of $\mathcal{A}$ involves vertical and horizontal Clifford variables, which follows from the fact that TM is decomposed into vertical and horizontal tangent spaces. The horizontal Clifford variables f_α are replaced by their dual dy_α, the Grassmannian. Therefore, to take into account these Clifford variables a rescaling operator φ_t is introduced which multiplies each dy_α by $1/\sqrt{t}$. Replace $\mathcal{A}$ by the rescaled Bismut's superconnection $\mathcal{A}_t = \sqrt{t}\varphi_t(\mathcal{A})$ in (1), then use the canonical order calculus and its interplay with the Clifford variable. Thus the bivariant character in (1) is simplified into a single heat operator composed with a collection of commutators.

The localization property of the heat operator allows us to work locally. Therefore, employing the explicit formula for the rescaled curvature $\mathcal{A}_t^2$ (which is a generalization of the Lichnerowicz formula) and then applying our techniques, the heat kernel for $\mathcal{A}_t^2$ is approximated by the rescaled heat kernel for the harmonic oscillator type operator which in turn is given by Mehler's formula. Using this, the local cyclic cycle formula is expressed in terms of the Chern roots of the fibration M/B and differential forms on B.

The paper is organized as follows: Section 1 starts with preliminary and background material. In Section 2, following Wu's method, we construct the bivariant Chern–Connes character map associated with the rescaled Bismut's superconnection. In an attempt to make the paper self-contained, we briefly review the canonical order calculus that was developed by B. Simon. Most of the computations are carried out in Section 3. The main result is in Section 4, where we express the bivariant Chern character in terms of the Chern roots of M/B and differential forms on the base B.

1. Preliminary and background material

1.1. Fibration of manifolds and connections

Let B be an m dimensional connected compact smooth Riemannian manifold and M is a $2n+m$ dimensional compact connected smooth manifold. Let $\pi : M \to B$ be a fibration where the fibres $\{M_y = \pi^{-1}(y) \mid y \in B\}$ are family of connected compact Riemannian manifolds of dimension $2n$. This family will be denoted by M/B and $T(M/B)$ will denote the bundle

of vertical tangent vectors, i.e. $T_x(M/B)$ is the tangent space at x to the fibre $M_{\pi(x)}$.

Taking the orthogonal bundle of $T(M/B)$ in TM with respect to any Riemannian metric, determines a smooth horizontal subbundle $T^H M$, i.e. $TM = T^H M \oplus T(M/B)$. Vector fields $X \in TB$ will be identified with their horizontal lifts $X \in T^H M$, moreover $T_x^H M$ is isomorphic to $T_{\pi(x)}B$ via π_*.

Let ∇^B denote the Levi–Civita connection on TB. The metric of TB lifts to a smooth inner product on $T^H M$. Assume that $T(M/B)$ is endowed with a smooth inner product, these inner products can be extended to TM by assuming that $T^H M$ and $T(M/B)$ are orthogonal. Now M is a Riemannian manifold. Let ∇^L be the corresponding Levi–Civita connection on M. This connection need not preserve the splitting of TM into horizontal and vertical subspaces. Therefore we define a second connection ∇ on TM. As in [Bi] there is a unique natural connection ∇ on TM satisfying the following:

1. $Y \in TB$, $Z \in TB$, $\nabla_Y Z = \nabla_Y^B Z$

2. $Y \in T(M/B)$, $Z \in TB$, $\nabla_Y Z = 0$

3. $Y \in M$, $Z \in T(M/B)$, $\nabla_Y Z = \nabla_Y^{M/B} Z = P_{M/B}(\nabla_Y^L Z)$

where $P_{M/B}$ is the projection operator from TM onto $T(M/B)$. Clearly ∇ preserves the metric and the splitting $TM = T^H M \oplus T(M/B)$. Moreover, for $y \in B$ the restriction of ∇ to $T(M_y)$ coincides with the Levi–Civita connection of the Riemannian submanifold M_y. The restriction of ∇ to $T(M/B)$ will be denoted by $\nabla^{M/B}$.

From now on we assume that the tangent bundle $T(M/B)$ over M is oriented and spin. Denote by $\mathcal{E}$ the superbundle of spinors associated with the Clifford bundle $Cl(T(M/B))$. This is a Z_2 graded Hermitian bundle over M whose fibre $\mathcal{E}_x$ at a point $x \in M$ is the space of vertical spinors.

The connection ∇ defines naturally a unitary connection on $\mathcal{E}$ which is still denoted by ∇. Let $R^{M/B}$ be the curvature tensor of ∇ restricted to $T(M/B)$ which lifts naturally as the curvature tensor of $\mathcal{E}$.

The spinor bundle $\mathcal{E}$ over M gives rise to an infinite dimensional superbundle $\mathcal{W}$ over B, whose fibre at $y \in B$ is the infinite dimensional space of sections $\mathcal{W}_y = \Gamma(M_y, \mathcal{E}_y)$, here $\mathcal{E}_y = \mathcal{E}|_{M_y}$. We define $\Gamma(B, \mathcal{W})$ as $\Gamma(M, \mathcal{E})$ (note that $\Gamma(-, -)$ will always mean smooth sections).

The Hermitian structure on $\mathcal{E}$ and the natural volume form on the fibres $\{M_y \mid y \in B\}$ gives rise to a $C^\infty(B)$-valued inner product on $\mathcal{W}$, which is defined by integration along the fibres;

$$\langle s_1, s_2 \rangle(y) = \int_{M_y} \langle s_1(x), s_2(x) \rangle dx, \quad s_1, s_2 \in \Gamma(B, \mathcal{W}).$$

Thus, each fibre $\mathcal{W}_y$ has a pre Hilbert space structure, let $\mathcal{H} = L^2(\Gamma(B, \mathcal{W}))$ be the Hilbert space.

The smooth family $D = \{D_y \mid y \in B\}$ of Dirac operators can be considered as an odd endomorphism of the infinite dimensional bundle $\mathcal{W}$, where each D_y acts fibrewise on $\mathcal{W}^\pm = \Gamma(M_y, \mathcal{E}^\pm)$. Moreover, each D_y is a self adjoint elliptic operator, thus the index bundle of the family D is well defined.

ATIYAH and SINGER in [At-S], showed that the difference bundle $\mathrm{Ker}\, D_{+,y} - \mathrm{Ker}\, D_{-,y}$ over B, is well defined in the sense of K-theory.

BISMUT [Bi] constructed a superconnection $\mathcal{A}$ (which is defined independent of the metric g_B) acting on $\wedge(B) \otimes \Gamma(B, \mathcal{W})$, we will refer to this superconnection as Bismut's superconnection. Its curvature is an elliptic second order differential operator $\mathcal{F} = \mathcal{A}^2$. Denote by $\mathcal{F}^y$ the restriction of $\mathcal{F}$ to M_y.

For each $t > 0$, the heat operator $e^{-t\mathcal{F}^y}$ is given by a smooth kernel $e^{-t\mathcal{F}^y}(x, x')$ which is C^∞ in (t, x, x', y), here $x, x' \in M_y$ (c.f. [Bi] Proposition 2.8). This kernel is a linear mapping from $\mathcal{E}_{x'}$ into $\wedge_y(B) \hat{\otimes} \mathcal{E}_x$ which is even. In particular, for $x \in M_y$, $tr_s e^{-t\mathcal{F}^y}(x, x)$ is an even element in $\wedge_y(B)$, here $\wedge(B)$ denotes the exterior algebra of T^*B.

Theorem 1.1 ([Bi] Theorem 3.4). *The C^∞ form over B*

(2)
$$\mathrm{Tr}_s(e^{-t\mathcal{F}^y}) = \int_{M_y} \mathrm{tr}_s(e^{-t\mathcal{F}^y}(x, x))dx$$

is closed and is a representative in cohomology of $\mathrm{ch}(\ker D_{+,y} - \ker D_{-,y})$

As in [Q1], we change the normalization constant in the definition of the Chern character. Namely, for a vector bundle V with connection μ and curvature C, we set $\mathrm{Ch}(V) = \mathrm{Tr}\exp(-C)$.

2. The bivariant Chern–Connes character

WU [W] constructed a bivariant Chern–Connes character by incorporating the JLO formula and the superconnection formalism of Quillen with values in the bivariant cyclic theory of Lott. Following Wu's method, we define the bivariant Chern character associated with Bismut's superconnection.

Let $C^1(M)$ be the Banach algebra which is the completion of $C^\infty(M)$ with respect to the norm $|f| := \|f\| + \|[D, f]\|$, for $f \in C^\infty(M)$. The commutator $[D, f]$ extends to a bounded operator on $\mathcal{H}$. The algebra $C^1(M)$ acts on $\mathcal{L}(\mathcal{H})$ (bounded operators on $\mathcal{H}$) by multiplication. Thus, $(\mathcal{H}, D)$ defines a θ-summable Fredholm module on $C(M)$. Denote by $\mathcal{V}$ the projective tensor product of the Banach algebras $C^1(M)$ and $C^\infty(B)$, i.e. $\mathcal{V} = C^\infty(B) \hat{\otimes} C^1(M)$, with the projective tensor product norm.

Let $\mathcal{M} = \wedge(B) \otimes \mathcal{H}$ be a $\mathcal{V} - C^\infty(B)$ bimodule, where $\mathcal{V}$ acts on the left of $\mathcal{M}$ by letting $C^\infty(B)$ act on $\wedge(B)$ by multiplication by left and $C^1(M)$ acts on $\mathcal{H}$ while $\wedge(B)$ acts on $\mathcal{M}$ by multiplication from right. And there is a continuous $C^\infty(B)$ valued inner product on $\mathcal{M}$.

Convention: By T, a bounded operator (or trace class operator) on $\mathcal{M}$, we mean that $T \in \wedge(B) \otimes \mathcal{L}(\mathcal{H})$ or $(T \in \wedge(B) \otimes \mathcal{L}^1(\mathcal{H}))$.

Before defining the bivariant Chern–Connes character, we briefly recall the bivariant cyclic theory which is discussed in LOTT (c.f. [L1] also [Q2]).

2.1. Bivariant Cyclic homology

Let $C_n(\mathcal{V}) = \mathcal{V} \otimes \bar{\mathcal{V}}^{\otimes n}$, where $\bar{\mathcal{V}} = \mathcal{V}/\mathbb{C}$ and $\otimes$ means the completed projective tensor product. There is a certain norm defined on the space $\oplus_{n \geq 0} C_n(\mathcal{V})$. Denote by $C_*^r(\mathcal{V})$ the completion of $\oplus_{n \geq 0} C_n(\mathcal{V})$ with respect to that norm, and $CE_*(\mathcal{V}) = \bigcup_{r > 0} C_*^r(\mathcal{V})$. Thus, $CE_*(\mathcal{V})$ is a Z_2 graded Fréchet space with boundary operator $b + B$ of degree -1, where b and B are Connes cyclic boundary operator. The homology of the complex $(CE_*(\mathcal{V}), b + B)$ is the entire cyclic homology of $\mathcal{V}$, which we denote it by $HE_*(\mathcal{V})$ (c.f. [Co3], [G-S]). The (noncommutative) de Rham homology $H_*^{dR}(C^\infty(B))$ of the differential graded algebra $(\wedge(B), d_B)$ over the algebra $C^\infty(B)$ is in fact the de Rham cohomology $H_{dR}^*(B, d_B)$ of B (c.f. [Ka]).

Consider the Z_2 graded Hom complex $CE(\mathcal{V}, \wedge(B)))$ of continuous linear maps from $CE_*(\mathcal{V})$ to $\wedge(B)$ with boundary operator ∂ defined by

$$(\partial \phi)(Z) = d(\phi(Z)) + (-1)^{|\phi|} \phi((b + B)(Z))$$

One can easily show that $\partial^2 = 0$ which follows from the properties of b and B. The bivariant cyclic homology $HE((\mathcal{V}, \wedge(B)), \partial)$ is the homology of the complex $CE(\mathcal{V}, \wedge(B))$ and any class $[\tau] \in HE((\mathcal{V}, \wedge(B)), \partial)$ defines a homomorphism $\tau : HE(\mathcal{V}) \rightarrow H_*^{dR}(B)$.

2.2. The bivariant character

For bounded operators A_i on $\mathcal{M}$, $i = 1, \ldots, 2k$. Define

$$\langle\langle A_0, A_1, \ldots, A_{2k}\rangle\rangle$$

(3)
$$:= \int_{\Delta_{2k}} \mathrm{Tr}_s\Big(A_0 e^{-t_1 \mathcal{F}} A_1 e^{-(t_2 - t_1)\mathcal{F}} \ldots A_{2k} e^{-(1 - t_{2k})\mathcal{F}}\Big) dt_1 \ldots dt_{2k}$$

where Δ_{2k} is the standard $2k$ simplex. The above integral is convergent in $\wedge(B) \otimes \mathcal{L}^1(\mathcal{H})$, since it is composition of trace class operators $e^{-(t_{i+1} - t_i)\mathcal{F}} \in \wedge(B) \otimes \mathcal{L}^1(\mathcal{H})$ and bounded operators $A_j's$ (c.f. [W]). Thus, $\langle\langle A_0, A_1, \ldots, A_{2k}\rangle\rangle \in \wedge(B)$.

Let $|A_i|$ denote the degree of A_i. The next proposition is a generalization of Lemma 2.2 of [G-S].

Proposition 2.1. *For A_i bounded operators on $\mathcal{M}$, $i = 1, \ldots, 2k$, and $\varepsilon_i = (|A_0| + \cdots + |A_{i-1}|)(|A_i| + \cdots + |A_{2k}|)$. Then we have*

1. $\langle\langle A_0, \ldots, A_{2k}\rangle\rangle = (-1)^\varepsilon \langle\langle A_i, \ldots, A_{2k}, A_0, \ldots, A_{i-1}\rangle\rangle$
2. $\langle\langle A_0, \ldots, A_{2k}\rangle\rangle = \sum_i (-1)^\varepsilon \langle\langle 1, A_i, \ldots, A_{2k}, A_0, \ldots, A_{i-1}\rangle\rangle$
3. $\sum_i (-1)^{|A_0| + \cdots + |A_{i-1}|} \langle\langle A_0, \ldots, [\mathcal{A}, A_i], \ldots, A_{2k}\rangle\rangle = 0$
4. $\langle\langle A_0, \ldots, [\mathcal{F}, A_i], \ldots, A_{2k}\rangle\rangle$
   $= \langle\langle A_0, \ldots, A_{i-1}A_i, \ldots, A_{2k}\rangle\rangle - \langle\langle A_0, \ldots, A_i A_{i+1}, \ldots, A_{2k}\rangle\rangle.$

PROOF. The proof of this proposition is the same as that of Lemma 2.2 of [G-S] together with the fact $\mathrm{Tr}_S[\mathcal{A}, e^{-\mathcal{F}}] = 0$ (see also [B-G-V] Chapter 9). $\square$

The Dirac operator D extends trivially to $\mathcal{V}$. Thus, for $h \otimes f \in \mathcal{V}$, then $h \otimes f \in C^\infty(B) \otimes \mathcal{L}(\mathcal{H})$ and the commutator $[D, h \otimes f] = h \otimes [D, f]$. Bismut's superconnection $\mathcal{A}$ acts on $\mathcal{M}$. Hence, the graded commutator $[\mathcal{A}, h \otimes f] \in \wedge(B) \otimes \mathcal{L}(\mathcal{H})$. Furthermore,

(4) $$[\mathcal{A}, h \otimes f] = d_B(h \otimes f) + h \otimes [D, f]$$

which follows from the definition of $\mathcal{A}$ (see Section 3.1).

Definition 2.2. The bivariant Chern–Connes character of the module $(\mathcal{M}, \mathcal{A})$ over the algebra $\mathcal{V}$ is given by

$$\mathrm{Ch}^{2k}(\mathcal{M}, \mathcal{A})\big(h_0 \otimes f_0, \ldots, h_{2k} \otimes f_{2k}\big)$$
$$:= \langle\langle h_0 \otimes f_0, [\mathcal{A}, h_1 \otimes f_1], \ldots, [\mathcal{A}, h_{2k} \otimes f_{2k}]\rangle\rangle.$$

And the total bivariant character is

$$\mathrm{Ch}(\mathcal{M}, \mathcal{A}) = \{\mathrm{Ch}^0(\mathcal{M}, \mathcal{A}), \mathrm{Ch}^1(\mathcal{M}, \mathcal{A}), \ldots, \mathrm{Ch}^{2k}(\mathcal{M}, \mathcal{A}), \ldots\}.$$

The bivariant character $\mathrm{Ch}(\mathcal{M}, \mathcal{A})$ is a linear map $\mathrm{Ch}(\mathcal{M}, \mathcal{A})$: $\oplus_{j \geq 0} C_j(\mathcal{V}) \to \wedge(B)$, which extends continuously to a map [W];

$$\mathrm{Ch}(\mathcal{M}, \mathcal{A}) : CE(\mathcal{V}) \to \wedge(B).$$

Hence, $\mathrm{Ch}(\mathcal{M}, \mathcal{A}) \in CE(\mathcal{V}, \wedge(B))$.

Theorem 2.3. *The bivariant Chern–Connes character* $\mathrm{Ch}^*(\mathcal{M}, \mathcal{A})$ *is closed and hence defines a homology class in* $HE_{ev}(\mathcal{V}, \wedge(B))$, *i.e.* $\mathrm{Ch}^*(\mathcal{M}, \mathcal{A})$ *is a homomorphism from the entire cyclic homology* $HE(\mathcal{V})$ *to the de Rham cohomology* $H^*_{dR}(B)$.

PROOF. This follows from the above observations, Proposition 2.1, and Theorem 1.1, see also [W].   □

2.3. Rescaling Bismut's superconnection

As in [Bi-F] for $t > 0$, let φ_t be the rescaling operator on $\wedge(B)$, which multiplies differential forms of degree p by $t^{-p/2}$. More explicitly, for $y \in B$ consider a fixed submanifold M_y. Then for $x, x' \in M_y$, φ_t is the homomorphism

$$(5) \qquad \varphi_t : \wedge_y(B) \,\hat{\otimes}\, \mathrm{Hom}(\mathcal{E}_{x'}, \mathcal{E}_x) \to \wedge_y(B) \,\hat{\otimes}\, \mathrm{Hom}(\mathcal{E}_{x'}, \mathcal{E}_x)$$

$$(6) \qquad \varphi_t : dy_\alpha h \mapsto \frac{1}{\sqrt{t}} dy_\alpha h, \quad h \in \mathrm{Hom}(\mathcal{E}_{x'}, \mathcal{E}_x).$$

Let $\mathcal{A}_t = \sqrt{t}\, \varphi_t(\mathcal{A})$ be the rescaled Bismut's superconnection with rescaled curvature $\mathcal{F}_t = \mathcal{A}_t^2 = t\varphi_t(\mathcal{F})$.

In [Bi] BISMUT proved that the cohomology class of $\text{Ch}(\mathcal{A}_t) = \text{Tr}_s(e^{-\mathcal{F}_t^y})$ is closed and independent of t and showed that the limit $\lim_{t\to 0} \text{Ch}(\mathcal{A}_t)$ exists. Furthermore

$$(7) \qquad \text{Tr}_s(e^{-\mathcal{F}_t^y}) = \int_{M_y} \text{tr}_s(e^{-\mathcal{F}_t^y}(x,x))dx = \text{Tr}_s\, e^{-t\varphi_t(\mathcal{A}^2)}$$

$$(8) \qquad\qquad = \text{Tr}_s(\varphi_t e^{-t\mathcal{A}^2}) = \varphi_t \int_{M_y} \text{tr}_s(e^{-t\mathcal{F}}(x,x))dx.$$

This leads to the following proposition (c.f. [Z] also [B–G–V] Chapter 10).

Proposition 2.4. *For all $t > 0$,*

$$\text{Tr}_s(e^{-\mathcal{F}_t^y}) = \int_{M_y} \varphi_t \, \text{tr}_s\left(e^{-t\mathcal{F}^y}(x,x)\right) dx$$

is a representative of $\text{Ch}(\ker D_{+,y} - \ker D_{-,y})$.

Theorem 2.5. *For $t > 0$, the Chern character $\text{Ch}^{2k}(\mathcal{M}, \mathcal{A}_t)$ is closed and defines a homology class which is independent of t, moreover $\lim_{t\to 0} \text{Ch}_*(\mathcal{M}, \mathcal{A}_t)$ exists.*

PROOF. An easy consequence of Theorem 2.3 and Proposition 2.4, see also [G-S]. □

Replace Bismut's superconnection $\mathcal{A}$ and curvature $\mathcal{F}$ in (3) by their rescaled $\mathcal{A}_t$ and $\mathcal{F}_t$. Then using $[\varphi_t(\mathcal{A}), h_i \otimes f_i] = \varphi_t[\mathcal{A}, h_i \otimes f_i]$ and the relation in (4), $\text{Ch}^{2k}(\mathcal{M}, \mathcal{A}_t)$ becomes;

$$\text{Ch}^{2k}(\mathcal{M}, \mathcal{A}_t)\big(h_0 \otimes f_0, h_1 \otimes f_1, \ldots, h_{2k} \otimes f_{2k}\big)$$

$$= \int_{\Delta_{2k}} \text{Tr}_s\Big(h_0 \otimes f_0 e^{-t_1 \mathcal{F}_t} \ldots [\mathcal{A}_t, h_{2k} \otimes f_{2k}]e^{-(1-t_{2k})\mathcal{F}_t}\Big)dt_1 \ldots dt_{2k}$$

$$= t^k \int_{\Delta_{2k}} \text{Tr}_s\Big(h_0 \otimes f_0 \varphi_t(e^{-t_1 t\mathcal{F}}) \ldots [\varphi_t(\mathcal{A}), h_{2k} \otimes f_{2k}]\varphi_t(e^{-(1-t_{2k})t\mathcal{F}})\Big)dt_1 \ldots dt_{2k}$$

$$(9) \qquad = \frac{1}{t^k} \int_{\Delta_{2k,t}} \varphi_t\Big[\text{Tr}_s\Big(h_0 \otimes f_0 e^{-s_0 \mathcal{F}}[\mathcal{A}, h_1 \otimes f_1]e^{-s_1 \mathcal{F}} \ldots [\mathcal{A}, h_{2k} \otimes f_{2k}]e^{-s_{2k}\mathcal{F}}\Big)\Big]ds$$

$$(10) \qquad = \frac{1}{t^k} \sum_{q=0}^{2k} \int_{\Delta_{2k,t}} \varphi_t\Big[\text{Tr}_s\Big(T_0 e^{-s_0 \mathcal{F}} T_1 e^{-s_1 \mathcal{F}} \ldots T_{2k} e^{-s_{2k}\mathcal{F}}\Big)\Big]ds_1 \ldots ds_{2k},$$

where T_i is either $d_B(h_i \otimes f_i)$ or $h_i \otimes [D, f_i]$ for $i = 1, \ldots, 2k$, and $T_0 = h_0 \otimes f_0$. Here q is the number of times the operator $d_B(h_i \otimes f_i)$ appears

in (10). In (9) we did a change of variables:

$$s_0 = t_1 t, \ s_j = (t_{j+1} - t_j)t \text{ for } j = 1, \ldots 2k-1, \text{ and } s_{2k} = (1-t_{2k})t.$$

Thus, $\sum_{i=1}^{2k} s_i \leq t$ and $s_0 = t - s_1 \cdots - s_{2k}$, and we are integrating over the $\Delta_{2k,t}$ simplex

$$\Delta_{2k,t} = \left\{ s = (s_1, \ldots, s_{2k}) \ni \sum_{i=1}^{2k} s_i \leq t \text{ with } 0 \leq s_i \leq t \right\}.$$

The rest of the paper deals with simplifying the above formula, and then expressing the bivariant character in terms of the chern roots and differential form.

As heat operator $e^{-s_j \mathcal{F}}$ does not commute with T_i, thus we can write it as

(11) $$e^{-s_j \mathcal{F}} T_i = T_i e^{-s_j \mathcal{F}} + [e^{-s_j \mathcal{F}}, T_i].$$

Therefore in (10) start from the far right and then apply (11) repeatedly. Continue this process until all the operators T_i are moved to the left while the heat operators to the right. At every step of this process some extra terms are produced these will be considered as error terms. Hence we have the following proposition.

Proposition 2.6.

(12) $$\frac{1}{t^k} \sum_{q=0}^{2k} \int_{\Delta_{2k,t}} \varphi_t \left[\mathrm{Tr}_s \left(T_0 e^{-s_0 \mathcal{F}} T_1 e^{-s_1 \mathcal{F}} \ldots T_{2k} e^{-s_{2k} \mathcal{F}} \right) \right] ds_1 \ldots ds_{2k}$$

$$= \frac{t^k}{(2k)!} \sum_{q=0}^{2k} \varphi_t \, \mathrm{Tr}_s \left(T_0 T_1 \ldots T_{2k} e^{-t\mathcal{F}} \right) + \varphi_t(R^{T_1}) \cdots + \varphi_t(R^{T_{2k}}),$$

where

$$R^{T_{2k-j}} = \frac{1}{t^k} \sum_q \int_{\Delta_{2k,t}} \mathrm{Tr}_s \left(T_0 e^{-s_0 \mathcal{F}} \ldots T_{2k-j-1}(B_{2k-j}) \right.$$

$$\left. \cdot \, e^{-s_{2k-j} \mathcal{F}} \ldots e^{-s_{2k} \mathcal{F}} \right) ds_1 \ldots ds_{2k}$$

for $j = 0, 1, \ldots, 2k - 1$, and

$$B_{2k-j} = \left[e^{-s_{2k-j-1}\mathcal{F}}, T_{2k-j}T_{2k-j+1} \ldots T_{2k} \right].$$

Next, we claim that the error terms $\varphi_t(R^{T_j})$ vanish as t tends to zero. The claim will be proved using canonical order calculus. For completeness we briefly define this calculus and mention some of its properties without any proof. The main reference is ([C-F-K-S] Chapter 12).

2.4. Canonical order calculus

Definition 2.7. A family of operators $\{P_t\}_{t>0}$ on $L^2(\mathbb{R}^{2n}, dx)$ [or on $L^2(N, dx)$ with N a compact Riemannian $2n$ dimensional manifold] is said to have canonical order $r \in \mathbb{R}$ if and only if:

1. for each $t > 0$, P_t maps $\mathbb{H}_{-\infty}$ to $\mathbb{H}_\infty$, here $\mathbb{H}_{-\infty} = \bigcup_s \mathbb{H}_s$, $\mathbb{H}_\infty = \bigcap_s \mathbb{H}_s$, where $\mathbb{H}_s$ is the s-Sobolev space.
2. for any $k \geq l \in \{0, \pm 1, \pm 2, \ldots\}$ there is a constant c, so that for $0 < t < 1$

$$||P_t u||_k \leq ct^{-a}||u||_l, \quad \text{with } a = \frac{k-l}{2} - r.$$

Remark 2.8. The definition is such that if P_t has canonical order r, it has canonical order q for any $q \leq r$. From this it follows that if P_t is a sum of operators, i.e. $P_t = B_t^1 + \cdots + B_t^r$, where each B_t^i is an operator of canonical order p_i, then P_t has canonical order p, where $p = \min_{1 \leq j \leq r}(p_j)$.

The whole point of the canonical order calculus is the information it yields on integral kernels.

Proposition 2.9. If the operator P_t has canonical order r, then P_t has an integral kernel $P_t(x, y)$ satisfying

$$\lim_{t \to 0} t^b \sup_{x,y} |P_t(x, y)| = 0, \quad \text{for any } b > n - r.$$

Theorem 2.10. Let Δ be the Laplacian on $\mathbb{R}^{2n}$ or on $\bigwedge(N)$. Then (assuming all the operators make sense)

1. $e^{-t\Delta}$ has canonical order zero,
2. the operators $\frac{\partial^p}{\partial x_i^p} e^{-t\Delta}$ and $e^{-t\Delta} \frac{\partial^p}{\partial y_i^p}$ have canonical order $-p/2$, where p is some non-negative integer.

The next theorem will be used frequently, it deals with composition of operators.

Theorem 2.11. *Let $P_t^0, P_t^1, \ldots, P_t^r$ be operators of canonical order $m_0, m_1, \ldots, m_r$ with $m_j > -1$. Then*

$$P_t = \int_{\sum_{i=1}^r s_i \leq t} P_{s_0}^0 P_{s_1}^1 \ldots P_{s_r}^r \, ds_1 ds_2 \ldots ds_r$$

is a convergent integral and P_t is an operator of canonical order $r + \sum_{j=0}^r m_j$, here $s_0 = t - s_1 \ldots - s_r$ with $0 < s_i \leq t$.

3. The error term and the harmonic oscillator

Choose a point $y \in B$. From now on all our computations will be restricted to the fibre M_y. Now fix a point $\xi \in M_y$. The asymptotic expansion of the heat kernel $e^{-t\mathcal{F}}(\xi, \exp_\xi \boldsymbol{x})$ in the limit $(\boldsymbol{x}, t) \to 0$ are local, in the sense that regions of M_y outside an ϵ - ball $B(\xi, \epsilon)$ contribute an exponentially vanishing amount to the heat kernel inside this ball (c.f. [G]) and this fact localizes the computation.

The tangent space $T_\xi M$ is decomposed into vertical and horizontal spaces as $T_\xi M = T_\xi(M_y) \oplus T_y B$. Let $U = \{\boldsymbol{x} \in T_\xi(M_y) \ni \|\boldsymbol{x}\| < \epsilon\}$ where ϵ is small. Then U is identified via the exponential map $\exp_\xi : T_\xi(M_y) \longrightarrow M_y; \boldsymbol{x} \to \exp_\xi \boldsymbol{x}$, with the ϵ ball $B(\xi, \epsilon) \subset M_y$. Using the parallel translation map along the geodesic from $x \in B(\xi, \epsilon)$ to ξ (which is defined with respect to the Levi–Civita connection on M_y) we identify $T_{\boldsymbol{x}}(M_y)$ with $T_\xi(M_y)$.

Let $e_1, \ldots, e_{2n}$ be a fixed orthonormal base of $T_\xi(M_y)$. Then $e_1, \ldots \ldots, e_{2n}$ generates the Clifford algebra $Cl(T_\xi M_y)$ which acts naturally on $\mathcal{E}$. We turn the e_i's into local frame E_i's by parallel translating the e_i's for all i along geodesics passing through ξ. The parallel translation is defined with respect to the connection $\nabla^{M/B}$, but restriction to a single fibre M_y implies that it is the Levi–Civita connection of M_y.

The exponential map gives rise to normal coordinates at ξ. Thus for any $x \in B(\xi, \epsilon)$, the *normal coordinates* $(x_1, x_2, \ldots, x_{2n})$ are defined by $\exp_\xi(\boldsymbol{x}) = x$, where $\boldsymbol{x} = \sum_{i=1}^{2n} x_i e_i$.

Let $dx_1, \ldots, dx_{2n}$ be an oriented orthonormal base of $T_\xi^*(M_y)$ dual to $e_1, \ldots, e_{2n}$ and let $f_1, \ldots f_\alpha, \ldots, f_m$ be a fixed oriented orthonormal base of $T_y B$ with dual $dy_1, \ldots, dy_m$. We denote by Z_I a local orthonormal frame of TM on U consisting of the union of $E_1, \ldots, E_{2n}$ and a fixed basis $f_1, \ldots, f_m$.

Note that $dx_1, \ldots, dx_{2n}, dy_1, \ldots, dy_m$ satisfies the usual anticommutation rules as elements of the exterior algebra $\bigwedge(M)$.

Remark 3.1. In what follows;

- all the summation signs will be omitted

- the subscripts α, β will be used for horizontal variables and the subscripts i, j for the vertical ones, i.e. the variables in $T(M/B)$.

- the subscripts I, J will be used to denote both vertical and horizontal variables.

- we identify the orthonormal basis $f_1, \ldots, f_m$ of $T_y B$ with their lift in $T_x^H M$ (for $x \in M_y$). Also $dy_1, \ldots, dy_m$ are considered as differential forms on M.

- the Clifford variable e_i's and the differential dy_α's satisfy the relation $e_i dy_\alpha + dy_\alpha e_i = 0$.

3.1. Explicit expression of $\mathcal{F}$

In [Bi] Proposition 3.3, Bismut proved that the superconnection $\mathcal{A}$ is independent of the metric g_B and showed that

$$(13) \qquad \mathcal{A} = D + \tilde{\nabla} + E = e_i \left[\nabla_{E_i} + \frac{1}{2} \Gamma_{ij}^\alpha e_j \, dy_\alpha + \frac{1}{4} \Gamma_{i\alpha}^\beta \, dy_\alpha dy_\beta \right]$$
$$+ dy_\alpha \left[\nabla_{f_\alpha} + \frac{1}{2} \Gamma_{\alpha i}^\beta e_i dy_\beta \right].$$

The Γ_{IJ}^K are the Christoffel symbols defined with respect to the Levi–Civita connection ∇^L on M, i.e. $\Gamma_{IJ}^K = \langle \nabla_{Z_I}^L Z_J, Z_K \rangle$. Here the Dirac operator $D = e_i \nabla_{E_i}$, $\tilde{\nabla}$ is a connection on $\wedge(B) \otimes \Gamma(B, \mathcal{W})$ given by $\tilde{\nabla} = dy_\alpha \nabla_{f_\alpha}$ and $E \in \wedge(B) \otimes \mathrm{End}(\mathcal{W})$.

Thus for $w \otimes s \in \wedge(B) \otimes \Gamma(B, \mathcal{W})$, the connection is defined by

$$\tilde{\nabla}(w \otimes s) = (dy_\alpha \nabla_{f_\alpha}^B w) \otimes s + (-1)^{|w|} w \otimes dy_\alpha \nabla_{f_\alpha} s$$
$$= (d_B w) \otimes s + (-1)^{|w|} w \otimes dy_\alpha \nabla_{f_\alpha} s,$$

$\wedge(B) \otimes \mathrm{End}(\Gamma(B, \mathcal{W}))$ acts naturally on $\wedge(B) \otimes \Gamma(B, \mathcal{W})$, namely if $s \in \mathrm{End}(\Gamma(B, \mathcal{W}))$, $r \in \Gamma(B, \mathcal{W})$ and $w_1, w_2 \in \wedge(B)$. Then we set $(w_1 s)(w_2 r) = (-1)^{|w_1||r|}(sr) w_1 \wedge w_2$. This implies that for $h \otimes f \in \mathcal{V}$, then the commutator

$$(14) \qquad\qquad [\tilde{\nabla}, h \otimes f] = d_B h \otimes f + h \otimes dy_\alpha f_\alpha(f).$$

Note that f is a smooth function on M, hence it is a function in $y \in B$ and $x \in M_y$ variables. Thus, by abuse of notation we denote $dy_\alpha f_\alpha(f)$ by $d_B f$, and we write $[\tilde{\nabla}, h \otimes f] = d_B(h \otimes f)$.

The commutator of Bismut's superconnection $\mathcal{A}$ with $h \otimes f$ has the form

$$(15) \qquad [\mathcal{A}, h \otimes f] = [D, h \otimes f] + [\tilde{\nabla}, h \otimes f]$$
$$= h \otimes [D, f] + d_B(h \otimes f).$$

Under the identification of $T(M_y)$ with its dual space $T^*(M_y)$, we can write $[D, f]$ as $[D, f] = c(\hat{d}f)$, i.e. Clifford multiplication by $\hat{d}f = dx_i E_i(f)$.

Lemma 3.2. *With respect to the normal coordinates at $\xi = 0$. The Christoffel symbols Γ_{IJ}^K's have the following Taylor expansion about ξ*

$$\Gamma_{IJ}^K = \frac{1}{2} \sum_r R_{IrJK}(0)x_r + o(|x|^2).$$

where $R_{IrJK} = -(R^{M/B}(Z_I, E_r)Z_J, Z_K)$ is the curvature operator defined with respect to $\nabla^{M/B}$ connection.

PROOF. This is a standard argument, one can consult for example [Bo]. We only need to realize that we are restricted to a fixed fibre M_y and the normal coordinates are defined with respect to the orthonormal frame Z_I. $\qquad\square$

Theorem 3.3. *With respect to the normal coordinates at ξ and orthonormal frame Z_I, Bismut's curvature $\mathcal{F}^y$ is given explicitly by*

$$(16) \qquad \mathcal{F}^y = -\Delta^y + \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{B},$$

where

$\Delta^y = \sum_{i=1}^{2n} \frac{\partial^2}{\partial x_i^2}$ *is the Laplacian along the fibre M_y of the bundle M/B*

$$\boldsymbol{a} = \frac{1}{4}R_{ijst}x_i \frac{\partial}{\partial x_j}e_s e_t + \frac{1}{2}R_{ijs\alpha}x_i \frac{\partial}{\partial x_j}e_s dy_\alpha + \frac{1}{4}R_{ij\alpha\beta}x_i \frac{\partial}{\partial x_j}dy_\alpha dy_\beta,$$

$$\boldsymbol{b} = \frac{1}{64} x_i x_j R_{irlk} R_{rjst} e_l e_k e_s e_t + \frac{1}{16} x_i x_j R_{irk\alpha} R_{rjst} e_k e_s e_t dy_\alpha$$

$$+ \frac{1}{32} x_i x_j R_{irlk} R_{rj\alpha\beta} e_l e_k dy_\alpha dy_\beta + \frac{1}{16} x_i x_j R_{irk\alpha} R_{rjl\beta} e_k dy_\alpha e_l dy_\beta$$

$$+ \frac{1}{16} x_i x_j R_{irl\alpha} R_{rj\lambda\mu} e_l dy_\alpha dy_\lambda dy_\mu + \frac{1}{64} x_i x_j R_{ir\alpha\beta} R_{rj\lambda\mu} dy_\alpha dy_\beta dy_\lambda dy_\mu.$$

And $\boldsymbol{B}$ is of the form $\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5$; where

$$\phi_1 = cx^4 e^3 dy, \quad \phi_2 = c\frac{\partial}{\partial x_j} x^2 e^2, \quad \phi_3 = cx^3 e^4$$

$$\phi_4 = cx^3 e dy^3, \quad \phi_5 = cx^3 dy^4.$$

The notation x^3 means $\sum x_i x_j x_k$ similarly dy^3 denotes $\sum dy_\alpha dy_\beta dy_\mu$, also the Clifford variable e^3 means $\sum e_r e_s e_t$ and c is some constant.

PROOF. This follows from Lemma 3.2 and the generalized Lichnerowicz formula given by BISMUT in [Bi], Theorem 3.5, see also [Z]. □

3.2. The heat kernel of Bismut's superconnection

The asymptotic of the heat kernel is given by

$$(17) \qquad e^{-t\mathcal{F}^y}(\xi, x) = \frac{e^{-\rho^2(\xi,x)/4t}}{(4\pi t)^n} \sum_{j=0}^{N} t^j U_j(\xi, x) + O(t^{N-n+1}).$$

Where $N > n + [\frac{m}{2}]$. $\rho(\xi, x)$ is the Riemannian distance between ξ and x, i.e. $x \in B(\xi, \epsilon)$ and $U_j(\xi, x) : \mathcal{E}_\xi \longrightarrow \wedge_y(B) \otimes \mathcal{E}_x$ are linear transformations satisfying some properties.

Additionally, each $U_j(\xi, x)$ has r_j Clifford variables $e_{i_1} \dots e_{i_{r_j}}$ and q_j differential forms $dy_{\alpha_1} \dots dy_{\alpha_{q_j}}$ (c.f. [B-G-V], Chapter 10) such that;

$$(18) \qquad\qquad \text{their total sum is } r_j + q_j \le 2j.$$

For any $x, x' \in B(\xi, \varepsilon)$, let $K_t^y(\boldsymbol{x}', \boldsymbol{x})$ denote the heat kernel $e^{-t\mathcal{F}^y}(x', x)$ transferred to U by means of parallel translation map, where $x = \exp_\xi \boldsymbol{x}$ and $x' = \exp_\xi \boldsymbol{x}'$. Moreover, $\rho^2(x, x')$ has the following Taylor expansion

about ξ (c.f. [D]).

$$(19) \quad \rho^2(x, x') = \sum_i (x'_i - x_i)^2$$

$$- 1/3 \sum_{ijkl} R_{ijkl}(\xi) x_j x_l (x'_i - x_i)(x'_k - x_k) + 0(x - x')^3$$

where the x_i's and x'_j's are normal coordinates of x and x' respectively at $\xi = 0$. Therefore the asymptotic expansion of the heat kernel $K_t(\boldsymbol{x}', \boldsymbol{x})$, with $\rho^2(x, x')$ replaced by its Taylor expansion, becomes

$$(20) \quad K_t^y(\boldsymbol{x}', \boldsymbol{x}) = \frac{e^{-\|\boldsymbol{x}' - x\|^2 / 4t}}{(4\pi t)^n} \left[1 + F(t, \boldsymbol{x}', \boldsymbol{x})\right] \sum_{j=0}^{N} t^j U_j(\boldsymbol{x}', \boldsymbol{x})$$

$$+ O(t^{N-n+1}),$$

where

$$F(t, \boldsymbol{x}', \boldsymbol{x}) = -1/3 \sum_{ijkl} \left(\frac{R_{ijkl}(\xi) x_j x_l (x'_i - x_i)(x'_k - x_k)}{4t} \right) + \left(\frac{0(x - x')^3}{4t} \right).$$

3.3. The error term

In this section we show that the error terms vanishes as t tends to zero. But first we observe the following facts.

Lemma 3.4. *Let φ_t be the rescaling operator as in (5) and U_j as in (17). Then*

1. $\varphi_t(U_j) = t^{-q_j/2} U_j$, $\varphi_t d_B(h_i \otimes f_i) = t^{-1/2} d_B(h_i \otimes f_i)$ *and* $\varphi_t(h_i \otimes [D, f_i]) = h_i \otimes [D, f_i]$,

2. *the commutator* $\varphi_t([U_j, T_1 \ldots T_p]) = t^{-(q_j/2 + r/2)} [U_j, T_1 \ldots T_p]$, *where r is the number of times $T_i = d_B(h_i \otimes f_i)$ and q_j as in (18).*

Lemma 3.5. *For x, x' in $B(\xi, \epsilon)$, then*

1. *the operator $(x'_i - x_i)^p e^{-t\Delta^y}$ has canonical order $p/2$, where p is some non-negative integer,*

2. *the operator $(x'_i - x_i)^p \frac{\partial^r}{\partial x_j^r} e^{-t\Delta^y}$ has canonical order $p/2 - r/2$.*

For the proof one can refer to [C-F-K-S] page 280.   □

Theorem 3.6. *For $R^{T_{2k-j}}$ as in Proposition 2.6. Then*

$$\varphi_t(R^{T_{2k-j}}) = o(t^\epsilon), \text{ for some } \epsilon > 0 \text{ and } j = 0, 1, \dots 2k-1.$$

PROOF. With all the previous assumptions and restricting all the computation to a fixed fibre M_y. Let $K^y_{t,2k-j}$ denote the kernel of the operator $\varphi_t(R^{T_{2k-j}})$ transferred to $U \subset T_\xi(M_y)$. For simplicity we consider one case only and leave the other cases which follows similarly. Consider the case $q = 2k$, i.e. when all the T_i's are of the form $d_B(h_i \otimes f_i)$. Expand each heat kernel $e^{-s_i \mathcal{F}}$ as in (20). Thus on the diagonal at the origin $\xi = 0$, the heat kernel $K^y_{t,2k-j}(\xi, \xi)$ becomes;

$$K^y_{t,2k-j}(\xi, \xi) \simeq t^{-k} \sum_{i_0, \dots i_{2k}} \mathrm{tr}_s\, \varphi_t\Big(T_0 P_{i_0} U_{i_0} T_1 \dots$$

$$\dots P_{i_{2k-j-1}}\Big[U_{i_{2k-j-1}}, T_{2k-j} \dots T_{2k}\Big] P_{i_{2k-j}} U_{i_{2k-j}} \dots P_{i_{2k}} U_{i_{2k}}\Big)(\xi, \xi)$$

where each $P_{i_k}(\boldsymbol{x}, \boldsymbol{z}) = s_k^{i_k} e^{-s_k \Delta}(\boldsymbol{x}, \boldsymbol{z})\big[1 + F(t, \boldsymbol{x}, \boldsymbol{z})\big]$. To have a nonzero supertrace we need at least $2n$ (which is the dimension of M_y) distinct Clifford variables. Hence a term with at least $2n$ Clifford variables in the above expansion will satisfy

$$(21) \qquad\qquad r_{i_0} + r_{i_1} + \cdots + r_{i_{2k}} = 2n$$

here r_{i_j} is the number of Clifford variables in U_{i_j} as in (18). Next we compute its canonical order. Each P_{i_k} has canonical order i_k which follows from Lemma 3.5 and Remark 2.8. Consequently, by Lemma 3.4 and Theorem 2.11 (since we are integrating the heat kernel over the simplex $\Delta_{2k,t}$), the canonical order of that term is

$$-2k - (q_0/2 + q_1/2 \cdots + q_{2k}/2) + (i_0 + i_1 + \cdots + i_{2k}) + 2k$$

$$> r_0/2 + r_1/2 + \cdots + r_{2k}/2 = n \quad \text{by (21)}$$

the second inequality holds since the commutator term satisfies $r_{i_{2k-j-1}} + q_{i_{2k-j-1}} < 2i_{2k-j-1}$ (see Lemma 1 in the appendix). Therefore by Proposition 2.8 it vanishes as t tends to zero. $\qquad\square$

Indeed we have shown that

$$\mathrm{Ch}^{2k}(\mathcal{M}_y, \mathcal{A}_t^y)\big(h_0 \otimes f_0, \dots, h_{2k} \otimes f_{2k}\big)$$

$$= \frac{t^k}{(2k)!} \sum_q \varphi_t \, \mathrm{Tr}_s\Big(T_0 T_1 \dots T_{2k} e^{-t\mathcal{F}^y}\Big) + o(t^\epsilon).$$

3.4. Approximation of The Heat Kernel $e^{-t\mathcal{F}^y}$

Replace the x_i's that appear in the local expression of $\mathcal{F}^y$ (as in Theorem 3.3) by bounded smooth functions $h_i(x)$ with bounded derivatives, so that $h_i(x) = x_i$ in a neighborhood of $\xi = 0$. We continue using the notation x_i to denote $h_i(x)$. Thus, applying Duhamel's expansion we get

$$\frac{t^k}{(2k)!} \sum_q \Big\{ \varphi_t \, \mathrm{tr}_s\Big(T_0 T_1 \dots T_{2k} e^{-t\mathcal{F}^y}\Big) - \varphi_t \, \mathrm{tr}_s\Big(T_0 T_1 \dots T_{2k} e^{-t\Delta^y}\Big) \Big\}(0,0)$$

(22)
$$= \sum_{q=0}^{2k} \frac{t^{k-q/2}}{(2k)!} \, \mathrm{tr}_s\Big(T_0 T_1 \dots T_{2k} \int_0^t e^{-(t-s_1)\Delta^y} \varphi_t(-\boldsymbol{a} - \boldsymbol{b} + \boldsymbol{B}) e^{-s_1\Delta^y} ds_1\Big)(0,0)$$

$$+ \sum_{q=0}^{2k} \frac{t^{k-q/2}}{(2k)!} \, \mathrm{tr}_s\Big(T_0 T_1 \dots T_{2k} \int_0^t \int_0^{s_1} e^{-(t-s_1-s_2)\Delta_y} \varphi_t(-\boldsymbol{a} - \boldsymbol{b} + \boldsymbol{B})$$

$$\times e^{-s_1\Delta^y} \varphi_t(-\boldsymbol{a} - \boldsymbol{b} + \boldsymbol{B}) e^{-s_2\Delta^y} ds_1 ds_2\Big)(0,0) + \dots$$

When we refer to the canonical order of any operator in the expansion (22), take for example the operator $\boldsymbol{a}$, then we actually mean the canonical order of the operator $e^{-(t-s_{j_1}\dots s_{j_l})\Delta^y}\boldsymbol{a}$.

Remarks 3.7. It follows from Lemma 3.5 that:

The canonical order of $\boldsymbol{a}$ is zero, of $\boldsymbol{b}$ is 1, whereas of ϕ_1 is 2, of ϕ_2 is $1/2$ and of ϕ_i is $3/2$ for $i = 3, 4, 5$.

For operators $\boldsymbol{a}$ and $\boldsymbol{b}$ there is a relation between the number of Clifford variables e_i and the number of differentials dy_α. In the case of the operator $\boldsymbol{a}$, their total number adds up to 2, whereas in the case of $\boldsymbol{b}$ their total number adds up to 4.

Let us elaborate on our technique of computing the canonical order and keeping track of Clifford variables. Take for example the operator $\boldsymbol{b}$, suppose it appears j times in the expansion (22), this will be denoted by $\boldsymbol{b}^j$. Each $\boldsymbol{b}$ contains r Clifford variables $0 \le r \le 4$ and q differentials dy_α's $0 \le q \le 4$, such that $r + q = 4$. Hence the total number of Clifford variables for $\boldsymbol{b}^j$ (operator $\boldsymbol{b}$ appearing j times) is jr and the total number of dy_α's

is jq. Thus applying the rescaling operator φ_t to $\boldsymbol{b}^j$ yields a $t^{-(jq)/2}$ power of t. Similar arguments work with the operator $\boldsymbol{a}$.

The canonical order of the operator $\boldsymbol{b}^j = \sum^j$ (canonical order of $\boldsymbol{b}$) $+$ $j = 2j$ (which follows from Theorem 2.11, note that in this case we are integrating j times). Therefore after applying the rescaling operator φ_t, the canonical order of $\varphi_t(\boldsymbol{b}^j) = 2j - (jq)/2$.

Next, we consider all different combinations of the operators $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{B}$ in the expansion (22) that would give us a total of at least $2n$ Clifford variables (to have a nonzero supertrace). After rescaling by φ_t, if their canonical order is greater than n, then by Proposition 2.9 they vanish. We will work out one case in detail. The rest follows in a similar way.

For simplicity consider the term with $q = 2k$. The notation $(\boldsymbol{a}^j, \boldsymbol{b}^k, \phi_1^l)$ denotes that the operator $\boldsymbol{a}$ appears j times, $\boldsymbol{b}$ appears k times and $\phi_1 \in B$ appears l times in the expansion (22), here $j > 0$, $k \geq 0$ and $l > 0$.

Each $\boldsymbol{a}$ has p Clifford variables $0 \leq p \leq 2$ and s differentials dy_α, $0 \leq s \leq 2$ such that $p + s = 2$, while $\boldsymbol{b}$ has r Clifford variables $0 \leq r \leq 4$ and q differentials $0 \leq q \leq 4$ such that $r + q = 4$, and ϕ_1 has 3 Clifford variables and one dy_α.

$$(23) \qquad \text{The total number of Clifford variables in } (\boldsymbol{a}^j, \boldsymbol{b}^k, \phi_1^l)$$
$$= jp + kr + 3l = 2n.$$

The rescaled canonical order of $\varphi_t(\boldsymbol{a}^j, \boldsymbol{b}^k, \phi_1^l)$ is

$$\varphi_t(\boldsymbol{a}^j) + \varphi_t(\boldsymbol{b}^k) + \varphi_t(\phi_1^l) = \left(j - \frac{js}{2} \right) + \left(2k - \frac{kq}{2} \right) + \left(3l - \frac{l}{2} \right)$$

$$= \frac{jp}{2} + \frac{kr}{2} + \frac{5l}{2} > n \quad \text{by (23)}.$$

Hence it vanishes as t tends to zero. Repeat the same process with the other terms, hence we have

$$(24) \qquad \sum_{q=0}^{2k} \frac{t^{k-q/2}}{(2k)!} \, \mathrm{tr}_s \left(T_0 T_1 \ldots T_{2k} \left[\varphi_t e^{-t\mathcal{F}^y} - \varphi_t e^{-t\Delta^y} \right](0,0) \right) = 0(t^\epsilon)$$

for some $\epsilon > 0$.

Theorem 3.8. *For each $0 \le q \le 2k$, we have*

$$t^{k-q/2} \operatorname{tr}_s \left(T_0 \dots T_{2k} \varphi_t(e^{-t\mathcal{F}^y}) \right)(0,0) -$$

$$t^{k-q/2} \operatorname{tr}_s \left(T_0 \dots T_{2k} \varphi_t(e^{-t(-\Delta^y + \boldsymbol{b})}) \right)(0,0) = 0(t^\epsilon)$$

for some $\epsilon > 0$.

PROOF. Using Duhamel's expansion, expand both operators $e^{-t\mathcal{F}^y}$ and $e^{-t(-\Delta^y + \boldsymbol{b})}$ as perturbations of $e^{-t\Delta^y}$. All terms that have the operator $\boldsymbol{b}$ will cancel out, and what remains are terms having operators $\boldsymbol{a}$ and ϕ_i's in $\boldsymbol{B}$. The above computation shows that terms with $2n$ Clifford variables have a rescaled canonical order greater than n. Consequently they vanish as t tends to zero. □

4. The main result

Let $\mathcal{L}^y$ denote the operator

$$(25) \qquad \mathcal{L}^y = -\Delta^y + \boldsymbol{b} = -\sum_{i=1}^{2n} \frac{\partial^2}{\partial x_i{}^2} + \frac{1}{16} \sum_{i,j=1}^{2n} x_i x_j C_{ij}^2$$

with $C_{ij}^2 = \sum_r C_{ir} C_{rj}$, and each $C_{ir} = -1/2 \sum R_{irIJ}(0) h_I h_J$. Here I, J etc, denotes the total subscripts i, j, α, etc. i.e. h_I is the total notation for e_i and dy_α.

Denote by Ω the curvature matrix of two forms over M

$$\Omega_{ir}^{M/B} = -1/2 \sum R_{irIJ}(0) dz_I dz_J$$

where $z = (x, y)$ and dz_I is the total notation for dx_i and dy_α. Clearly $\Omega = (\Omega_{ij}^{M/B})$ is the curvature matrix of two forms for the connection $\nabla^{M/B}$ of the vector bundle $T(M/B)$ over M. Without loss of generality assume the matrix Ω is of the block diagonal form

$$\Omega^{M/B} = \begin{pmatrix} 0 & v_1 & & & \\ -v_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & v_n \\ & & & -v_n & 0 \end{pmatrix}$$

where the v_j's are the 2-form Chern roots. Define

$$L = -\sum_i \frac{\partial^2}{\partial x_i{}^2} - \frac{1}{16} \sum_{i,j=1}^{2n} x_i x_j \big(\Omega_{ij}^{M/B}\big)^2$$

$$= -\sum_i \frac{\partial^2}{\partial x_i{}^2} + \frac{1}{16} \sum_{j=1}^{n} (x_{2j-1}^2 + x_{2j}^2) v_j^2.$$

Then L is a harmonic oscillator type operator; its fundamental solution $e^{-tL}(x,0)$ is given by Mehler's formula in the neighborhood of $\xi = 0$, (for details consult [B-G-V]).

$$e^{-tL}(x,0)$$

$$= \frac{1}{(4\pi t)^n} \left(\prod_{j=1}^{n} \frac{iv_j t/2}{\sinh(iv_j t/2)}\right) \exp\left(\frac{-iv_j t}{2} \frac{(x_{2j-1}^2 + x_{2j}^2)}{4t} \coth\left(\frac{iv_j t}{2}\right)\right).$$

Therefore, on the diagonal at $\xi = 0$

$$e^{-tL}(0,0) = \frac{1}{(4\pi t)^n} \left(\prod_{j=1}^{n} \frac{iv_j t/2}{\sinh(iv_j t/2)}\right).$$

Each v_j is a two form, thus φ_t applied to v_j will produce either $t^{-1/2}$, t^{-1} or no powers of t. Apply φ_t to the expansion of $e^{-tL}(0,0)$ and then take the limit as t tends to zero and use the fact that the complexified Clifford algebra $Cl(T_\xi M_y)$ is identified (as vector space) with the complexified exterior algebra $\wedge(T_\xi M_y)$, we get for each q

$$\lim_{t\to 0} \frac{t^{k-q/2}}{(2k)!} \operatorname{tr}_s\left(T_0 \ldots T_{2k} \varphi_t(e^{-t\mathcal{L}^y})\right)(0,0)$$

$$(26) \quad = \lim_{t\to 0} \frac{t^{k-q/2}}{(2k)!} (2/i)^n \left(\tilde{T}_0 \ldots \tilde{T}_{2k} \varphi_t(e^{-tL}(0,0))\right)_{2n} = \frac{1}{(2k)!} \frac{1}{(2\pi i)^n}$$

$$\times \left[\tilde{T}_0 \tilde{T}_1 \ldots \tilde{T}_{2k} \sum_{p=0}^{[m/2]-[q/2]} \left(\prod_{j=1}^{n} \frac{iv_j/2}{\sinh(iv_j/2)}\right)(n - \widehat{k+q/2}+p)\right]_{2n}$$

where $\left(\frac{iv_j/2}{\sinh(iv_j/2)}\right)(\hat{r})$ is the r-th term in the Taylor series expansion with respect to t about $t = 0$. The $(2/i)^n$ appears since $\operatorname{tr}_s(e_1 \ldots e_{2n}) = (2/i)^n$

and $(\ )_{2n}$ stands for terms which are multiples of $dx_1 \ldots dx_{2n}$. Also $\tilde{T}_i$ is either $d_B(h_i \otimes f_i)$ or $h_i \otimes \hat{d}f_i$ (see Section 3.1).

From Theorems 3.6, 3.8 and (26) it follows that

$$(27) \qquad \lim_{t \to 0}\Big(\mathrm{Ch}^{2k}(\mathcal{M}, \mathcal{A}_t), (h_0 \otimes f_0, h_1 \otimes f_1, \ldots, h_{2k} \otimes f_{2k})\Big)$$

$$= \lim_{t \to 0} \sum_{q=0}^{2k} \frac{t^{k-q/2}}{(2k)!} \,\mathrm{Tr}_s\Big(T_0 T_1 \ldots T_{2k}\varphi_t e^{-t\mathcal{F}}\Big)$$

$$= \sum_{q=0}^{2k} \int_{M/B} \lim_{t \to 0} \frac{t^{k-q/2}}{(2k)!} \,\mathrm{tr}_s\Big(T_0 \ldots T_{2k}\varphi_t(e^{-t\mathcal{L}^y})\Big)(\xi, \xi)$$

$$= \sum_{q=0}^{2k} \int_{M/B} \frac{1}{(2k)!} \frac{1}{(2\pi i)^n} \tilde{T}_0 \tilde{T}_1 \ldots \tilde{T}_{2k}\hat{A}(i\Omega^{M/B})$$

where $\hat{A}(i\Omega^{M/B}) = \prod_{j=1}^{n} \frac{iv_j/2}{\sinh(iv_j/2)}$.

Theorem 4.1. *The $2k$-th component of the bivariant Chern–Connes character of the module $(\mathcal{M}, \mathcal{A})$ over $\mathcal{V}$ is given by:*

$$\Big(\mathrm{Ch}^{2k}(\mathcal{M}, \mathcal{A}), (h_0 \otimes f_0, h_1 \otimes f_1, \ldots, h_{2k} \otimes f_{2k})\Big)$$

$$= \frac{1}{(2k)!} \frac{1}{(2\pi i)^n} \sum_{q=0}^{2k} \int_{M/B} \tilde{T}_0 \tilde{T}_1 \ldots \tilde{T}_{2k}\hat{A}(i\Omega^{M/B}),$$

where $h_j \otimes f_j$'s $\in \mathcal{V}$ and q is the number of times the operator $\tilde{T}_i$ is equal to $d_B(h_i \otimes f_i)$ and $2n$ is the dimension of the fibre M_y.

Remarks 4.2. Note that when the manifold B is a single point y, then the fibre M_y is M and $\mathcal{V} = C^1(M)$. The Hilbert space becomes $L^2(\Gamma(M, \mathcal{E}))$, moreover Bismut's superconnection $\mathcal{A}$ is reduced to the Dirac operator D. The bivariant Chern–Connes character in Theorem 4.1 becomes

$$\Big(\mathrm{Ch}^{2k}(\mathcal{M}, D), (f_0, f_1, \ldots f_{2k})\Big) = \frac{1}{(2k)!} \frac{1}{(2\pi i)^n} \int_M f_0 df_1 \ldots df_{2k}\hat{A}(i\Omega^M)$$

which is the Chern character formula for the Dirac operator on the spinor bundle $\mathcal{E}$.

APPENDIX

A. Clifford algebra and supertrace

Let $e_1, \ldots, e_{2n}$ be the canonical oriented orthonormal base for the Euclidean space $\mathbb{R}^{2n}$. The Clifford algebra $\mathrm{Cl}(\mathbb{R}^{2n})$ of $\mathbb{R}^{2n}$ is the algebra generated by $1, e_1, \ldots, e_{2n}$ subject to these relations

$$(28) \qquad e_i^2 = -1, \quad e_i e_j = -e_j e_i, \quad i \neq j.$$

$\mathrm{Cl}(\mathbb{R}^{2n})$ carries a natural Z_2 grading and as a vector space it is isomorphic to the exterior algebra $\bigwedge(\mathbb{R}^{2n})$.

Let $f_1 \ldots, f_\alpha, \ldots, f_m$ be an oriented orthonormal base in $\mathbb{R}^m$ with dual $dy_1, \ldots, dy_\alpha, \ldots, dy_m$. The graded tensor product of the Z_2 graded algebras $\mathrm{Cl}(\mathbb{R}^{2n})$ and $\bigwedge(\mathbb{R}^m)$ is denoted by E, i.e. $E = \mathrm{Cl}(\mathbb{R}^{2n}) \,\hat{\otimes}\, \bigwedge(\mathbb{R}^m)$.

Elements of E will be written without the graded tensor sign $\hat{\otimes}$. For example $e_i dy_\alpha$ is a well defined element of E, and this relation holds

$$(29) \qquad e_i dy_\alpha + dy_\alpha e_i = 0.$$

Lemma A.1. *Let* $Q_p = e_{i_1} \ldots e_{i_p}$, $P_r = dy_{\alpha_1} \ldots dy_{\alpha_r}$ *and* $C_q = dy_{\beta_1} \ldots dy_{\beta_q}$, *then*

1. *The commutator* $[Q_p P_r, C_q]$ *is zero, unless the sum* $p + r$ *is odd and* q *is odd.*

2. *The number of Clifford variables in the commutator* $[Q_p, Q_q]$ *is at most* $p + q - 2$, *where* $Q_q = e_{j_1} \ldots e_{j_q}$.

PROOF. Direct computation which follows from (28) and (29). □

The complexified Clifford algebra $\mathrm{Cl}(\mathbb{R}^{2n}) \otimes \mathbb{C}$ can be identified with $\mathrm{End}(\mathcal{S})$, where $\mathcal{S}$ is the Z_2 graded 2^n dimensional complex vector space of spinors.

There is a natural supertrace on the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{2n})$ defined by $\mathrm{Tr}_s(a) = \mathrm{Tr}(\tau a)$, where τ is the grading operator $\tau = (\sqrt{-1})^n e_1 \ldots e_{2n}$. And the explicit formula for the supertrace $\mathrm{Tr}_s \in \mathrm{End}(\mathcal{S})$ is given by

Lemma A.2.

$$\mathrm{Tr}_s(e_{i_1} \ldots e_{i_p}) = 0, \qquad p < 2n$$

and

$$\mathrm{Tr}_s(e_1 e_2 \ldots e_{2n}) = \left(\frac{2}{\sqrt{-1}} \right)^n.$$

The space $\mathrm{End}(\mathcal{S})$ of endomorphism of $\mathcal{S}$ is a superalgebra. Let $K = \mathrm{End}(\mathcal{S}) \hat{\otimes} \bigwedge \mathbb{R}^m$, then K has a natural grading and the supertrace Tr_s can be extended to K with values in $\bigwedge(\mathbb{R}^m)$. Thus if $\omega \in \bigwedge(\mathbb{R}^m)$ and $k \in K$, then

$$\mathrm{Tr}_s(\omega k) = \omega \, \mathrm{Tr}_s(k).$$

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