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On some special Finsler immersions in a Minkowski space

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Abstract. We prove that totally umbilical proper Finsler submanifolds of a Minkowski space are totally geodesic. We also prove that minimal proper Finsler surfaces with respect to the Finsler connection induced by the Berwald connection of a Minkowski space must be totally geodesic.

1. Introduction

Studying the geometry of Finsler submanifolds is one of the most difficult aspects of Finsler geometry. This is because, in general, the induced Finsler connection on the submanifold does not necessarily coincide with its intrinsic Finsler connection. For the latest results on the theory of Finsler submanifolds we refer the reader to our recent monograph (A. Bejancu and H. R. Farran [3]).

One natural approach to this theory is to consider some "nice" Finsler immersions (see for example L. M. Abatangelo [1], A. Bejancu [2], S. Dragomir [4] and M. Matsumoto [5]). The present paper is a step in this direction. Here we investigate totally umbilical and minimal Finsler immersions. We deal with proper Finsler immersions which are characterized by the property that their Cartan tensor field is nowhere zero. First, we prove that totally umbilical proper Finsler submanifolds of a Minkowski space must be totally geodesic (Theorems 3.3 and 3.4). We should stress that this result is true for both the Berwald and Cartan connections of a Minkowski space. Minimal Finsler submanifolds are introduced by using

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the h-second fundamental form induced by the Berwald connection of the ambient Minkowski space. Then we prove that minimal proper Finsler surfaces of a Minkowski space are totally geodesic (Theorem 4.1). The question whether this result is true for submanifolds of arbitrary dimension is still open.

2. Induced geometric objects on a Finsler submanifold

Let $\widetilde{\mathbb{F}}^{m+n} = (\widetilde{M}, \widetilde{F})$ be a real $(m+n)$-dimensional Finsler manifold, where $\widetilde{F}$ is the fundamental function of $\widetilde{\mathbb{F}}^{m+n}$. Denote by θ the zero section of the tangent bundle $T\widetilde{M}$ of $\widetilde{M}$ and set $T\widetilde{M}_0 = T\widetilde{M} \backslash \theta(\widetilde{M})$. We take $(x^i, y^i), i \epsilon \{1, \ldots, m+n\}$ as local coordinates on $T\widetilde{M}_0$, where (x^i) are the local coordinates on $\widetilde{M}$. Then there exists a Riemannian metric $\widetilde{g}$ on the vertical vector bundle $VT\widetilde{M}_0$ over $T\widetilde{M}_0$ whose local components are given by

$$\widetilde{g}_{ij}(x, y) = \widetilde{g}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \frac{1}{2}\frac{\partial^2 \widetilde{F}^2}{\partial y^i \partial y^j}.$$

When $\widetilde{M} = \mathbb{R}^{m+n}$, and $\widetilde{F}$ is a function that depends on $(y^1, \ldots, y^{m+n})$ alone, we say that $\widetilde{\mathbb{F}}^{m+n} = (\mathbb{R}^{m+n}, \widetilde{F})$ is a *Minkowski space*.

Throughout the paper we use the following ranges for indices, $i, j, k, \ldots \in \{1, \ldots, m+n\}$; $\alpha, \beta, \gamma, \ldots \in \{1, \ldots, m\}$; $a, b, c, \ldots \in \{m+1, \ldots \ldots, m+n\}$. We also use the Einstein convention, that is, repeated indices with one upper index and one lower index denote summation over their range. We denote the algebra of smooth functions on $\widetilde{M}$ by $\mathcal{F}(\widetilde{M})$, and the $\mathcal{F}(\widetilde{M})$-module of smooth sections of a vector bundle E over $\widetilde{M}$ by $\Gamma(E)$. Similar notations will be used for any other manifold. For terminology in general, notations and basic results see A. Bejancu and H. R. Farran [3].

Now, we consider a real m-dimensional submanifold M of $\widetilde{M}$ given locally by the equations:

$$x^i = x^i(u^1, \ldots, u^m); \quad \text{rank}[B^i_\alpha] = m; \quad B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}. \tag{2.1}$$

Then the differential of the immersion of M in $\widetilde{M}$ carries a point (u^α, v^α) of TM_0 into a point $(x^i(u), y^i(u, v))$ of $T\widetilde{M}_0$, where we set

$$y^i(u, v) = B^i_\alpha v^\alpha. \tag{2.2}$$

In order to simplify the equations involved in the study we use the notations:

$$B^{ij\dots}_{\alpha\beta\dots} = B^i_\alpha B^j_{\beta\dots}; \quad B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}; \quad B^i_{\alpha 0} = B^i_{\alpha\beta} v^\beta.$$

As a consequence of (2.1) and (2.2) we obtain

$$\text{(2.3)} \qquad (a) \quad \frac{\partial}{\partial u^\alpha} = B^i_\alpha \frac{\partial}{\partial x^i} + B^i_{\alpha 0} \frac{\partial}{\partial y^i}; \qquad (b) \quad \frac{\partial}{\partial v^\alpha} = B^i_\alpha \frac{\partial}{\partial y^i},$$

where $\{\partial/\partial u^\alpha, \partial/\partial v^\alpha\}$ and $\{\partial/\partial x^i, \partial/\partial y^i\}$ are the natural frame fields on TM_0 and $T\widetilde{M}_0$ respectively. From (2.3b) we deduce that the vertical vector bundle VTM_0 over TM_0 is a vector subbundle of $VT\widetilde{M}_{0_{|TM_0}}$. Hence the Riemannian metric $\widetilde{g}$ on $VT\widetilde{M}_0$ induces a Riemannian metric g on VTM_0 whose local components are given by

$$\text{(2.4)} \qquad g_{\alpha\beta}(u,v) = g\left(\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial v^\alpha}\right) = \widetilde{g}_{ij}(x(u), y(u,v)) B^{ij}_{\alpha\beta}.$$

On the other hand, the fundamental function $\widetilde{F}$ of $\widetilde{\mathbb{F}}^{m+n}$ induces on TM_0 the function F locally given by

$$F(u,v) = \widetilde{F}(x(u), y(u,v)).$$

Then it is easy to check that $\mathbb{F}^m = (M, F)$ is a Finsler manifold whose Riemannian metric on VTM_0 is exactly $g = (g_{\alpha\beta})$ given by (2.4). Thus we are entitled to say that $\mathbb{F}^m$ is a *Finsler submanifold* of $\widetilde{\mathbb{F}}^{m+n}$.

The orthogonal complementary vector bundle to VTM_0 in $VT\widetilde{M}_{0_{|TM_0}}$ is denoted by $VTM_0^\perp$ and called the *Finsler normal bundle* of the Finsler submanifold $\mathbb{F}^m$. Thus we may consider a local field of orthonormal frames $\{N_a = N^i_a \partial/\partial y^i\}$ in $VTM_0^\perp$ with respect to $\widetilde{g}$, i.e., we have:

$$\text{(2.5)} \qquad (a) \quad \widetilde{g}_{ij} B^i_\alpha N^j_a = 0; \qquad (b) \quad \widetilde{g}_{ij} N^i_a N^j_b = \delta_{ab}.$$

Then we denote by $[\widetilde{B}^\alpha_i \widetilde{N}^a_i]$ the inverse of the matrix $[B^i_\alpha N^i_a]$.

A complementary vector bundle $HT\widetilde{M}_0$ to $VT\widetilde{M}_0$ in $TT\widetilde{M}_0$ is called a non-linear connection on $\widetilde{\mathbb{F}}^{m+n}$. It is noteworthy that on $\widetilde{\mathbb{F}}^{m+n}$ there exists a non-linear connection $GT\widetilde{M}_0 = (\widetilde{G}^i_j)$ constructed from the fundamental function $\widetilde{F}$ as follows

$$\text{(2.6)} \qquad \widetilde{G}^i_j = \frac{\partial \widetilde{G}^i}{\partial y^j}; \quad \widetilde{G}^i = \frac{1}{4}\widetilde{g}^{ih}\left(\frac{\partial^2 \widetilde{F}^2}{\partial y^h \partial x^j} y^j - \frac{\partial \widetilde{F}^2}{\partial x^h}\right).$$

We call $GT\widetilde{M}_0$ the *canonical non-linear connection* of $\widetilde{\mathbb{F}}^{m+n}$. The *induced non-linear connection* on $\mathbb{F}^m$ by $GT\widetilde{M}_0$ is $HTM_0 = (F_\alpha^\beta)$, where we set

$$F_\alpha^\beta = \widetilde{B}_i^\beta \left(B_{\alpha 0}^i + \widetilde{G}_j^i B_\alpha^j \right). \tag{2.7}$$

Then according to the decomposition

$$TTM_0 = HTM_0 \oplus VTM_0,$$

we obtain a local frame field $\{\delta/\delta u^\alpha, \partial/\partial v^\alpha\}$ where $\delta/\delta u^\alpha \in \Gamma(HTM_0)$, and it is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - F_\alpha^\beta \frac{\partial}{\partial v^\beta}. \tag{2.8}$$

It is important to note that HTM_0 is a vector subbundle of $HT\widetilde{M}_{0|TM_0} \oplus VTM_0^\perp$. More precisely, we have

$$\frac{\delta}{\delta u^\alpha} = B_\alpha^i \frac{\delta}{\delta x^i} + H_\alpha^a N_a, \tag{2.9}$$

where we set

$$\text{(2.10)} \quad (a) \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \widetilde{G}_i^j \frac{\partial}{\partial y^j}; \qquad (b) \quad H_\alpha^a = \widetilde{N}_i^a \left(B_{\alpha\circ}^i + B_\alpha^j \widetilde{G}_j^i \right).$$

Also, we stress that, in general, the induced non-linear connection does not coincide with the canonical non-linear connection $GMT_0 = (G_\alpha^\beta)$ of $\mathbb{F}^m$. Actually, we have

$$G_\alpha^\beta = F_\alpha^\beta + g_a{}^\beta{}_\alpha H_\gamma^a v^\gamma,$$

where we set

$$g_a{}^\beta{}_\alpha = g^{\beta\gamma}\, \widetilde{g}_{ijk} N_a^i B_{\gamma\alpha}^{jk}; \quad \widetilde{g}_{ijk} = \frac{1}{2} \frac{\partial \widetilde{g}_{ij}}{\partial y^k}.$$

This is going to be the main difficulty in studying the geometry of a Finsler immersion. The tensor field $\widetilde{g}_{ijk}$ is known as the *Cartan tensor field* of $\widetilde{\mathbb{F}}^{m+n}$.

Now, we consider a Finsler connection $\widetilde{FC} = (\widetilde{\nabla}, GT\widetilde{M}_0) = (\widetilde{F}_i{}^k{}_j, \widetilde{C}_i{}^k{}_j, \widetilde{G}_i^k)$, where $\widetilde{\nabla}$ is a linear connection on $VT\widetilde{M}_0$ whose local coefficients are defined by

$$\widetilde{\nabla}_{\delta/\delta x^j} \frac{\partial}{\partial y^i} = \widetilde{F}_i{}^k{}_j \frac{\partial}{\partial y^k} \quad \text{and} \quad \widetilde{\nabla}_{\partial/\partial y^j} \frac{\partial}{\partial y^i} = \widetilde{C}_i{}^k{}_j \frac{\partial}{\partial y^k}.$$

In the literature, we frequently meet one of the following well known connections (cf. A. Bejancu and H. R. Farran [3], pp. 39, 41).

– the Cartan connection $(\widetilde{F}_i{}^k j, \widetilde{g}_i{}^k{}_j, \widetilde{G}_i^k)$
– the Berwald connection $(\widetilde{G}_i{}^k{}_j, 0, \widetilde{G}_i^k)$
– the Rund connection $(\widetilde{F}_i{}^k{}_j, 0, \widetilde{G}_i^k)$,

where we set

$$(2.11)\qquad (a)\quad \widetilde{F}_i{}^k{}_j = \frac{1}{2}\widetilde{g}^{kh}\left(\frac{\delta\widetilde{g}_{hi}}{\delta x^j} + \frac{\delta\widetilde{g}_{hj}}{\delta x^i} - \frac{\delta\widetilde{g}_{ij}}{\delta x^h}\right);$$
$$(b)\quad \widetilde{g}_i{}^k{}_j = \widetilde{g}^{kh}\widetilde{g}_{hij};\qquad (c)\quad \widetilde{G}_i{}^k j = \frac{\partial G_i^k}{\partial y^j}.$$

The Finsler connection $\widetilde{FC}$ induces on $\mathbb{F}^m$ a Finsler connection $IFC = (\nabla, HTM_0) = (F_\alpha{}^\gamma{}_\beta, C_\alpha{}^\gamma{}_\beta, F_\alpha^\gamma)$, where ∇ is a linear connection on VTM_0 with local coefficients $(F_\alpha{}^\gamma{}_\beta, C_\alpha{}^\gamma{}_\beta)$ given by

$$(2.12)\qquad (a)\quad F_\alpha{}^\gamma{}_\beta = \widetilde{B}_i^\gamma\left(B^i_{\alpha\beta} + \widetilde{F}_j{}^i{}_k B^{jk}_{\alpha\beta} + \widetilde{C}_j{}^i{}_k B^j_\alpha H^a_\beta N^k_a\right);$$
$$(b)\quad C_\alpha{}^\gamma{}_\beta = \widetilde{B}_i^\gamma \widetilde{C}_j{}^i{}_k B^{jk}_{\alpha\beta}.$$

We call IFC the *induced Finsler connection* by $\widetilde{FC}$. Also, $\widetilde{FC}$ induces two vectorial Finsler connections:

$$\overline{FC} = (\overline{\nabla}, HTM_0) = (\overline{F}_i{}^k{}_\alpha, \overline{C}_i{}^k{}_\alpha, F_\alpha^\beta) \quad \text{on} \quad VT\widetilde{M}_{0_{|TM_0}}$$
$$FC^\perp = (\nabla^\perp, HTM_0) = (F_a{}^b{}_\alpha, C_a{}^b{}_\alpha, F_\alpha^\beta) \quad \text{on} \quad VTM_0^\perp,$$

where $\overline{\nabla}$ and $\nabla^\perp$ are linear connections on the vector bundles $VT\widetilde{M}_{0_{|TM_0}}$ and $VTM_0^\perp$ whose local coefficients are given by

$$(2.13)\qquad \overline{F}_i{}^k{}_\alpha = \widetilde{F}_i{}^k{}_j B^j_\alpha + \widetilde{C}_i{}^k{}_j N^j_a H^a_\alpha;\qquad (b)\quad \overline{C}_i{}^k{}_\alpha = \widetilde{C}_i{}^k{}_j B^j_\alpha,$$

and

$$(a)\quad F_a{}^b{}_\alpha = \widetilde{N}^b_k\left(\frac{\delta N^k_a}{\delta u^\alpha} + N^i_a\overline{F}_i{}^k{}_\alpha\right);$$
$$(b)\quad C_a{}^b{}_\alpha = \widetilde{N}^b_k\left(\frac{\partial N^k_a}{\partial v^\alpha} + N^i_a\overline{C}_i{}^k{}_\alpha\right),$$

respectively.

Usually, Finsler geometry deals with mixed Finsler tensor fields T with local components $T^{\alpha ia}_{\beta jb}$ satisfying

$$T^{\alpha ia}_{\beta jb}\frac{\partial \bar{u}^{\mu}}{\partial u^{\alpha}}\,\frac{\partial \bar{x}^{h}}{\partial x^{i}}A^{d}_{a}=\overline{T}^{\mu hd}_{\gamma kc}\frac{\partial \bar{u}^{\gamma}}{\partial u^{\beta}}\,\frac{\partial \bar{x}^{k}}{\partial x^{j}}A^{c}_{b},$$

with respect to the transformations of coordinates $\overline{u}^{\mu}=\overline{u}^{\mu}(u^{\alpha})$ and $\overline{x}^{h}=\overline{x}^{h}(x^{i})$ on M and $\widetilde{M}$ respectively, and with respect to the transformations $\overline{N}_{b}=A^{c}_{b}N_{c}$ of orthonormal fields of frames on $VTM_{0}^{\perp}$. Using $IFC, \overline{FC}$ and $FC^{\perp}$ we may define the *relative h-covariant derivative* and the *relative v-covariant derivative* of T as follows:

$$\text{(2.15)}\qquad \begin{aligned} T^{\alpha ia}_{\beta jb|\gamma}=&\frac{\delta T^{\alpha ia}_{\beta jb}}{\delta u^{\gamma}}+T^{\varepsilon ia}_{\beta jb}F_{\varepsilon}{}^{\alpha}{}_{\gamma}+T^{\alpha ka}_{\beta jb}\overline{F}_{k}{}^{i}\gamma+T^{\alpha ic}_{\beta jb}F_{c}{}^{a}{}_{\gamma}\\ &-T^{\alpha ia}_{\varepsilon jb}F_{\beta}{}^{\varepsilon}{}_{\gamma}-T^{\alpha ia}_{\beta kb}\overline{F}_{j}{}^{k}{}_{\gamma}-T^{\alpha ia}_{\beta jc}F_{b}{}^{c}{}_{\gamma}, \end{aligned}$$

and

$$\begin{aligned} T^{\alpha ia}_{\beta jb\|\gamma}=&\frac{\partial T^{\alpha ia}_{\beta jb}}{\partial v^{\gamma}}+T^{\varepsilon ia}_{\beta jb}C_{\varepsilon}{}^{\alpha}{}_{\gamma}+T^{\alpha ka}_{\beta jb}\overline{C}_{k}{}^{i}{}_{\gamma}+T^{\alpha ic}_{\beta jb}C_{c}{}^{a}{}_{\gamma}\\ &-T^{\alpha ia}_{\varepsilon jb}C_{\beta}{}^{\varepsilon}{}_{\gamma}-T^{\alpha ia}_{\beta kb}\overline{C}_{j}{}^{k}{}_{\gamma}-T^{\alpha ia}_{\beta jc}C_{b}{}^{c}{}_{\gamma}, \end{aligned}$$

respectively.

Finally, $\widetilde{FC}$ induces two *second fundamental forms*:

$$\text{(2.17)}\qquad \begin{aligned} &(a)\quad H^{a}{}_{\alpha\beta}=\widetilde{N}^{a}_{i}\left(B^{i}_{\alpha\beta}+\widetilde{F}_{j}{}^{i}{}_{k}B^{jk}_{\alpha\beta}+\widetilde{C}_{j}{}^{i}{}_{k}B^{j}_{\alpha}H^{b}_{\beta}N^{k}_{b}\right)\\ &(b)\quad V^{a}{}_{\alpha\beta}=\widetilde{N}^{a}_{i}\widetilde{C}_{j}{}^{i}{}_{k}B^{jk}_{\alpha\beta}, \end{aligned}$$

and two *shape operators*:

$$\text{(2.18)}\qquad \begin{aligned} &(a)\quad H'^{\beta}_{a}{}_{\alpha}=-\widetilde{B}^{\beta}_{k}\left(\frac{\delta N^{k}_{a}}{\delta u^{\alpha}}+N^{i}_{a}\overline{F}_{i}{}^{k}{}_{\alpha}\right)\\ &(b)\quad V'^{\beta}_{a}{}_{\alpha}=-\widetilde{B}^{\beta}_{k}\left(\frac{\partial N^{k}_{a}}{\partial v^{\alpha}}+N^{i}_{a}\overline{C}_{i}{}^{k}{}_{\alpha}\right). \end{aligned}$$

These are related by

$$\text{(2.19)}\qquad \begin{aligned} &(a)\quad H'_{a\alpha\beta}=H_{a\alpha\beta}+\widetilde{g}_{ij|\beta}B^{i}_{\alpha}N^{j}_{a};\\ &(b)\quad V'_{a\alpha\beta}=V_{a\alpha\beta}+\widetilde{g}_{ij\|\beta}B^{i}_{\alpha}N^{j}_{a}, \end{aligned}$$

where raising and lowering indices is done by using $\delta_{ab}, \delta^{ab}, g_{\alpha\beta}$ and $g^{\alpha\beta}$.

By using the above geometric objects we may write down all Gauss–Codazzi–Ricci equations of the Finsler immersion of $\mathbb{F}^m$ in $\widetilde{\mathbb{F}}^{m+n}$. We only recall here the equations we need in the next sections. First, from the B-Codazzi equations we recall (cf. A. Bejancu and H. R. Farran [3], p. 87).

$$B^i_\alpha \overline{P}_i{}^h{}_{\beta\gamma} \widetilde{N}^a_h = H^a{}_{\alpha\beta\|\gamma} - V^a{}_{\alpha\gamma|\beta} + H^a{}_{\alpha\mu} C_\beta{}^\mu{}_\gamma + V^a{}_{\alpha\mu} P^\mu{}_{\beta\gamma}, \tag{2.20}$$

where we set

$$\overline{P}_i{}^h{}_{\beta\gamma} = \widetilde{P}_i{}^h{}_{jk} B^{jk}_{\beta\gamma} + \widetilde{S}_i{}^h{}_{jk} N^j_a H^a_\beta B^k_\gamma, \tag{2.21}$$

$\widetilde{P}_i{}^h{}_{jk}$ and $\widetilde{S}_i{}^h{}_{jk}$ being the hv-curvature and v-curvature Finsler tensor fields of $\widetilde{FC}$ respectively. Also, we recall (cf. Bejancu–Farran [3], p. 92)

$$H^a_{\alpha\|\beta} + H^a_\gamma C_\alpha{}^\gamma{}_\beta - H^a{}_{\beta\alpha} = \bar{P}^i{}_{\alpha\beta} \widetilde{N}^a_i, \tag{2.22}$$

where we set

$$\text{(2.23)} \quad (a) \quad \bar{P}^i{}_{\alpha\beta} = \widetilde{P}^i{}_{jk} B^{jk}_{\alpha\beta} + \widetilde{S}^i{}_{jk} N^j_a B^k_\beta H^a_\alpha;$$

$$(b) \quad \widetilde{P}^i{}_{jk} = \frac{\partial \widetilde{G}^i_j}{\partial y^k} - \widetilde{F}_j{}^k{}_i; \qquad (c) \quad \widetilde{S}^i{}_{jk} = \widetilde{C}_j{}^i{}_k - \widetilde{C}_k{}^i{}_j.$$

3. Totally umbilical Finsler immersions in a Minkowski space

In the remaining part of the paper we take a Minkowski space $\widetilde{\mathbb{F}}^{m+n} = (\mathbb{R}^{m+n}, \widetilde{F})$ as ambient space. Then, since $\widetilde{F}$ depends on $(y^1, \dots, y^{m+n})$ only, we deduce from (2.6) and (2.11) that

$$\text{(3.1)} \quad (a) \quad \widetilde{G}^k_i = 0; \qquad (b) \quad \widetilde{F}_i{}^k{}_j = 0; \qquad (c) \quad \widetilde{G}_i{}^k{}_j = 0.$$

Suppose that $\widetilde{\mathbb{F}}^{m+n}$ is endowed with the Berwald connection, which actually, in this case, coincides with the Rund connection. Then we have:

$$\text{(3.2)} \quad (a) \quad \widetilde{P}_i{}^h{}_{jk} = \frac{\partial \widetilde{G}{}_{i}{}^h_j}{\partial y^k} - \frac{\delta \widetilde{C}_i{}^h{}_k}{\delta x^j} + \widetilde{G}_i{}^r{}_j \widetilde{C}_r{}^h{}_k - \widetilde{C}_i{}^r{}_k \widetilde{G}_r{}^h{}_j + \widetilde{G}_j{}^r{}_k \widetilde{C}_i{}^h{}_r = 0$$

$$(b) \quad \widetilde{S}_i{}^h{}_{jk} = \frac{\partial \widetilde{C}_i{}^h{}_j}{\partial y^k} - \frac{\partial \widetilde{C}_i{}^h{}_k}{\partial y^j} + \widetilde{C}_i{}^r{}_j \widetilde{C}_r{}^h{}_k - \widetilde{C}_i{}^r{}_k \widetilde{C}_r{}^h{}_j = 0.$$

Also, taking into account (3.1), from (2.23) we obtain

$$\bar{P}^i{}_{\alpha\beta} = 0. \tag{3.3}$$

Next, we consider a Finsler submanifold $\mathbb{F}^m = (M, F)$ of $\widetilde{\mathbb{F}}^{m+n}$. Then from (2.12b) and (2.17b) we infer that

$$(a) \quad C_\alpha{}^\gamma{}_\beta = 0 \quad \text{and} \quad (b) \quad V^a{}_{\alpha\beta} = 0, \tag{3.4}$$

since $\widetilde{C}_i{}^k{}_j = 0$. Therefore, by using (3.2)–(3.4) and (2.21) we see that (2.20) and (2.22) become

$$H^a{}_{\alpha\beta\|\gamma} = 0, \tag{3.5}$$

and

$$H^a_{\alpha\|\beta} = H^a{}_{\beta\alpha}, \tag{3.6}$$

respectively.

Next, as in case of Riemannian immersions we say that $\mathbb{F}^m$ is a *B-totally umbilical Finsler submanifold* if on any coordinate neighbourhood $\mathcal{U}$ of TM_0 there exist smooth functions ρ^a such that

$$H^a{}_{\alpha\beta} = \rho^a g_{\alpha\beta}. \tag{3.7}$$

If any geodesic of $\mathbb{F}^m$ is a geodesic of $\widetilde{\mathbb{F}}^{m+n}$ then we say that $\mathbb{F}^m$ is a *totally geodesic Finsler submanifold*. In the sequel we need the following results.

Theorem 3.1 (A. Bejancu and H. R. Farran [3], p. 134)**.** *$\mathbb{F}^m$ is a totally geodesic Finsler submanifold of $\widetilde{\mathbb{F}}^{m+n}$ if and only if*

$$H^a{}_{\alpha\beta} = 0, \ \text{for any } a \in \{m+1, \dots, m+n\} \text{ and } \alpha, \beta \in \{1, \dots, m\}. \tag{3.8}$$

Theorem 3.2 (A. Bejancu and H. R. Farran [3], p. 136)**.** *Let $\mathbb{F}^m = (M, F)$ be a totally geodesic Finsler submanifold of a Minkowski space $\widetilde{\mathbb{F}}^{m+n} = (\mathbb{R}^{m+n}, \widetilde{F})$. Then M is $\mathbb{R}^m$ or an open submanifold of $\mathbb{R}^m$.*

A Finsler manifold with nowhere vanishing Cartan tensor field is called a *proper Finsler manifold*. Thus a proper Finsler manifold is not a Riemannian manifold. Now, we prove the following

Theorem 3.3. *Any B-totally umbilical proper Finsler submanifold of a Minkowski space is totally geodesic.*

Proof. Take the relative v-covariant derivative of (3.7) and by using (3.5) obtain

$$\rho^a{}_{\|\gamma} g_{\alpha\beta} + 2\rho^a g_{\alpha\beta\gamma} = 0, \tag{3.9}$$

where $g_{\alpha\beta\gamma}$ is the Cartan tensor field of $\mathbb{F}^m$. Contracting (3.9) by v^α and taking into account that $g_{\beta\gamma}$ are positive homogeneous functions of degree zero with respect to $(y^1, \dots, y^{m+n})$ we deduce that

$$\rho^a{}_{\|\gamma} g_{\alpha\beta} v^\alpha = 0,$$

at any point $(u, v) \in TM_0$. As the zero section is not in $\Gamma(TM_0)$ we conclude that $\rho^a{}_{\|\gamma} = 0$. Thus (3.9) becomes

$$\rho^a g_{\alpha\beta\gamma} = 0,$$

for any $a \in \{m+1, \dots, m+n\}$ and $\alpha, \beta \in \{1, \dots, m\}$. But $g_{\alpha\beta\gamma}$ is nowhere zero on a coordinate neighborhood $\mathcal{U}$ of TM_0. Hence $\rho^a = 0$ for any $a \in \{m+1, \dots, m+n\}$ and thus (3.7) becomes (3.8). Finally, by Theorem 3.1 we have the assertion of our theorem. □

It is interesting to note that Theorem 3.3 is also true when we take the Cartan connection on $\widetilde{\mathbb{F}}^{m+n}$. In this case we denote by $H^a_{*\alpha\beta}$ the local components of the h-second fundamental form induced by the Cartan connection. Then we say that $\mathbb{F}^m$ is *C-totally umbilical* if on any coordinate neighborhood $\mathcal{U}$ of TM_0 there exist smooth functions η^a such that

$$H^a_{*\alpha\beta} = \eta^a g_{\alpha\beta}. \tag{3.7′}$$

Then we prove the following result.

Theorem 3.4. *Any C-totally umbilical Finsler submanifold of a Minkowski space is B-totally umbilical.*

Proof. By using (2.17a) for both $H^a{}_{\alpha\beta}$ and $H^a_{*\alpha\beta}$ and taking into account (3.1b) and (3.1c) we deduce that

$$H^a_{*\alpha\beta} = H^a{}_{\alpha\beta} + \widetilde{N}^a_i \widetilde{g}_j{}^i{}_k B^j_\alpha H^b_\beta N^k_b. \tag{3.10}$$

From (3.7′) it follows that $H^a_{*\alpha\beta}$ are symmetric mixed Finsler tensor fields with respect to $(\alpha\beta)$. Also from (2.17a) it follows that $H^a{}_{\alpha\beta}$ induced by the Berwald connection of $\widetilde{\mathbb{F}}^{m+n}$ are symmetric with respect to $(\alpha\beta)$. Thus from (3.10) we obtain

$$\widetilde{N}^a_i \widetilde{g}_j{}^i{}_k B^j_\alpha H^b_\beta N^k_b = \widetilde{N}^a_i \widetilde{g}_j{}^i{}_k B^j_\beta H^b_\alpha N^k_b. \tag{3.11}$$

Then contracting (3.11) by v^β and taking into account that $\widetilde{g}_j{}^i{}_k y^j = 0$, we infer that

$$\widetilde{N}^a_i \widetilde{g}_j{}^i{}_k B^j_\alpha H^b_\beta v^\beta N^k_b = 0.$$

Hence, by contracting (3.10) by v^β we obtain

$$H^a_{*\alpha\beta} v^\beta = H^a{}_{\alpha\beta} v^\beta. \tag{3.12}$$

As both $H^a_{*\alpha\beta}$ and $H^a{}_{\alpha\beta}$ are symmetric mixed Finsler tensor fields we also have

$$H^a_{*\alpha\beta} v^\alpha = H^a{}_{\alpha\beta} v^\alpha. \tag{3.13}$$

Now, we take the relative v-covariant derivative of (3.13) induced by the Berwald connection of $\widetilde{\mathbb{F}}^{m+n}$, and by using (3.5) we deduce that

$$H^a_{*\gamma\beta} + v^\alpha H^a_{*\alpha\beta\|\gamma} = H^a{}_{\gamma\beta}. \tag{3.14}$$

By using (3.7′) and taking into account that $v^\alpha g_{\alpha\beta\gamma} = 0$, (3.14) becomes

$$H^a_{*\gamma\beta} + v^\alpha g_{\alpha\beta} \eta^a{}_{\|\gamma} = H^a{}_{\gamma\beta}. \tag{3.15}$$

Contracting (3.15) by v^β and taking into account (3.12) we infer that

$$v^\alpha g_{\alpha\beta} v^\beta \eta^a{}_{\|\gamma} = 0,$$

which implies that $\eta^a{}_{\|\gamma} = 0$, since $v^\alpha g_{a\beta} v^\beta = F^2 \neq 0$ on $\mathcal{U}$. Thus from (3.15) we obtain that $H^a_{*\alpha\beta} = H^a{}_{\alpha\beta}$. Hence (3.7′) implies (3.7), that is, $\mathbb{F}^m$ is B-totally umbilical. □

Next, we recall that the condition (3.7) for a totally umbilical Riemannian submanifold is equivalent to a condition on the shape operators of the submanifold. We find the same equivalence for Finsler submanfolds of a Minkowski space endowed the Cartan connection. Indeed, in this case, by using (2.15), (2.9), (2.13) and (2.16) we deduce that

$$\begin{aligned} \widetilde{g}_{ij|_*\alpha} &= \frac{\delta\widetilde{g}_{ij}}{\delta u^\alpha} - \widetilde{g}_{hj}\bar{F}_i{}^h{}_\alpha - \widetilde{g}_{ih}\bar{F}_j{}^h{}_\alpha \\ &= \frac{\delta\widetilde{g}_{ij}}{\delta x^k}B^k_\alpha + H^a_\alpha N^k_a \widetilde{g}_{ij\|*k} = 0, \end{aligned} \tag{3.16}$$

since we have

$$\frac{\delta\widetilde{g}_{ij}}{\delta x^k} = 0 \quad \text{and} \quad \widetilde{g}_{ij\|_*k} = 0.$$

Here, and in the sequel, all the covariant derivatives induced by the Cartan connection will have a " $*$ " after the corresponding bars. By using (3.16) in (2.19a) we obtain

$$H'_{*a}{}^\alpha{}_\beta = g^{\alpha\gamma}\delta_{ab}H^b_{*\gamma\beta}.$$

Thus the condition (3.7′) is equivalent to

$$H'_{*a}{}^\alpha{}_\beta = \eta_a\delta^\alpha_\beta, \tag{3.7''}$$

as in case of Riemannian submanifolds.

Finally, we analyse a condition similar to (3.7″) for the shape operators induced by the Berwald connection of $\widetilde{\mathbb{F}}^{m+n}$.

Suppose that the shape operators $H'^\alpha_a{}_\beta$ induced by the Berwald connection of $\widetilde{\mathbb{F}}^{m+n}$ are proportional to the indentity operator, that is, on any coordinate neighbourhood $\mathcal{U}$ we have

$$H'^\alpha_a{}_\beta = \rho_a\delta^\alpha_\beta. \tag{3.7'''}$$

Then we examine (2.19a). By calculations similar to those performed for (3.16) we obtain

$$\widetilde{g}_{ij|\beta} = 2H^b_\beta N^k_b\widetilde{g}_{ijk},$$

where, this time, the relative h-covariant derivative from the left side is induced by the Berwald connection of $\widetilde{\mathbb{F}}^{m+n}$. Hence (2.19a) becomes

$$H'_{a\alpha\beta} = H_{a\alpha\beta} + 2H^b_\beta N^k_b\widetilde{g}_{ijk}B^i_\alpha N^j_a. \tag{3.17}$$

Starting with (3.17) instead of (3.10) and following the same steps as in the proof of Theorem 3.4, we deduce that any Finsler submanifold whose shape operators induced by the Berwald connection satisfy (3.7‴) is B-totally umbilical.

The above results enable us to state that any proper Finsler submanifold $\mathbb{F}^m = (M, F)$ of a Minkowski space $\widetilde{\mathbb{F}}^{m+n} = (\mathbb{R}^{m+n}, \widetilde{F})$ satisfying one of the conditions (3.7) (3.7′), (3.7″) or (3.7‴) is totally geodesic. Moreover, by Theorem 3.2 such a submanifold must by $\mathbb{R}^m$ or an open submanifold of $\mathbb{R}^m$. This generalizes the results obtained in [3] for hypersufaces.

4. Minimal Finsler immersions in a Minkowski space

Let $\widetilde{\mathbb{F}}^{m+n} = (\mathbb{R}^{m+n}, \widetilde{F})$ be a Minkowski space endowed with the Berwald connection. Suppose $\mathbb{F}^m = (M, F)$ is a Finsler submanifold whose h-second fundamental form satisfies

$$H^a{}_{\alpha\beta} g^{\alpha\beta} = 0, \quad \forall a \in \{m+1, \dots, m+n\}. \tag{4.1}$$

Then, according to the terminology from Riemannian submanifolds we may say that $\mathbb{F}^m$ is a *B-minimal Finsler submanifold*. Clearly, by Theorem 3.1 it follows that any totally geodesic Finsler submanifold is B-minimal. In the present section we prove that the converse of the above assertion is true, provided $\mathbb{F}^m$ is a proper Finsler submanifold of lowest dimension.

When $m = 2$, we have a *Finsler surface* $\mathbb{F}^2 = (M, F)$ immersed in a Minkowski space $\widetilde{\mathbb{F}}^{2+n}$. The *Berwald frame* $\{\ell, m\}$ of $\mathbb{F}^2$ is an orthonormal basis of $\Gamma(VTM_0)$ with respect to the Riemannian metric $g = (g_{\alpha\beta})$ on the vertical vector bundle VTM_0 (cf. Bejancu–Farran [3], p. 209). Locally, we set

$$\ell = \ell^\alpha \frac{\partial}{\partial v^\alpha} \quad \text{and} \quad m = m^\alpha \frac{\partial}{\partial v^\alpha}, \quad \alpha \in \{1, 2\}.$$

The intrinsic geometry of $\mathbb{F}^2$ is controled by the *main scalar* $\boldsymbol{I}$ which is incorporated in the expression of the Cartan tensor field of $\mathbb{F}^2$ as follows

$$g_{\alpha\beta\gamma} = \frac{\boldsymbol{I}}{F} m_\alpha m_\beta m_\gamma. \tag{4.2}$$

The v-covariant derivatives of the two vector fields from the Berwald frame with respect to the Cartan connection are given by (cf. Bejancu–Farran [3], p. 211)

$$(a) \quad \ell^\alpha{}_{\|*\beta} = \frac{1}{F} m_\beta m^\alpha; \qquad (b) \quad m^\alpha{}_{\|*\beta} = -\frac{1}{F} m_\beta \ell^\alpha. \tag{4.3}$$

Then by direct calculations using (2.16), (3.4a), (4.3) and (4.2) we obtain

$$(4.4)\qquad (a)\quad \ell^{\alpha}{}_{\|\beta} = \frac{1}{F} m_{\beta} m^{\alpha}; \qquad (b)\quad m^{\alpha}{}_{\|\beta} = -\frac{1}{F} m_{\beta}(\ell^{\alpha} - \boldsymbol{I} m^{\alpha}),$$

where in the left part we have the relative v-covariant derivatives induced on $\mathbb{F}^2$ by the Berwald connection of $\widetilde{\mathbb{F}}^{2+n}$. Now we can prove the following

Theorem 4.1. *Let $\mathbb{F}^2$ be a B-minimal proper Finsler surface of a Minkowski space $\widetilde{\mathbb{F}}^{2+n}$. Then $\mathbb{F}^2$ is totally geodesic.*

Proof. Take the relative v-covariant derivative of (4.1) induced by the Berwald connection of $\widetilde{\mathbb{F}}^{2+n}$, and by using (3.5) and (2.16) we obtain

$$(4.5)\qquad H^{a}{}_{\alpha\beta}\, g^{\alpha\mu} g^{\beta\nu} g_{\mu\nu\gamma} = 0, \quad \forall a \in \{3, \dots, 2+n\}.$$

By using (4.2) in (4.5) we deduce that

$$(4.6)\qquad H^{a}{}_{\alpha\beta}\, m^{\alpha} m^{\beta} = 0,$$

since $\boldsymbol{I} \neq 0$ and $m \neq 0$ on the coordinate neighbourhood $\mathcal{U}$ of TM_0. Next, take the relative v-covariant derivative of (4.6) and by using (3.5), (4.4b) and (4.6) we infer that

$$(4.7)\qquad H^{a}{}_{\alpha\beta}\, \ell^{\alpha} m^{\beta} = 0.$$

Finally, we take the relative v-covariant derivative of (4.7) and by using (3.5), (4.4), (4.6) and (4.7) we obtain

$$(4.8)\qquad H^{a}{}_{\alpha\beta}\, \ell^{\alpha} \ell^{\beta} = 0.$$

Since $\{\ell, m\}$ is a basis for $\Gamma(VTM_0)$, from (4.6)–(4.8) we have $H^{a}{}_{\alpha\beta} = 0$ for any $a \in \{3, \dots, 2+n\}$ and $\alpha, \beta \in \{1, 2\}$. Hence by Theorem 3.1, $\mathbb{F}^2$ is a totally geodesic Finsler submanifold. □

The question whether Theorem 4.1 is true for Finsler submanifolds of arbitrary dimension is still open.

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