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A criterion for various additive models of the analysis of variance

By L. TAR (Debrecen)

Dedicated to Professor Lajos Tamássy on his 70th birthday

1. Introduction

First time such kind of examinations were performed by Béla Gyires. He has proved the following theorem for the randomized blocks designs ([4], p.285, Theorem 2). *The expectations of the sample elements can be decomposed into the sum of two quantities corresponding to the block-effect and to the treatment-effect, respectively, if and only if the expectations of the random errors are zero.*

After this the author was unable to obtain the corresponding criterion for the Latin square design employing the Gyires's method ([5], [6]). But this method was applied successfully for the fixed effects one-way analysis of variance model ([7]).

Recently we were able to find a newer method to prove the beforementioned criterion for various anova models ([8]). This is founded on a theorem well-known for the solution of the homogeneous and nonhomogeneous linear matrix equation ([3], pp.199-209).

The following theorem may be proved for the various models of the analysis of variance. *If the expectations of the random variables occuring in an anova model can be decomposed into the sum of corresponding quantities then the expectations of the random errors are zero.*

Our aim is to reverse this kind of theorems.

In the present paper we will use the following notations: x_{jk}, e_{jk}, l_j, $m_k \dots$ random variables; $x, y_1, y_2, z \dots$ matrix-valued random variables with m rows and n columns, that is matrices of dimension $m \times n$. Their elements are random variables having expectations; E is identity matrix of order m or $n; O$ is generally a zero matrix of dimension $m \times n$; S_1, S_2 are stochastic and idempotent matrices of order m and n, respectively;

the transpose of A is A^*; its inverse is A^{-1}; W_1, W_2 orthogonal matrices; $M(x_{jk})$ the expectation of x_{jk}; $M(x)$ consists of the expectations of the elements of x;

$$A = \begin{pmatrix} a_{11}, & a_{12}, & \dots, & a_{1n} \\ \dots & & & \\ a_{m1}, & a_{m2}, & \dots, & a_{mn} \end{pmatrix}$$

is a matrix given by its elements; $A = \|a_{jk}\|_{m\times n}$ is a matrix given by its general element; a_0 is an m–dimensional column vector having identical components $1; b_0$ is an n–dimensional column vector with identical components $1; a_0^*$ is the transpose of a_0; $0_{m\times 1}$ is the notation of zero vector with dimension m; instead of $j = 1, 2, \dots, m$ we introduce the shorter notation $j = \overline{1, m}$; if it is necessary, we indicate the dimension of a vector or a matrix in the following forms: $a_0^* = (1, \dots, 1)_{1\times m}, a_{0,m\times 1}$, $A_{m\times n}$, $A = \|a_{jk}\|_{m\times n}$; γ is a constant, the so-called overall mean; λ_j is the effect due to the j–th level of the first systematic factor; μ_k is the k–th differential effect of the k–th level of the second nonrandom factor.

We assume that the random variables or matrices have expectations. We shall denote this in the forms: $x_{jk} \in M$ and $x \in M$. We also suppose that the random variables $e_{jk}(j = \overline{1, m}; k = \overline{1, n})$ are independent, identically distributed (i.i.d.) normal random variables with mean 0 and unknown variance σ^2. A notation for the last fact is $e_{jk} \in N(0, \sigma^2)$. To obtain unique least-squares estimators for the unkown parameters we require the usual side conditions

$$(*) \qquad \sum_{j=1}^{m} \lambda_j = 0 \quad \text{and} \quad \sum_{k=1}^{n} \mu_k = 0\,.$$

In this paper we shall deal with the following five models.

1. *The fixed effects one-way analysis of variance model with equal numbers of observations.* We suppose that the number of experiments is n at each level of the single factor. This one-way model is

$$(1) \qquad x_{jk} = \gamma + \lambda_j + e_{jk} \quad (j = \overline{1, m};\ k = \overline{1, n})$$

where γ is the common part of the expectations, λ_j is a quantity corresponding to the j–th level of the systematic factor and $e_{jk} \in N(0, \sigma^2)$ $(j = \overline{1, m};\ k = \overline{1, n})$. Moreover the random variables $e_{jk}(j = \overline{1, m};$ $k = \overline{1, n})$ are independent ones. On the basis of our assumptions

$$(2) \qquad M(x_{jk}) = \gamma + \lambda_j \quad \text{and} \quad D^2(x_{jk}) = \sigma^2\,.$$

In the case of model (1) the task is to obtain the least-squares estimators of the unknown parameters and test null hypotheses

$$H_0 : \lambda_j = 0 \quad (j = \overline{1, m})\,.$$

2. *The random effects one-way analysis of variance model with equal numbers of observations.* The number of experiments is n at each level. This model has the form

$$x_{jk} = \gamma + l_j + e_{jk} \quad (j = \overline{1,m};\ k = \overline{1,n})\,, \tag{3}$$

where γ is the overall mean, $l_j\,(j = \overline{1,m})$ is a random variable corresponding to the j–th level of the random factor, $l_j\ (j = \overline{1,m})$ are independent and identically distributed with distribution $N(0, \sigma_l^2)$, where σ_l^2 is an unkown parameter. $e_{jk}\ (j = \overline{1,m};\, k = \overline{1,n})$ are i.i.d. normal random variables with distribution $N(0, \sigma^2)$. In this model $l_j\,(j = \overline{1,m})$ and $e_{jk}\,(j = \overline{1,m};\, k = \overline{1,n})$ are assumed to be jointly independent random variables. From (3)

$$M(x_{jk}) = \gamma \quad \text{and} \quad D^2(x_{jk}) = \sigma_l^2 + \sigma^2\,. \tag{4}$$

Now the unknown parameters are γ, σ_l and σ. The task is to estimate them and to test the hypothesis $H_0 : \quad \sigma_l = 0$.

3. *The unreplicated fixed effects two-way layout with no interaction.* This model is said to be additive. The observations take the form

$$x_{jk} + \gamma + \lambda_j + \mu_k + e_{jk} \quad (j = \overline{1,m};\, k = \overline{1,n})\,. \tag{5}$$

Here γ is the overall mean, λ_j is the j–th differential (or main) effect of the first systematic factor, μ_k is the effect due to the k–th level of the second nonrandom factor. $e_{jk}\ (j = \overline{1,m};\, k = \overline{1,n})$ are independent and identically distributed with distribution $N(0, \sigma^2)$. For $\lambda_j\ (j = \overline{1,m})$ and $\mu_k\ (k = \overline{1,n})\,(*)$ is true. In this model

$$M(x_{jk}) = \gamma \quad \text{and} \quad D^2(x_{jk}) = \sigma^2\,. \tag{6}$$

4. *The unreplicated mixed two-way layout with no interaction.* This is the *unreplicated randomized blocks design,* where the first factor has fixed effects and the second one has random effects on the results of the experiment and the factors have no common effect. In this case

$$x_{jk} = \gamma + \lambda_j + m_k + e_{jk}\ (j = \overline{1,m};\, k = \overline{1,n})\,, \tag{7}$$

where γ is a constant, λ_j shows the effect of the j–th level of the systematic factor, m_k is a random variable corresponding to the k–th level of the random factor, m_k represents the k–th block-effect, they are i.i.d. random variables and $m_k \in N(0, \sigma_m^2)\,(k = \overline{1,n})$, $e_{jk} \in N(0, \sigma^2)\,(j = \overline{1,m};$ $k = \overline{1,n})$ and these are also i.i.d. random variables, moreover $m_k\,(k = \overline{1,n})$ and $e_{jk}\ (j = \overline{1,m};\, k = \overline{1,n})$ are jointly independent. So

$$M(x_{jk}) = \gamma + \lambda_j \quad \text{and} \quad D^2(x_{jk}) = \sigma_m^2 + \sigma^2\,. \tag{8}$$

We require the assumption $\sum\limits_{j=1}^{m} \lambda_j = 0$.

5. *The unreplicated random effects two-way layout with no interaction.* The additive model is

$$x_{jk} = \gamma + l_j + m_k + e_{jk} \quad (j = \overline{1,m};\ k = \overline{1,n})\,, \tag{9}$$

where γ is a constant — overall mean — , l_j is a random variable due to the j–th level of the first random factor, m_k is also a random variable corresponding to the k–th level of the second random factor and $e_{jk} \in N(0,\sigma^2)$ as at the above-mentioned models. Here $l_j \in N(0,\sigma_l^2)$ $(j = \overline{1,m})$ $m_k \in N(0,\sigma_m^2)\,(k = \overline{1,n})$ and they are i.i.d. random variables. In this model all random variables are assumed to be jointly independent. From (9)

$$M(x_{jk}) = \gamma \quad \text{and} \quad D^2(x_{jk}) = \sigma_l^2 + \sigma_m^2 + \sigma^2\,. \tag{10}$$

Further details in connection with these models can be found in the special literature, for example in B. J. Winer, "Statistical principles in experimental design" (McGraw–Hill, New York San Francisco Toronto London, 1962).

The following theorems are valid for these models.

Theorem 1. *If the model has the form* (1), $x_{jk} \in M$ $(j = \overline{1,m}$; $k = \overline{1,n})$ *and* $\sum_{j=1}^{m} \lambda_j = 0$ *then*

$$M(x_{jk} - \bar{x}_{j\cdot}) = 0\,, \tag{11}$$

where

$$\bar{x}_{j\cdot} = \frac{1}{n}\sum_{k=1}^{n} x_{jk} \tag{12}$$

is one of the marginal means.

Remark 1. The left side of (11) is *the expectation of the random error* and (12) is *a marginal mean.*

Theorem 2. *If* (3) *is valid and* $x_{jk} \in M$ $(j = \overline{1,m};\ k = \overline{1,n})$ *then the expectation of the random error is zero.*

Theorem 3. *If* (5) *is true,* $x_{jk} \in M$ $(j = \overline{1,m};\ k = \overline{1,n})$ *and* $(*)$ *is valid for* λ_j $(j = \overline{1,m})$ *and* μ_k $(k = \overline{1,n})$ *then*

$$M(x_{jk} - \bar{x}_{j\cdot} - \bar{x}_{\cdot k} + \bar{x}) = 0\,. \tag{13}$$

Remark 2. According to (13) the expectation of the random error is zero under certain conditions in model (5). The means are defined by (12) and the following formulae:

$$\bar{x}_{\cdot k} = \frac{1}{m}\sum_{j=1}^{m} x_{jk} \quad \text{and} \quad \bar{x} = \frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n} x_{jk}\,. \tag{14}$$

Theorem 4. *If* (7) *is given,* $x_{jk} \in M$ $(j = \overline{1,m};\ k = \overline{1,n})$ *and* $\sum_{j=1}^{m} \lambda_j = 0$ *then* (13) *is true.*

Theorem 5. *If* x_{jk} *is defined by* (9) *and* $x_{jk} \in M$ $(j = \overline{1,m};\ k = \overline{1,n})$ *then* (13) *is valid.*

Remark 3. The above-mentioned theorems can be proved by the help of the corresponding model taking into account the side conditions. Similar theorems are valid — not only in the additive case — for the other anova models having more than two factors.

In this paper we deal with the proofs of the converse statements of the Theorems 1–5. For that purpose we shall give a matrix generalization of the former models and prove the suitable criterions in the generalized models. From these theorems can be obtained the reversed theorems for the special models. In the second section we give the Jordan normal form of a special idempotent metrix. This will be applied in the next sections at the solutions of the homogeneous and nonhomogeneous linear matrix equations. The third section contains the generalized forms of the fixed and random effects one-way analysis of variance models having equal numbers of the observations in each cell. The fourth section treats the unreplicated two-way layouts with no interaction.

2. The Jordan normal form of an idempotent matrix

The Jordan normal form of a special idempotent matrix will be applied at the solution of the homogeneous and nonhomogeneous linear matrix equations. In the solutions of these equations S_1 and S_2 will play an important role. Since they are similar to one another therefore we deal only with S_1.

The Jordan normal form of $S_1 = \|m^{-1}\|_{m\times m}$ is

$$S_1 = W_1 \begin{pmatrix} 1, & 0, & \ldots, & 0 \\ 0, & 0, & \ldots, & 0 \\ \ldots & & & \\ 0, & 0, & \ldots, & 0 \end{pmatrix}_{m\times m} W_1^* \,, \tag{15}$$

where $W_{1,m\times m}$ is the following orthogonal matrix:

$$\begin{pmatrix} m^{-1/2}, & [(m-1)/m]^{1/2}, & 0, & \ldots, & 0 \\ m^{-1/2}, & -[(m-1)m]^{-1/2}, & [(m-2)/(m-1)]^{1/2}, & \ldots, & 0 \\ m^{-1/2}, & -[(m-1)m]^{-1/2}, & -[(m-2)(m-1)]^{-1/2}, & \ldots, & 0 \\ m^{-1/2}, & -[(m-1)m]^{-1/2}, & -[(m-2)(m-1)]^{-1/2}, & \ldots, & 0 \\ \vdots & & & & \vdots \\ & & & & 0 \\ m^{-1/2}, & -[(m-1)m]^{-1/2}, & -[(m-2)(m-1)]^{-1/2}, & \ldots, & 2^{-1/2} \\ m^{-1/2}, & -[(m-1)m]^{-1/2}, & -[(m-2)(m-1)]^{-1/2}, & \ldots, & -2^{-1/2} \end{pmatrix}. \tag{16}$$

How can one obtain (15)? S_1 is singular and idempotent with characteristic roots either unity or zero. Its rank is equal to the trace of S_1. So the rank of S_1 is 1. Therefore its minimum dyadical representation — being an Hermitian matrix — is

$$S_1 = \begin{pmatrix} m^{-1/2} \\ m^{-1/2} \\ \vdots \\ m^{-1/2} \end{pmatrix}_{m\times 1} \left(m^{-1/2}, m^{-1/2}, \dots, m^{-1/2}\right)_{1\times m}, \tag{17}$$

that is $S_1 = m^{-1} a_0 a_0^*$, or with the notation $w_1^{(1)} = m^{-1/2} a_0$ it may be written in the form $S_1 = w_1^{(1)} w_1^{(1)*}$.

(15) was obtained by the help of the following theorem which gives the Jordan normal form of an idempotent matrix.

If the idempotent matrix P of dimension $m \times m$ and rank r $(1 \leq r$, $r \leq m)$ *has the minimum dyadical representation*

$$P = \sum_{k=1}^{r} u_k v_k^* = UV^*, \tag{18}$$

and the so-called complementary idempotent matrix $E - P$ has the minimum dyadical representation

$$E - P = \sum_{l=1}^{m-r} w_l z_l^* = WZ^*, \tag{19}$$

then the Jordan normal form of P with the characteristic vectors of (18) *and* (19) *is*

$$P = (u_1, \dots, u_r, w_1, \dots, w_{m-r}) \cdot \\ \cdot \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & (0) & \\ & & 1 & & & \\ \hline & & & 0 & & \\ & (0) & & & \ddots & \\ & & & & & 0 \end{array}\right)_{m\times m} \begin{pmatrix} v_1^* \\ \vdots \\ v_r^* \\ z_1^* \\ \vdots \\ z_{m-r}^* \end{pmatrix},$$

and here the number of characteristic roots 1 is r.

On the basis of this theorem the rank of $E - S_1$ is $m-1$ and $E - S_1$ is also an Hermitian matrix. So it can be decomposed into the sum of $m-1$ Hermitian dyads with a minimum dyadical representation.

The m–dimensional column vectors of Hermitian dyads of $E - S_1$ are as follows:

$$w_2^{(1)} = \begin{pmatrix} [(m-1)/m]^{1/2} \\ -[(m-1)m]^{-1/2} \\ \vdots \\ -[(m-1)m]^{-1/2} \end{pmatrix}, w_3^{(1)} = \begin{pmatrix} 0 \\ [(m-2)/(m-1)]^{1/2} \\ -[(m-2)(m-1)]^{-1/2} \\ \dots \\ -[(m-2)(m-1)]^{-1/2} \end{pmatrix},$$

$$\dots, w_m^{(1)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 2^{-1/2} \\ -2^{-1/2} \end{pmatrix}.$$

Finally — according to the above-formulated theorem — the Jordan normal form of S_1 is (15). (16) can be written in the form

$$W_{1;m\times m} = \left(w_1^{(1)}, w_2^{(1)}, w_3^{(1)}, \dots, w_m^{(1)}\right)$$

with the column vectors $w_j^{(1)}$ $(j = \overline{1,m})$.

In conformity with an EGERVÁRY's theorem ([2], X. tétel) the row- and column vectors of the dyads at a minimum dyadical representation form a biorthogonal vector system which is not a complete one, but it may be changed into a complete system by the help of the above-formulated theorem. So — on the basis of EGERVÁRY's theorem — W_1 is an orthogonal matrix.

3. A criterion for the one-way analysis of the variance models with equal numbers of observations

First we consider the fixed effects anova model (1).

Let $x = \|x_{jk}\|_{m\times n}$, where x_{jk} $(j = \overline{1,m};\ k = \overline{1,n})$ is defined by (1) and $x_{jk} \in M$ $(j = \overline{1,m};\ k = \overline{1,n})$. So a matrixical generalization of (1) is

$$x = \|\gamma\|_{m\times n} + \|\lambda_j\|_{m\times n} + \|e_{jk}\|_{m\times n}\,, \tag{20}$$

where $M(\|e_{jk}\|) = O_{m\times n}$. From (20) in consequence of (2)

$$M(x) = \gamma a_0 b_0^* + \lambda b_0^* \tag{21}$$

with $\lambda^* = (\lambda_1, \lambda_2, \dots, \lambda_m)$, $a_0^* = (1,1,\dots,1)_{1\times m}$ and $b_0^* = (1,1,\dots,1)_{1\times n}$. Let S_1 be a stochastic and idempotent matrix of order m having identical elements $\frac{1}{m}$. Let S_2 be a stochastic and projector matrix of order n

consisting of identical elements $\frac{1}{n}$. Then

$$S_1 = \frac{1}{m} a_0 a_0^*, \quad S_2 = \frac{1}{n} b_0 b_0^* . \tag{22}$$

Let us further define the following matrix-valued random variables of dimension $m \times n$:

$$y_2 = x S_2^*, \; z = S_1 \times S_2^* . \tag{23}$$

In this case

$$y_2 = \|\bar{x}_{j\cdot}\|_{m\times n}, \; z = \|\bar{x}\|_{m\times n} .$$

By the help of these formulae *the matrix of the random errors* is

$$x - y_2 = \|\bar{x}_{jk} - \bar{x}_{j\cdot}\|_{m\times n} ,$$

and *the matrix of the discrepancies between the effects due to the levels of the systematic factor* is given by

$$y_2 - z = \|\bar{x}_{j\cdot} - \bar{x}\|_{m\times n} .$$

The following theorem is valid for the last matrix.

Theorem 6. *Let* (21) *be true. In this case* $M(y_2 - z) = O_{m\times n}$ *if and only if* $\lambda = c a_0$, *where* c *is a constant.*

PROOF. 1. *If* $\lambda = c a_0$ *then* $M(y_2 - z) = O_{m\times n}$.
In consequence of (23) $M(y_2 - z) = (E - S_1) M(x) S_2^*$. By (21)

$$M(y_2 - z) = \gamma (E - S_1) a_0 (S_2 b_0)^* + (E - S_1) \lambda (S_2 b_0)^* .$$

Since $(E - S_1) a_0 = 0_{m\times 1}$ and $S_2 b_0 = b_0$, we get

$$M(y_2 - z) = (E - S_1) \lambda b_0^* . \tag{24}$$

Taking into account $\lambda = c a_0$ and $(E - S_1) a_0 = 0_{m\times 1}$ we get from (24)

$$M(y_2 - z) = O_{m\times n} .$$

2. *If* $M(y_2 - z) = O_{m\times n}$ *then* $\lambda = c a_0$.
According to (24)

$$(E - S_1) \lambda b_0^* = O_{m\times n} .$$

But this is possible only in the case $(E - S_1)\lambda = 0_{m\times 1}$. The last formula is true if $\lambda = c a_0$.

This completes the proof of Theorem 6.

Now we formulate the following theorem for the matrix of the random errors.

Theorem 7. *The decomposition* (21) *is valid if and only if*

$$M(x-y_2)=O_{m\times n}\,. \tag{25}$$

The proof of Theorem 7 is given in [8] (pp. 117–119).

Remark 4. The proof of Theorem 7 may be fulfilled a bit simpler than it was put through in [8]. This method will be applied at the proof of Theorem 8.

Further we examine the random effects one-way analysis of variance model (3). At this the differences $x_{jk}-\bar{x}_{j\cdot}$ ($j=\overline{1,m}$; $k=\overline{1,n}$) are also *the* so-called *random errors*. According to Theorem 2 their expectations are zero.

We introduce the generalized form of model (3) to prove the reverse of Theorem 2.

Let x be such a matrix of dimension $m\times n$ where the general element x_{jk} ($j=\overline{1,m}$; $k=\overline{1,n}$) is defined by (3). So the *generalized model* is

$$x=\|\gamma\|_{m\times n}+\|l_j\|_{m\times n}+\|e_{jk}\|_{m\times n}\,, \tag{26}$$

where $M(\|l_j\|)=O_{m\times n}$ and $M(\|e_{jk}\|)=O_{m\times n}$. So from (26)

$$M(x)=\|\gamma\|_{m\times n},\ \text{that is } M(x)=\gamma a_0 b_0^*\,, \tag{27}$$

where $a_0^*=(1,1,\dots,1)_{1\times m}$ and $b_0^*=(1,1,\dots,1)_{1\times n}$. Let S_1 and S_2 be given by (22). S_1 and S_2 are stochastic and idempotent matrices of order m and n, respectively. Then we can also define the matrix-valued random variables y_2 and z with (23). So

$$y_2=\|\bar{x}_{j\cdot}\|_{m\times n} \quad \text{and} \quad z=\|\bar{x}\|_{m\times n}\,.$$

Therefore *the matrix of the random errors* is

$$x-y_2=\|x_{jk}-\bar{x}_{j\cdot}\|_{m\times n}\,.$$

The next theorem corresponds to Theorem 7 at model (26).

Theorem 8. *Let us assume that the random variables* x_{jk} ($j=\overline{1,m}$; $k=\overline{1,n}$) *have expectations. Then* (27) *is valid if and only if*

$$M(x-y_2)=O_{m\times n}\,. \tag{28}$$

Proof. 1. *From* (27) *comes* (28). The left side of (28) — in consequence of the theorems valid for the expectation — is

$$M(x)-M(y_2)\,.$$

So (28) is true if $M(x)=M(y_2)$. But $M(y_2)=M(\|\bar{x}_{j\cdot}\|)$ and from (27) $M(\bar{x}_{j\cdot})=\gamma$. Therefore $M(y_2)=M(x)$. So

$$M(x)=M(y_2)\,.$$

2. *From* (28) *we get* (27). (28) is equivalent to

$$M(x) - M(x)S_2 = O_{m\times n}\,, \tag{29}$$

where S_2 has a simple structure. Since (29) is a homogeneous linear matrix equation of form $AX - XB = O$ therefore we can apply a well-known theorem to solve it ([3], p.202, Satz 1). The Jordan normal form of S_2 is similar to (15). Then

$$S_2 = W_2 \begin{pmatrix} 1, & 0, & \dots, & 0 \\ 0, & 0, & \dots, & 0 \\ \dots \\ 0, & 0, & \dots, & 0 \end{pmatrix}_{n\times n} W_2^*\,, \tag{30}$$

where W_2 is an orthogonal matrix of dimension $n \times n$. W_2 can be obtained from W_1 substituting n for m. So to solve (29) we can use the above-mentioned theorem ([3], p.202, Satz 1). Substituting (30) in (29) we get

$$M(x) - M(x)W_2 \begin{pmatrix} 1, & 0, & \dots, & 0 \\ 0, & 0, & \dots, & 0 \\ \dots \\ 0, & 0, & \dots, & 0 \end{pmatrix}_{m\times n} W_2^* = O_{m\times n}\,. \tag{31}$$

Post-multiplying (31) by W_2 and introducing the notation

$$\tilde{M}(x) = M(x)W_2 \tag{32}$$

we obtain from (31)

$$\tilde{M}(x) \begin{pmatrix} 0, & 0, & \dots, & 0 \\ 0, & 1, & \dots, & 0 \\ \dots \\ 0, & 0, & \dots, & 1 \end{pmatrix}_{m\times n} = O_{m\times n}\,. \tag{33}$$

Let $\tilde{M}(x) = \|\tilde{m}_{jk}\|_{m\times n}$. Then on the basis of (33) for the elements of $\tilde{M}(x)$

$$\begin{pmatrix} 0, & \tilde{m}_{12}, & \dots, & \tilde{m}_{1n} \\ 0, & \tilde{m}_{22}, & \dots, & \tilde{m}_{2n} \\ \dots \\ 0, & \tilde{m}_{m2}, & \dots, & \tilde{m}_{mn} \end{pmatrix} = O_{m\times n}\,.$$

So

$$\tilde{M}(x)_{m\times n} = \begin{pmatrix} \tilde{m}_{11}, & 0, & \dots, & 0 \\ \tilde{m}_{21}, & 0, & \dots, & 0 \\ \dots \\ \tilde{m}_{m1}, & 0, & \dots, & 0 \end{pmatrix}.$$

Hence $\tilde{M}(x)$ involves m free parameters which differ from zero. In consequence of (32) taking into account the form of W_2 which is similar to (16)

$$M(x) = n^{-1/2}\|\tilde{m}_{j1}\|_{m\times n}\,. \tag{34}$$

This means that $M(x)$ consists of identical elements in each row. Let us introduce the notation

$$\tilde{\lambda}_j = n^{-1/2}\tilde{m}_{j1} \quad (j = \overline{1,m}) \,. \tag{35}$$

If there exists an l index ($l = \overline{1,m}$) for wich $\tilde{\lambda}_l \neq 0$, then the minimum dyadical decomposition of $M(x)$ on the basis of [1] or [6] is as follows:

$$M(x) = \frac{1}{\tilde{\lambda}_l}\begin{pmatrix}\tilde{\lambda}_1\\ \vdots \\ \tilde{\lambda}_m\end{pmatrix}\left(\tilde{\lambda}_l, \tilde{\lambda}_l, \dots, \tilde{\lambda}_l\right)_{1\times n} ,$$

or in a simplier form we get from (29)

$$M(x) = \begin{pmatrix}\tilde{\lambda}_1\\ \vdots \\ \tilde{\lambda}_m\end{pmatrix}(1, 1, \dots, 1)_{1\times n} \,. \tag{36}$$

If $\tilde{\lambda}_j = \gamma$ ($j = \overline{1,m}$) then from (36)

$$M(x) = \gamma a_0 b_0^* \,,$$

that is (27) is valid.

With this the proof of Theorem 8 is finished.

Remark 5. Theorem 8 may be considered as that special case of Theorem 7 when $\lambda = 0_{m\times 1}$.

Remark 6. The selection $\tilde{\lambda}_j = \gamma$ ($j = \overline{1,m}$) *is possible.* Then the elements of the first column of $\tilde{M}(x)_{m\times n}$ are $mn^{-1/2}\gamma$, that is

$$\tilde{m}_{j1} = mn^{-1/2}\gamma \ (j = \overline{1,m}) \,. \tag{37}$$

Summing over both sides of (37) one can get

$$\gamma = n^{1/2}m^{-2}\sum_{j=1}^{m}\tilde{m}_{j1} \,. \tag{38}$$

The following theorem is valid for the fixed effects one-way analysis of variance model on the basis of Theorem 7 in the special case $m = n = 1$.

Criterion 1. *Let us assume that* (1) *is true and* $x_{jk} \in M$ ($j = \overline{1,m}$; $k = \overline{1,n}$). *Then* $M(x_{jk}) = \gamma + \lambda_j$ *if and only if* $M(x_{jk} - \bar{x}_{j\cdot}) = 0$.

One can get the next theorem for the random effects one-way analysis of variance model from Theorem 8 in the case $m = n = 1$.

Criterion 2. *Let us assume that* (3) *is valid for* x_{jk} *and* $x_{jk} \in M$ ($j = \overline{1,m}$; $k = \overline{1,n}$). *Then* $M(x_{jk}) = \gamma$ *if and only if* $M(x_{jk} - \bar{x}_{j\cdot}) = 0$.

4. A criterion for the unreplicated two-way analysis of variance models with no interaction

The models with no interaction are the so-called *additive models.* At unreplicated case the number of observations is one in each cell.

In the first place we consider the fixed (nonrandom) effects two-way analysis of variance model (5) for which Theorem 3 is true. Our aim to prove the reversed statement of Theorem 3 introducing a generalized model. We shall prove a criterion for this model applying the results valid for the general solution of the nonhomogeneous linear matrix equation $AX - XB = F$ ([3], pp.199–209). This criterion contains the statement of Theorem 3 and its reverse in the special case $m = n = 1$.

Let us consider the matrix

$$x = \|x_{jk}\|_{m\times n} \tag{39}$$

where x_{jk} is given by (5) and $x_{jk} \in M$ $(j = \overline{1,m};\ k = \overline{1,n})$. Then

$$x = \|\gamma\|_{m\times n} + \|\lambda_j\|_{m\times n} + \|\mu_k\| + \|e_{jk}\|_{m\times n}\,. \tag{40}$$

So — in consequence of $M(\|e_{jk}\|) = O_{m\times n}$ —

$$M(x) = \gamma a_0 b_0^* + \lambda b_0^* + a_0\mu^*\,, \tag{41}$$

where $a_0^* = (1,1,\ldots,1)_{1\times m}$, $b_0^* = (1,1,\ldots,1)_{1\times n}$, $\lambda^* = (\lambda_1,\lambda_2,\ldots,\lambda_m)$ (λ_j corresponds to the j–th level of the first factor) and $\mu^* = (\mu_1,\ldots,\mu_n)$ (μ_k is the effect due to the k–th level of the second nonrandom factor). Let S_1 and S_2 be stochastic and idempotent matrices of order m and n, respectively. Suppose that S_1 has identical elements $\frac{1}{m}$ and S_2 has identical elements $\frac{1}{n}$. It is well-known that they have 1 as a simple eigenvalue ([4], p.284, Corollary 4). Let y_2 and z defined by (23). Let us define a newer random variable

$$y_1 = S_1 x\,. \tag{42}$$

The marginal and grand means of the sample elements are given by (12) and (14). Then

$$y_1 = \|\bar{x}_{\cdot k}\|_{m\times n},\ y_2 = \|\bar{x}_{j\cdot}\|_{m\times n} \quad\text{and}\quad z = \|\bar{x}\|_{m\times n}\,.$$

The differences $\bar{x}_{j\cdot} - \bar{x}$ $(j = \overline{1,m})$ and $\bar{x}_{\cdot k} - \bar{x}$ $(k = \overline{1,n})$ are *the discrepancies between rows* and *the discrepancies between columns*, respectively. The quantities $x_{jk} - \bar{x}_{j\cdot} - \bar{x}_{\cdot k} + \bar{x}$ $(j = \overline{1,m};\ k = \overline{1,n})$ are *the random errors.* Since

$$\begin{gathered} y_1 - z = \|\bar{x}_{\cdot k} - \bar{x}\|_{m\times n}\,, \\ y_2 - z = \|\bar{x}_{j\cdot} - \bar{x}\|_{m\times n} \quad\text{and} \\ x - y_1 - y_2 + z = \|x_{jk} - \bar{x}_{j\cdot} - \bar{x}_{\cdot k} + \bar{x}\|\,, \end{gathered} \tag{43}$$

therefore we call the matrix $y_1 - z$ *the matrix of the discrepancies between columns*, the matrix $y_2 - z$ the matrix of the discrepancies between rows and the matrix $x - y_1 - y_2 + z$ is the so-called *random error matrix*.

Theorem 9. *Let* $x_{jk} \in M$ $(j = \overline{1,m};\ k = \overline{1,n})$. *Then* (41) *is true if and only if*

$$M(x - y_1 - y_2 + z) = O_{m\times n}\,.$$

The proof can be found in [8] (pp.121–125).

For the model (40) the following theorems are also valid.

Theorem 10. *Let* (41) *be true. Then*

$$M(y_2 - z) = O_{m\times n} \tag{44}$$

if and only if $\lambda = c_1 a_0$*, where* c_1 *is an arbitrary constant.*

Theorem 11. *Let us assume that* (41) *is valid for* $M(x)$*. Then*

$$M(y_1 - z) = O_{m\times n}$$

if and only if $\mu = c_2 b_0$*, where* c_2 *is a constant.*

Remark 7. The proof of Theorem 10 and Theorem 11 may be completed in similar way. Therefore we shall deal only with the proof of Theorem 10.

PROOF of Theorem 10. 1. *From* (44) *comes* $\lambda = c_1 a_0$*, where* c_1 *is a constant.* On the basis of (23) (44) may be written in the form

$$(E - S_1)M(x)S_2^* = O_{m\times n}\,. \tag{45}$$

Substituting $M(x)$ from (41) into (45) our matrix equation is

$$\gamma(E-S_1)a_0(S_2b_0)^* + (E-S_1)\lambda(S_2b_0)^* + (E-S_1)a_0(S_2\mu)^* = O_{m\times n}\,. \tag{46}$$

Since S_1 is a stochastic matrix of order m having identical elements $\frac{1}{m}$

$$S_1a_0 = 1a_0\,. \tag{47}$$

For S_2

$$S_2b_0 = 1b_0\,. \tag{48}$$

On the basis of (47) $(E - S_1)a_0 = 0a_0$. Taking into account this and (48) we obtain from (46)

$$\gamma 0_{m\times 1}b_0^* + (E - S_1)\lambda b_0^* + 0_{m\times 1}(S_2\mu)^* = O_{m\times n}\,,$$

that is

$$(E - S_1)\lambda b_0^* = O_{m\times n}\,. \tag{49}$$

This is true only in the case $(E - S_1)\lambda = 0_{m\times 1}$, or equivalently $S_1\lambda = \lambda$. Therefore (52) is valid if $\lambda = c_1 a_0$.

2. *If* $\lambda = c_1 a_0$ *then* $M(y_2 - z) = O_{m\times n}$ *assuming that* (44) *is satisfied.* On the basis of (41)

$$M(y_2 - z) = (E - S_1)\lambda b_0^* .$$

If $\lambda = c_1 a_0$ then

$$M(y_2 - z) = c_1(E - S_1)a_0 b_0 .$$

But in consequence of (47) $(E - S_1)a_0 = 0_{m\times 1}$. So $M(y_2 - z) = O_{m\times n}$.

This completes the proof of Theorem 10.

Remark 8. According to Theorem 10 the null hypothesis that the expectations of the discrepancies between rows are zero is equivalent to the null hypothesis that the quantities λ_j $(j = \overline{1,m})$ corresponding to the row-effects are equal to a constant c_1 at each j $(j = \overline{1,m})$.

Remark 9. On the basis of Theorem 11 the null hypothesis that the expectations of the discrepancies between columns are zero is equivalent to the one that the quantities μ_k $(k = \overline{1,n})$ — representing the column-effects — are equal to a constant c_2.

In the second place the author deals with *the unreplicated mixed two-way analysis of additive variance model* which is given by (7). For this model Theorem 4 is true. Model (7) is *the random blocks design.* We shall prove the reversed statement of Theorem 4 for a generalization of (7). In this case we shall also use the results well-known for the general solution of the nonhomogeneous linear matrix equation $AX - XB = F$ ([3], pp.199-209). Therefore one can obtain a criterion for this generalized model. From this it may be seen that Theorem 4 and its reversed statement is true.

Let us now consider the matrix of $x_{jk} \in M$ $(j = \overline{1,m};\ k = \overline{1,n})$ where x_{jk} is defined by (7)

$$x = \|x_{jk}\|_{m\times n}$$

Then

$$x = \|\gamma\|_{m\times n} + \|\lambda_j\|_{m\times n} + \|m_k\| + \|e_{jk}\|_{m\times n} . \tag{50}$$

From this

$$M(x) = \gamma a_0 b_0^* + \lambda b_0^* \tag{51}$$

according to the assumptions at (7) and on the basis of the theorems valid for the expectations. Let S_1 and S_2 be stochastic matrices given by (22).

Let y_1, y_2 and z be defined by (42) and (23). The formulae of the marginal and grand means are given by (12) and (14). The random variables $x_{jk} - \bar{x}_{j\cdot} - \bar{x}_{\cdot k} + \bar{x}$ $(j = \overline{1,m};\ k = \overline{1,n})$ are *the random errors* at model (7).

Remark 10. According to (51) $M(x)$ is the sum of two dyads. (51) can be obtained from (41) substituting $\mu = 0_{n\times 1}$ into it. If (53) is true then $y_1 = \|\bar{x}_{\cdot k}\|_{m\times n}$, $y_2 = \bar{x}_{j\cdot}\|_{m\times n}$ and $z = \|\bar{x}\|_{m\times n}$. From these

$$y_1 - z = \|\bar{x}_{\cdot k} - \bar{x}\|_{m\times n}\,,$$
$$y_2 - z = \|\bar{x}_{j\cdot} - \bar{x}\|_{m\times n} \quad \text{and}$$
$$x - y_1 - y_2 + z = \|x_{jk} - \bar{x}_{j\cdot} - \bar{x}_{\cdot k} + \bar{x}\|_{m\times n}\,.$$

Here x_{jk} is given by (7). $y_1 - z$ is *the matrix of the discrepancies between columns,* $y_2 - z$ is *the matrix of the discrepancies between rows.*

For our generalized model (50) the theorem corresponding to Theorem 9 is the next one.

Theorem 12. *Let* $x_{jk} \in M$ $(j = \overline{1,m};\ k = \overline{1,n})$. *Then* (51) *is fulfilled if and only if*

$$M(x - y_1 - y_2 + z) = O_{m\times n}\,.$$

PROOF. 1. *If* (51) *is true then the expectation of the random error matrix is the zero matrix.*

By the help of (23) and (42)

$$M(x - y_1 - y_2 + z) = (E - S_1)M(x)(E - S_2)^*\,. \tag{52}$$

The right side of (52) using (51) is the following expression:

$$\gamma(E - S_1)a_0[(E - S_2)b_0]^* + (E - S_1)\lambda[(E - S_2)b_0]^*\,.$$

So in consequence of $(E - S_1)a_0 = 0_{m\times 1}$ and $(E - S_1)b_0 = 0_{n\times 1}$

$$M(x - y_1 - y_2 + z) = O_{m\times n}\,.$$

2. *From* $M(x - y_1 - y_2 + z) = O_{m\times n}$ *we get* $M(x) = \gamma a_0 b_0^* + \lambda b_0^*$. According to (52) the matrix equation which must be solved

$$(E - S_1)_{m\times m} M(x)(E - S_2)^*_{n\times n} = O_{m\times n}\,. \tag{53}$$

Let us introduce the notation

$$\bar{M}(x) = (E - S_1)_{m\times m} M(x)\,. \tag{54}$$

Then (53) may be written in the form

$$\bar{M}(x) - \bar{M}(x)S_2 = O_{m\times n}\,. \tag{55}$$

This is a homogeneous linear matrix equation. It is similar to (32.) But in (55) occurs $\bar{M}(x)$ instead of $M(x)$. So the solution (55) using (36) is

$$\bar{M}(x) = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} (1, 1, \ldots, 1)_{1\times n}\,. \tag{56}$$

Hence for $M(x)$ can be obtained the following nonhomogeneous linear matrix equation from (54):

$$S_1 M(x) - M(x) = -\bar{M}(x)\,. \tag{57}$$

The general solution of (57) is the sum of the general solution of the corresponding homogeneous linear matrix equation

$$S_1 M(x) - M(x) = O_{m\times n} \tag{58}$$

and a particular solution of (57) ([3], pp. 208–209).

Now we give the general solution of (58). The Jordan normal form of S_1 is given by formulae (15) and (16). Substituting them into (58)

$$W_1 \begin{pmatrix} 1, & 0, & \dots, & 0 \\ 0, & 0, & \dots, & 0 \\ \dots & \dots & \dots & \dots \\ 0, & 0, & \dots, & 0 \end{pmatrix}_{m\times m} W_1^* M(x) - M(x) = O_{m\times n}\,. \tag{59}$$

Pre-multiplying (59) by W_1^* and considering the orthogonality of W_1 we get

$$\begin{pmatrix} 1, & 0, & \dots, & 0 \\ 0, & 0, & \dots, & 0 \\ \dots & \dots & \dots & \dots \\ 0, & 0, & \dots, & 0 \end{pmatrix}_{m\times m} W_1^* M(x) - W_1^* M(x) = O_{m\times n}\,. \tag{60}$$

Introducing the notation

$$\tilde{M}(x) = W_1^* M(x)$$

we obtain from (60)

$$\begin{pmatrix} 0, & 0, & \dots, & 0 \\ 0, & 1, & \dots, & 0 \\ \dots & \dots & \dots & \dots \\ 0, & 0, & \dots, & 1 \end{pmatrix}_{m\times m} \tilde{M}(x) = O_{m\times n}\,. \tag{62}$$

Let $\tilde{M}(x) = \|\tilde{m}_{jk}\|_{m\times n}$. So from (65)

$$\tilde{M}(x)_{m\times n} = \begin{pmatrix} \tilde{m}_{11}, & \tilde{m}_{12}, & \dots, & \tilde{m}_{1n} \\ 0, & 0, & \dots, & 0 \\ \dots & \dots & \dots & \dots \\ 0, & 0, & \dots, & 0 \end{pmatrix},$$

that is $\tilde{M}(x)$ contains n free parameters. From (61) $M(x) = W_1\tilde{M}(x)$. Calculating $W_1\tilde{M}(x)$

$$M(x) = \|m^{-1/2}\tilde{m}_{1k}\|_{m\times n}\,,$$

that is $M(x)$ consists of columnwise identical elements. If we introduce the notations $\tilde{\mu}_k = m^{-1/2}\tilde{m}_{1k}$ then

$$M(x) = \|\tilde{\mu}_k\|_{m\times n}\,. \tag{63}$$

Let us suppose that there is an l for which $\tilde{\mu}_l \neq 0$ ($l = \overline{1,n}$). Then the minimum dyadical representation of (63) is

$$M(x) = \frac{1}{\tilde{\mu}_l}\begin{pmatrix}\tilde{\mu}_l\\ \vdots\\ \tilde{\mu}_l\end{pmatrix}_{m\times m}(\tilde{\mu}_l, \tilde{\mu}_2, \ldots, \tilde{\mu}_n)\ ,$$

that is

$$M(x) = \begin{pmatrix}1\\1\\ \vdots\\ 1\end{pmatrix}_{m\times m}(\tilde{\mu}_l, \tilde{\mu}_2, \ldots, \tilde{\mu}_n)\ . \tag{64}$$

From this with the notation $\tilde{\mu}_k = \gamma\,(k = \overline{1,n})$

$$M(x) = \gamma a_0 b_0^*\,. \tag{65}$$

Now we prove that $M(x) = \lambda b_0^*$ *is a particular solution* of the matrix equation (57) if $\bar{M}(x)$ is given by the formula (56). Substituting $M(x){=}\lambda b_0^*$ and $\bar{M}(x) = \lambda b_0^*$ into (57)

$$S_1\lambda b_0^* - \lambda b_0^* = -\lambda b_0^*\,. \tag{66}$$

Since $\sum\limits_{j=1}^{m}\lambda_j = 0$ and

$$S_1\lambda = \frac{1}{m}\begin{pmatrix}\sum\limits^{m}\lambda_j\\ \vdots\\ \sum\limits^{m}\lambda_j\end{pmatrix}_{m\times 1}$$

hence $S_1\lambda = 0_{m\times 1}$. So (66) is true. Finally the general solution of the nonhomogeneous linear matrix equation (57) is

$$M(x) = \gamma a_0 b_0^* + \lambda b_0^*\,.$$

This completes the proof of theorem.

For the generalization of the unreplicated mixed two-way analysis of additive variance model is true a criterion corresponding to Theorem 10. This kind of theorem is valid only for the nonrandom factor of the mixed models.

Theorem 13. *Let* (51) *is true for* $M(x)$. *In this case*

$$M(y_2 - z) = O_{m\times n} \tag{67}$$

if and only if $\lambda = ca_0$, *where* c *is constant.*

Proof. The proof of this theorem is similar to that of Theorem 10.

Remark 11. According to Theorem 13 the null hypothesis that the expectations of the discrepancies between treatments are zero is equivalent to the hypothesis that the quantities λ_j ($j = \overline{1,m}$) — representing the j–th level of the treatment effects — are equal to an identical constant c.

Finally in this section we deal with *the unreplicated random effects two-way analysis of additive variance model.* This model is defined by (9) and for it Theorem 5 is valid. We shall prove a criterion for the generalized form of (9). From this theorem — in a special case — one may get Theorem 5 and the reversed statement of it.

In the proof of the above-mentioned criterion we shall use as earlier the theorems which are true for the general solution of a nonhomogeneous linear matrix equation ([3], pp.208–209).

Let us consider the matrix

$$x = \|\gamma + l_j + m_k + e_{jk}\|_{m\times n}\,, \tag{68}$$

where $x_{jk} \in M$ ($j = \overline{1,m}$; $k = \overline{1,n}$) and they are defined by (9). So

$$x = \|\gamma\|_{m\times n} + \|l_j\|_{m\times n} + \|m_k\|_{m\times n} + \|e_{jk}\|_{m\times n}\,.$$

In consequence of the assumptions

$$M(x) = \gamma a_0 b_0^*\,, \tag{69}$$

where $a_0^* = (1,1,\ldots,1)_{1\times m}$ and $b_0^* = (1,1,\ldots,1)_{1\times n}$. This means that $M(x)$ consists of a dyad. S_1 and S_2 are the earlier defined stochastic and idempotent matrices having dimensions $m\times m$ and $n\times n$, respectively. The rank of S_1 and S_2 is 1. The minimum dyadical representation of S_1 is given by (17). The minimum dyadical representation of S_2 is similar to (17) but in it m is substituted by n. y_1, y_2 and z are defined by formulae (42) and (23). The marginal and total means are given by (12) and (14). Since

$$y_1 - z = \|\bar{x}_{\cdot k} - \bar{x}\|_{m\times n}\,,$$

$$y_2 - z = \|\bar{x}_{j\cdot} - \bar{x}\|_{m\times n} \quad \text{and}$$

$$x - y_1 - y_2 + z = \|x_{jk} - \bar{x}_{j\cdot} - \bar{x}_{\cdot k} + \bar{x}\|_{m\times n}\,,$$

therefore the matrix $y_1 - z$ is *the matrix of the discrepancies between columns,* the matrix $y_2 - z$ is *the matrix of the discrepancies between rows* and the matrix $x - y_1 - y_2 + z$ is *the random error matrix.*

The following theorem is true for the generalization of the unreplicated random effects two-way analysis of variance model with no interaction. (The generalized model is given by (68) and (69).)

Theorem 14. *Let* $x_{jk} \in M$ $(j = \overline{1, m};\ k = \overline{1, n})$. *Then* (69) *is valid if and only if* $M(x - y_1 - y_2 + z) = O_{m\times n}$.

Proof. The method is similar to the proof of Theorem 9 and Theorem 12.

1. *If* (69) *is valid then* $M(x - y_1 - y_2 + z) = 0_{m\times n}$. Substituting y_1, y_2 and z on the basis of (42) and (23) into the expectation of the random error matrix

$$M(x - y_1 - y_2 + z) = (E - S_1)M(x)(E - S_2)^* \,. \tag{70}$$

This is formally identical with (52). Substituting (69) into (70)

$$M(x - y_1 - y_2 + z) = \gamma(E - S_1)a_0[(E - S_2)b_0]^* \,.$$

In consequence of $(E - S_1)a_0 = 0_{m\times 1}$ and $(E - S_2)b_0 = 0_{n\times 1}$

$$M(x - y_1 - y_2 + z) = \gamma 0_{m\times 1} 0^*_{n\times 1} \,,$$

that is

$$M(x - y_1 - y_2 + z) = O_{m\times n} \,.$$

With this the first part of Theorem 14 is proved.

2. *In the case of* $M(x - y_1 - y_2 + z) = O_{m\times n}$ (69) *is fulfilled for* $M(x)$. Using (70) our matrix equation is

$$(E - S_1)M(x)(E - S_2)^* = O_{m\times n} \,. \tag{71}$$

Introducing the notation

$$\bar{M}(x) = (E - S_1)_{m\times m} M(x) \tag{72}$$

(71) can be written in the form

$$\bar{M}(x) - \bar{M}(x)S_2 = O_{m\times n} \,. \tag{73}$$

This is a homogeneous linear matrix equation for unknown $\bar{M}(x)$. (73) is formally similar to (28). So its solution on the basis of (36) is

$$\bar{M}(x) = \begin{pmatrix} \tilde{\gamma}_1 \\ \vdots \\ \tilde{\gamma}_m \end{pmatrix} (1, 1, \ldots, 1)_{1\times n} \,. \tag{74}$$

If $\tilde{\gamma}_j = \tilde{\gamma}$ $(j = \overline{1, m})$ then

$$\bar{M}(x) = \tilde{\gamma} a_0 b_0^* \,. \tag{75}$$

From (72)

$$M(x) - S_1 M(x) = \bar{M}(x) \,,$$

that is

$$S_1M(x) - M(x) = -\bar{M}(x)\,. \tag{76}$$

This is similar to the nonhomogeneous linear matrix equation (57). So its general solution is the sum of the general solution of the corresponding homogeneous linear matrix equation and a particular solution of (76).

Let the general solution of

$$S_1M(x) - M(x) = O_{m\times n} \tag{77}$$

be

$$M(x) = \gamma' a_0 b_0^*\,, \tag{78}$$

where γ' is an arbitrary constant. Let a particular solution of (76) be

$$\hat{M}(x) = \tilde{\gamma} a_0 b_0^*\,.$$

Substituting $\hat{M}(x)$ into (76) taking into account (75) we get

$$S_1\tilde{\gamma} a_0 b_0^* - \tilde{\gamma} a_0 b_0^* = -\tilde{\gamma} a_0 b_0^*\,,$$

that is $S_1\tilde{\gamma} a_0 b_0^* = O_{m\times n}$. Since $S_1 a_0 = a_0$ therefore $\tilde{\gamma} a_0 b_0^* = O_{m\times n}$. From this $\tilde{\gamma} = 0$. So the general solution of (76) is given by (78). If we select as a particular solution of (76)

$$\hat{M}(x) = \gamma_1 a_0 b_0^*$$

then substituting it into (76) and applying (78) we obtain

$$\gamma_1 S_1 a_0 b_0^* - \gamma_1 a_0 b_0^* = -\tilde{\gamma} a_0 b_0^*\,.$$

In consequence of $S_1 a_0 = a_0$ the left side of this equation is a null matrix. So $\tilde{\gamma} a_0 b_0^* = O_{m\times n}$. From this $\tilde{\gamma} = 0$. Finally the general solution of (76) with the notation $\gamma' = \gamma$ is

$$M(x) = \gamma a_0 b_0^*\,. \tag{79}$$

In the case of particular solution $\hat{M}(x) = \gamma_1 a_0 b_0^*$ (76) will be a homogeneous equation also having the general solution (79).

The criterions for the three models considered in this section are as follows.

Criterion 3. *Let us assume that* (5) *is true and* $x_{jk} \in M$ $(j = \overline{1,m};\ k = \overline{1,n})$. *Then* $M(x_{jk}) = \gamma + \lambda_j + \mu_k$ *if and only if*

$$M(x_{jk} - \bar{x}_{j\cdot} - \bar{x}_{\cdot k} + \bar{x}) = 0\,.$$

Criterion 4. *Let* (7) *be valid and* $x_{jk} \in M\,(j=\overline{1,m};\ k=\overline{1,n})$. *So* $M(x_{jk})=\gamma+\lambda_j$ *if and only if*

$$M(x_{jk}-\bar{x}_{j\cdot}-\bar{x}_{\cdot k}+\bar{x})=0\,.$$

Criterion 5. *Let* (9) *be true for* x_{jk} *and* $x_{jk} \in M\ (j=\overline{1,m};\ k=\overline{1,n})$. *So* $M(x_{jk})=\gamma$ *if and only if*

$$M(x_{jk}-\bar{x}_{j\cdot}-\bar{x}_{\cdot k}+\bar{x})=0\,.$$

Remark 12. This criterions may be obtained from Theorem 9, Theorem 12 and Theorem 14, respectively. They may be get from the abovementioned theorems in the special case $m=n=1$.

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L. TAR
KOSSUTH LAJOS UNIVERSITY
MATHEMATICAL INSTITUTE
H–4010 DEBRECEN, P.O.BOX 12
HUNGARY

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