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Ruled fronts and developable surfaces

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Abstract. A developable surface in $\mathbb{R}^3$ has a Legendrian lift to the projective cotangent bundle over $\mathbb{R}^3$. In this paper we show that the converse assertion holds for singular ruled surfaces. We call such a surface *a ruled front*.

1. Introduction

Developable surfaces in $\mathbb{R}^3$ are a classical subject in differential geometry. It is, however, paid attention to in several other areas again (e.g., Projective differential geometry [5], Computer graphics [4], [6] containing industrial design etc.). A developable surface is a surface with the vanishing Gaussian curvature on the regular part and is also a ruled surface. It might be said that developable surfaces are quite special surfaces in $\mathbb{R}^3$ from the view-point of Euclidian differential geometry.

On the other hand, we can show that a developable surface has a Legendrian lift into the projective cotangent bundle over $\mathbb{R}^3$ (cf., Proposition 3.1). Theorem 3.2 asserts that a singular ruled front is a developable surface. Proposition 3.3 gives a condition that the Legendrian lift of a developable surface is non-singular. One of the typical examples of developable surfaces is the tangent developable surface of a regular space curve. In [2], Cleave has shown that the tangent developable surface of a regular space curve $\boldsymbol{\gamma}$ is locally diffeomorphic to the cuspidal cross cap if $\tau(s_0) = 0$ and $\tau'(s_0) \neq 0$, where $\tau(s)$ is the torsion of $\gamma(s)$ and the *cuspidal cross cap* is a singular surface which is defined by $\{(u^3, u^3v^3, v^2) \mid (u,v) \in \mathbb{R}^2\}$. It has been known that the cuspidal cross cap has a singular Legendrian

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lift and never can be lifted to a Legendrian immersion. The condition in Proposition 3.3 corresponds to the conditon that $\tau(s_0) \neq 0$ if we consider the tangent developable surface of a regular space curve. Although all arguments in this paper are classical and elementary, these assertions clarify the interesting properties of developable surfaces from the view-point of contact geometry.

All curves and maps considered here are of class C^∞ unless stated otherwise.

2. Basic notions

In this section we review basic concepts and properties of ruled surfaces and developable surfaces in $\mathbb{R}^3$. The classical theory has been given in [1], [3]. *A ruled surface* in $\mathbb{R}^3$ is (locally) the image of the map $F_{(\gamma,\delta)} : I \times J \to \mathbb{R}^3$ defined by $F_{(\gamma,\delta)}(t,u) = \boldsymbol{\gamma}(t) + u\boldsymbol{\delta}(t)$ where $\boldsymbol{\gamma} : I \to \mathbb{R}^3$, $\boldsymbol{\delta} : I \to \mathbb{R}^3 \setminus \{\mathbf{0}\}$ are smooth mappings and I, J are open intervals. We usually call $F_{(\gamma,\delta)}$ the *ruled surface* instead of the image. If $\boldsymbol{\delta}$ has a constant direction, then it is a cylindrical surface. Therefore, the ruled surface $F_{(\gamma,\delta)}$ is said to be *non-cylindrical* provided $\boldsymbol{\delta}' \wedge \boldsymbol{\delta}$ never vanishes, where $\wedge$ denotes *the exterior product* of vectors in $\mathbb{R}^3$. In this paper we only consider non-cylindrical ruled surfaces in order to avoid complicated situations. For any non-cylindrical ruled surface $F_{(\gamma,\delta)}(t,u)$ with $\|\boldsymbol{\delta}(t)\| \equiv 1$, there exists a unique smooth curve $\boldsymbol{\sigma} : I \longrightarrow \mathbb{R}^3$ such that $\mathrm{Image}\, F_{(\gamma,\delta)} = \mathrm{Image}\, F_{(\sigma,\delta)}$ and $\langle \boldsymbol{\sigma}'(t), \boldsymbol{\delta}'(t) \rangle = 0$, where $\langle , \rangle$ denotes the canonical inner product on $\mathbb{R}^3$. The curve $\boldsymbol{\sigma}(t)$ is called the *line of striction* of $F_{(\gamma,\delta)(t,u)}$ (cf., [1], [3]). The following lemma specifies the place where the singularities of the ruled surface are located [1], [3].

Lemma 2.1. *Let $F_{(\sigma,\delta)}$ be a non-cylindrical ruled surface with the line of striction $\boldsymbol{\sigma}$ and $\|\boldsymbol{\delta}(t)\| \equiv 1$. If $x_0 = F_{(\sigma,\delta)}(t_0,u_0)$ is a singular point of the ruled surface $F_{(\sigma,\delta)}$ then $u_0 = 0$ (i.e., $x_0 \in \mathrm{Image}\, \boldsymbol{\sigma}$). Moreover, if $\boldsymbol{\sigma}'(t_0) \neq \mathbf{0}$, then the ruling through $\boldsymbol{\sigma}(t_0)$ is tangent to $\boldsymbol{\sigma}$ at t_0.*

We say that $F_{(\gamma,\delta)}$ is *a developable surface* if the Gaussian curvature on the regular part of $F_{(\gamma,\delta)}$ vanishes. This condition is equivalent to the condition that $\det(\boldsymbol{\gamma}'(t)\, \boldsymbol{\delta}(t)\, \boldsymbol{\delta}'(t)) = 0$ for any $t \in I$. Under the assumption that $F_{(\gamma,\delta)}$ is non-cylindrical, $F_{(\gamma,\delta)}$ is a developable surface if and only if there exist smooth functions $\mu, \lambda : I \longrightarrow \mathbb{R}$ such that $\boldsymbol{\gamma}'(t) = \mu(t)\boldsymbol{\delta}(t) + \lambda(t)\boldsymbol{\delta}'(t)$.

3. Ruled fronts

In this section we consider the Legnedrian lift of a developable surface into the projective cotangent bundle $\pi : PT^*\mathbb{R}^3 \longrightarrow \mathbb{R}^3$. First, we review geometric properties of this space. Consider the tangent bundle $\tau : TPT^*\mathbb{R}^3 \to PT^*(\mathbb{R}^3)$ and the differential map $d\pi : TPT^*\mathbb{R}^3 \to T\mathbb{R}^3$ of π. For any $X \in TPT^*\mathbb{R}^3$, there exists an element $\alpha \in T^*_x\mathbb{R}^3$ such that $\tau(X) = [\alpha]$. For an element $V \in T_x\mathbb{R}^3$, the property $\alpha(V) = \mathbf{0}$ does not depend on the choice of representative of the class $[\alpha]$. Thus we have the following hyperplane field on $PT^*\mathbb{R}^3$:

$$K = \{X \in TPT^*\mathbb{R}^3 \mid \tau(X)(d\pi(X)) = 0\}.$$

We call it the *canonical contact structure* of $PT^*\mathbb{R}^3$.

Because of the trivialisation $PT^*\mathbb{R}^3 \cong \mathbb{R}^3 \times P(\mathbb{R}^2)^*$, we call $((x_1, x_2, x_3), [\xi_1 : \xi_2 : \xi_3])$ *a homogeneous coordinate*, where $[\xi_1 : \xi_2 : \xi_3]$ is the homogeneous coordinate of the dual projective space $P(\mathbb{R}^2)^*$.

It is easy to show that $X \in K_{(x,[\xi])}$ if and only if $\sum_{i=1}^3 \mu_i\xi_i = 0$, where $d\tilde{\pi}(X) = \sum_{i=1}^3 \mu_i \frac{\partial}{\partial x_i}$. An immersion $i : L \to PT^*\mathbb{R}^3$ is said to be *a Legendrian immersion* if $\dim L = 3$ and $di_q(T_qL) \subset K_{i(q)}$ for any $q \in L$. For a subset $i : L \subset PT^*\mathbb{R}^3$, it is called *a Legendrian subset* if i is a Legendrian immersion on the regular part of L. We also call the set $W(i) = \operatorname{image} \pi \circ i$ *a wave front* of i, and i (or, the image of i) is called *a Legendrian lift* of $W(i)$. If i is a Legendrian immersion, we say that the wave front $W(i)$ has *a regular Legendrian lift*. Otherwise, we say that the wave front $W(i)$ has *a singular Legendrian lift*. We now define the notion of ruled fronts. We say that a surface in $\mathbb{R}^3$ is *a ruled front* if it is a ruled surface and the Legendrian lift on the regular part of the surface can be ciontinuously extended into the singular part of the surface.

For any non-singular and non-cylindrical ruled surface $F_{(\gamma,\delta)} : I \times J \longrightarrow \mathbb{R}^3$, the Legendrian lift is given by

$$\widetilde{L}_{(\gamma,\delta)}(t,u) = (F_{(\gamma,\delta)}(t,u), [\boldsymbol{\gamma}(t) \wedge \boldsymbol{\delta}(t) + u\boldsymbol{\delta}(t) \wedge \boldsymbol{\delta}'(t)]),$$

where we denote $[\boldsymbol{v}] = [v_1 : v_2 : v_3] \in P(\mathbb{R}^2)^*$ for any non-zero vector $\boldsymbol{v} = (v_1, v_2, v_3)$. Since the vector $\boldsymbol{\gamma}'(t) \wedge \boldsymbol{\delta}(t) + u\boldsymbol{\delta}(t) \wedge \boldsymbol{\delta}'(t)$ is a normal vector at $F_{(\gamma,\delta)}(t,u)$, $\widetilde{L}_{(\gamma,\delta)}$ is the Legendrian lift of $F_{(\gamma,\delta)}$.

For any non-cylindrical developable surface $F_{(\gamma,\delta)} : I \times J \longrightarrow \mathbb{R}^3$, we have $[\boldsymbol{\gamma}(t) \wedge \boldsymbol{\delta}(t) + u\boldsymbol{\delta}(t) \wedge \boldsymbol{\delta}'(t)] = [\boldsymbol{\delta}(t) \wedge \boldsymbol{\delta}'(t)]$. Therefore the Legendrian lift $\widetilde{L}_{(\gamma,\delta)}$ can be rewritten as $L_{(\gamma,\delta)}(t,u) = (F_{(\gamma,\delta)}(t,u), [\boldsymbol{\delta}(t) \wedge \boldsymbol{\delta}'(t)])$ which makes sense even if $F_{(\gamma,\delta)}$ has singularities. Thus we have shown the following proposition.

Proposition 3.1. *Any non-cylindrical developable surface is a ruled front.*

We consider the converse assertion of the above proposition. If we consider Hyperboloid of one sheet, it is a non singular ruled surface and it is not a developable surface. This example shows that the converse assertion of the above proposition does not hold in general. We can, however, show the converse assertion of the above proposition for singular ruled fronts.

Theorem 3.2. *If a non-cylindrical ruled front has singular points, then it is a developable surface around the singularities.*

Proof. Without loss of generality, we assume that $F_{(\gamma,\delta)}$ is a ruled front such that $\boldsymbol{\gamma}$ is the line of striction and $\|\boldsymbol{\delta}(t)\| \equiv 1$. By Lemma 2.1, the singularities are located on the line of striction $\boldsymbol{\gamma}$. Let $x_0 = F_{(\gamma,\delta)}(t_0, 0)$ be a singular point. If $\boldsymbol{\gamma}'(t_0) = \mathbf{0}$, then the direction of the normal $u(\boldsymbol{\delta}'(t_0) \wedge \boldsymbol{\delta}(t_0))$ at (t_0, u) is constant along the ruling through x_0.

It also follows from Lemma 2.1 that the ruling through x_0 is tangent to $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(t_0)$ if $\boldsymbol{\gamma}'(t_0) \neq \mathbf{0}$.

In this case the direction of the normal vector of $F_{(\gamma,\delta)}$ at x_0 is also given by $\boldsymbol{\delta}'(t_0) \wedge \boldsymbol{\delta}(t_0)$.

On the other hand, if there exists a sequence $\{t_n\}_{n=1}^{\infty}$ convergent to t_0 such that $F_{(\gamma,\delta)}$ is non singular at each $(t_n, 0)$, then we have the normal vector $\boldsymbol{\gamma}'(t_n) \wedge \boldsymbol{\delta}(t_n)$ of the surface $F_{(\gamma,\delta)}$ at $(t_n, 0)$, so that we have

$$\begin{aligned}\langle \boldsymbol{\delta}'(t_n) \wedge \boldsymbol{\delta}(t_n), \boldsymbol{\gamma}'(t) \wedge \boldsymbol{\delta}(t_n)\rangle &= \langle \boldsymbol{\delta}'(t_n), \boldsymbol{\gamma}'(t_n)\rangle\langle \boldsymbol{\delta}(t_n), \boldsymbol{\delta}(t_n)\rangle \\ &\quad - \langle \boldsymbol{\delta}'(t_n), \boldsymbol{\delta}(t_n)\rangle\langle \boldsymbol{\delta}(t_n), \boldsymbol{\gamma}'(t_n)\rangle = 0.\end{aligned}$$

This means that the direction of $\boldsymbol{\gamma}'(t_n) \wedge \boldsymbol{\delta}(t_n)$ is always orthogonal to the direction of $\boldsymbol{\delta}'(t_n) \wedge \boldsymbol{\delta}(t_n)$. If we consider the limit position of the direction $\boldsymbol{\gamma}'(t_n) \wedge \boldsymbol{\delta}(t_n)$ as $n \longrightarrow \infty$, then it is also orthogonal to $\boldsymbol{\delta}'(t_0) \wedge \boldsymbol{\delta}(t_0)$. This means that we cannot continuously extend the normal direction of $F_{(\gamma,\delta)}$ to $(t_0, 0)$. This contradicts to the assumption that $F_{(\gamma,\delta)}$ is a ruled front.

Hence, the singular set $S = \{t \in I \mid F_{(\gamma,\delta)} \text{ is singular at } (t, 0)\ \}$ is an open subset in I. Since the singular set S is a closed subset in a connected set I, the surface $F_{(\gamma,\delta)}$ is singular along $\boldsymbol{\gamma}$. By the previous arguments, the surface $F_{(\gamma,\delta)}$ is a tangent developable along $\boldsymbol{\gamma}$ where $\boldsymbol{\gamma}'(t) \neq \mathbf{0}$. In this case the normal direction is constant along the ruling through $x = F_{(\gamma,\delta)}(t, 0)$. As we already mentioned that the normal direction is also constant along the ruling through $x = F_{(\gamma,\delta)}(t, 0)$ if $\boldsymbol{\gamma}'(t) = \mathbf{0}$. This is the condition that the ruled surface $F_{(\gamma,\delta)}$ is a developable surface. □

We have the following condition that the developable surface has the regular Legendrian lift.

Proposition 3.3. *Under the same notations as in the previous paragraph, $F_{(\gamma,\delta)}$ has a regular Legendrian lift at (t_0, u_0) if and only if* $\det(\boldsymbol{\delta}(t_0)\ \boldsymbol{\delta}'(t_0)\ \boldsymbol{\delta}''(t_0)) \neq 0$.

Proof. We define $D_{ij}(t) = \delta_i(t)\delta_j'(t) - \delta_j(t)\delta_i'(t)$ for $i, j = 1, 2, 3$. Then $\boldsymbol{\delta}(t) \wedge \boldsymbol{\delta}'(t) = (D_{23}, -D_{13}, D_{12})$. Without loss of generality, we may assume that $D_{23}(t_0) \neq 0$. In that case the local representation of $L_{(\gamma,\delta)}$ in the affine coordinate of $PT^*\mathbb{R}^3$ is given by

$$L_{(\gamma,\delta)}(t,u) = \left(\boldsymbol{\gamma}(t) + u\boldsymbol{\delta}(t), -\frac{D_{13}(t)}{D_{23}(t)}, \frac{D_{12}(t)}{D_{23}(t)}\right).$$

Therefore we have

$$\frac{\partial L_{(\gamma,\delta)}}{\partial t}(t,u) = \left(\boldsymbol{\gamma}'(t) + u\boldsymbol{\delta}'(t), \frac{\begin{vmatrix} D_{23}(t) & -D_{13}(t) \\ D_{23}'(t) & -D_{13}'(t) \end{vmatrix}}{(D_{23}(t))^2}, \frac{\begin{vmatrix} D_{23}(t) & D_{12}(t) \\ D_{23}'(t) & D_{12}'(t) \end{vmatrix}}{(D_{23}(t))^2}\right)$$

$$\frac{\partial L_{(\gamma,\delta)}}{\partial u}(t,u) = (\boldsymbol{\delta}(t), 0, 0).$$

It follows that $\operatorname{rank}\begin{pmatrix} \frac{\partial L_{(\gamma,\delta)}}{\partial t}(t_0,u_0) \\ \frac{\partial L_{(\gamma,\delta)}}{\partial u}(t_0,u_0) \end{pmatrix} = 2$ if and only if

$$\left(\begin{vmatrix} -D_{23}(t_0) & D_{13}(t_0) \\ -D_{23}'(t_0) & D_{13}'(t_0) \end{vmatrix}, \begin{vmatrix} D_{12}(t_0) & D_{23}(t_0) \\ D_{12}'(t_0) & D_{23}'(t_0) \end{vmatrix}\right) \neq (0,0).$$

Concerning the other cases, we can state that $L_{(\gamma,\delta)}$ is an immersion at (t_0, u_0) if and only if

$$\left(\begin{vmatrix} -D_{13}(t_0) & D_{12}(t_0) \\ -D_{13}'(t_0) & D_{12}'(t_0) \end{vmatrix}, \begin{vmatrix} D_{12}(t_0) & D_{23}(t_0) \\ D_{12}'(t_0) & D_{23}'(t_0) \end{vmatrix}, \begin{vmatrix} D_{23}(t_0) & D_{13}(t_0) \\ D_{23}'(t_0) & D_{13}'(t_0) \end{vmatrix}\right) \neq (0,0,0).$$

Since

$$\boldsymbol{\delta}(t_0) \wedge \boldsymbol{\delta}'(t_0) = (D_{23}(t_0), -D_{13}(t_0), D_{12}(t_0))$$

and

$$\boldsymbol{\delta}(t_0) \wedge \boldsymbol{\delta}''(t_0) = (D_{23}'(t_0), -D_{13}'(t_0), D_{12}'(t_0)),$$

this condition means that $(\boldsymbol{\delta}(t_0) \wedge \boldsymbol{\delta}'(t_0)) \wedge (\boldsymbol{\delta}(t_0) \wedge \boldsymbol{\delta}''(t_0)) \neq \mathbf{0}$.

On the other hand, we can easily show that $(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge (\boldsymbol{a} \wedge \boldsymbol{c}) = \det(\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c})\boldsymbol{a}$ for any vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$. So the above condition is equivalent to the condition that $\det(\boldsymbol{\delta}(t_0)\ \boldsymbol{\delta}'(t_0)\ \boldsymbol{\delta}''(t_0)) \neq 0$. This completes the proof. □

Remark. Theorem 3.2 and Proposition 3.3 clarify the feature of developable surfaces from the view-point of contact geometry.

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