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Weakly-Berwald spaces

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Abstract. We have two notions of Landsberg spaces and Douglas spaces as generalizations of Berwald spaces. Z. Shen introduced the notion of weakly affine spray ([12]), and in accordance this the first author gave the definition of a weakly-Berwald space ([4]) as another generalization of Berwald spaces. In this paper we will study the weakly-Berwald spaces.

In Sections 1 and 2, we shall summarize the properties of Landsberg spaces, Douglas spaces, projectively flat Finsler spaces and two-dimensional Finsler spaces respectively. In Section 3 we shall define weakly-Berwald spaces and investigate the three generalizations of Berwald spaces. Our main result is Corollary of Theorem 4. In Section 4, we shall show some examples of weakly-Berwald spaces. Especially, it is remarkable that the condition (4.6) for Randers spaces to be weakly-Berwald spaces is very simple.

1. Landsberg spaces, Douglas spaces and projectively flat Finsler spaces

Let M^n be an n-dimensional differential manifold and let $F^n{=}(M^n, L)$ be an n-dimensional Finsler space where L is a fundamental function. Let $g_{ij} = \dot\partial_i\dot\partial_j L^2/2$ be the fundamental tensor, where the symbol $\dot\partial_i$ means $\partial/\partial y^i$ and we define G_i as

$$G_i = \{y^r(\partial_r\dot\partial_i L^2) - \partial_i L^2\}/4,$$

and $G^i = g^{ij}G_j$ where the symbol ∂_i means $\partial/\partial x^i$ and (g^{ij}) is the inverse matrix of (g_{ij}). The coefficients $({G_j}^i{}_k, G^i{}_j)$ of the Berwald connection $B\Gamma$

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are defined as $G^i{}_j = \dot\partial_j G^i$ and $G_j{}^i{}_k = \dot\partial_k G^i{}_j$. The h- and v-covariant derivations with respect to $B\Gamma$ are denoted by (;) and ($\|$) respectively.

The Ricci formulas which show the commutative law of covariant differentiation are written as follows:

$$(1.1)\qquad \begin{cases} X^i_{;j;k} - X^i_{;k;j} = X^m H_m{}^i{}_{jk} - X^i_{\|m} R_j{}^m{}_k, \\ X^i_{;j\|k} - X^i_{\|k;j} = X^m G_m{}^i{}_{jk}, \\ X^i_{\|j\|k} - X^i_{\|k\|j} = 0. \end{cases}$$

The $(h)v$-torsion $R_j{}^h{}_k$ and the h- and hv-curvature tensors $H_i{}^h{}_{jk}$ are given by

$$(1.2)\qquad \begin{cases} R_j{}^h{}_k = A_{(jk)}\{\partial_k G^h{}_k - G_j{}^h{}_r G^r{}_k\}, \\ H_i{}^h{}_{jk} = R_j{}^h{}_{k\|i}, \\ G_i{}^h{}_{jk} = \dot\partial_i G_j{}^h{}_k, \end{cases}$$

where $A_{(jk)}$ means the interchange of the indices j, k and subtraction. We introduce two tensors $H_{ij} = H_i{}^r{}_{jr}$ and $G_{ij} = G_i{}^r{}_{jr}$, which are called the h- and hv-Ricci tensor respectively.

The C-tensor C_{ijk} is defined by $C_{ijk} = (\dot\partial_k g_{ij})/2$. The symbol ($|$) means the h-covariant derivation with respect to the Cartan connection. If a Finsler space satisfies the equations $C_{ijk|0} = C_{ijk|s} y^s = 0$, we call it Landsberg space. Using the second formula in (1.1), we get the equation $2C_{ijk|0} = -y_r G_i{}^r{}_{jk}$. Therefore, Landsberg spaces are also characterized by the equations $y_r G_i{}^r{}_{jk} = 0$.

Let us define a Douglas space. A Finsler space is said to be of Douglas type or a Douglas space, if $D^{ij} = G^i y^i - G^j y^i$ are homogeneous polynomials in (y^i) of degree three. The Douglas tensor is defined as follows ([5]):

$$(1.3)\qquad D_i{}^h{}_{jk} = G_i{}^h{}_{jk} - [G_{ij\|k} y^h + \{G_{ij}\delta^h{}_k + (i,j,k)\}]/(n+1)$$

where (i, j, k) indicate the terms obtained from the preceding term by cyclic permutation of the indices i, j, k.

The first author and M. Matsumoto proved ([3]):

A Finsler space is a Douglas space if and only if the Douglas tensor vanishes identically.

We now define a projectively flat Finsler space.

We consider two Finsler spaces $F^n = (M^n, L)$ and $\bar{F}^n = (M^n, \bar{L})$ on a common underlying manifold M^n. If any geodesic on F^n is also a geodesic on $\bar{F}^n$, the change $L \to \bar{L}$ of the metric is said to be projective. It is well-known ([5]) that $L \to \bar{L}$ is a projective change if and only if there exists a (1)p-homogeneous Finsler scalar field $P(x, y)$ on M^n satisfying

$$\bar{G}^i(x, y) = G^i(x, y) + P(x, y)y^i.$$

The scalar field P is called the projective factor.

If there exists a projective change of a Finsler space $F^n = (M^n, L)$ to $\bar{F}^n = (M^n, \bar{L})$ such that the Finsler space $\bar{F}^n$ is a locally Minkowski space, F^n is called projectively flat.

The Weyl tensor is given by

$$W^h{}_{ij} = R_i{}^h{}_j + A_{(ij)}\{y^h H_{ij} + \delta^h{}_i H_j\}/(n+1)$$

where $H_i = (nH_{0i} + H_{i0})/(n-1)$. It is well-known that the Douglas tensor and the Weyl tensor are projectively invariant. In a Minkowski space, the Douglas tensor and the Weyl tensor vanish identically.

2. Two-dimensional Finsler spaces

Let $F^2 = (M^2, L)$ be a two-dimensional Finsler space with the fundamental function L. Let (l^i, m^i) be a Berwald frame of the space F^2 which satisfies the following equations:

$$l^r l_r = 1, \qquad m^r m_r = \varepsilon,$$

where $\varepsilon = \pm 1$. There exists a scalar I which satisfies the equation $LC_{ijk} = Im_i m_j m_k$. We call the scalar I a main scalar of the space.

For a scalar field S we adopt the notions

$$S_{;1} = S_{;i} l^i, \qquad S_{;2} = \varepsilon S_{;i} m^i$$

$$S_{.1} = LS_{\|i} l^i, \qquad S_{.2} = \varepsilon LS_{\|i} m^i.$$

It is noted that $S_{.i}$ vanishes for a (0)p-homogeneous scalar S.

The h-curvature tensor $R_i{}^h{}_{jk}$ of $C\Gamma$ is written ([5]) as follows:

$$R_i{}^h{}_{jk} = R(l_i m^h - l^h m_i)(l_j m_k - l_k m_j), \tag{2.1}$$

where the scalar R is called the h-scalar curvature. The $(v)h$-torsion tensor $R_i{}^h{}_j(= R_0{}^h{}_{jk})$ of $C\Gamma$ coincides with that of $B\Gamma$, so that by (2.1) we get ([5])

$$R_j{}^h{}_k = LRm^h(l_j m_k - l_k m_j). \tag{2.2}$$

We have ([3])

$$LG_i{}^r{}_{jk} = (-2I_{;1}l^r + I_2 m^r)m_i m_j m_k, \tag{2.3}$$

where $I_2 = I_{;2} + I_{;1.2}$. By (2.3) we obtain

$$LG_{ij} = \varepsilon I_2 m_i m_j. \tag{2.4}$$

Applying the v-derivative $\|k$ with respect to the Berwald connection $B\Gamma$ to both sides of (2.4), we get

$$\begin{aligned} L^2 G_{ij\|k} = {} & \varepsilon(2\varepsilon I I_2 + I_{2.2})m_i m_j m_k - \\ & - \varepsilon I_2(l_i m_j m_k + m_i l_j m_k + m_i m_j l_k). \end{aligned} \tag{2.5}$$

Substituting (2.3), (2.4) and (2.5) in (1.3), we get ([3], [5])

$$3LD_i{}^h{}_{jk} = -[6I_{;1} + (2II_2 + \varepsilon I_{2.2})]l^h m_i m_j m_k. \tag{2.6}$$

Next we consider the curvature tensor $H_i{}^h{}_{jk}$. Since $H_i{}^h{}_{jk} = R_j{}^h{}_{k\|i}$, using (2.2) we get ([5])

$$H_{ijk}^h = \{R(l_i m^h - m_i l^h) + R_{.2} m_i m^h\}(l_j m_k - l_k m_j).$$

Thus we get

$$H_{ij} = \varepsilon R(l_i l_j + \varepsilon m_i m_j) + \varepsilon R_{.2} m_i l_j.$$

Since $H_i = 2H_{0i} + H_{i0}$, we get

$$H_i = \varepsilon L(3Rl_i + R_{.2} m_i).$$

In virtue of the equation mentioned above, we get

$$\begin{aligned} H_{i;j} = {} & \varepsilon L(3R_{;1} l_i l_j + 3R_{;2} l_i m_j \\ & + R_{.2;1} m_i l_j + R_{.2;2} m_i m_j - \varepsilon R_{.2} I_{;1} m_i m_j). \end{aligned}$$

Since $K_{ij} = H_{i;j} - H_{j;i}$, we get

$$K_{ij} = \varepsilon L(3R_{;2} - R_{.2;1})(l_i m_j - m_i l_j). \tag{2.7}$$

Applying the h-derivation $|r$ with respect to the Cartan connection $C\Gamma$ to the equation $LC_{ijk} = Im_i m_j m_k$, we get

$$LC_{ijk|r} = (I_{;1} l_r + I_{;2} m_r) m_i m_j m_k. \tag{2.8}$$

Transvecting y^r to (2.8), we get

$$C_{ijk|0} = I_{;1} m_i m_j m_k. \tag{2.9}$$

3. The relation between Berwald spaces and their three generalizations, and weakly-Berwald spaces with some conditions

A Berwald space is a space which satisfies the condition $G_i{}^h{}_{jk} = 0$, that is to say, whose coefficients $G_i{}^h{}_j$ of the Berwald connection are functions of the position (x^i) alone. Therefore the equations $y_r G_i{}^r{}_{jk} = 0$ hold. $2G^i = G_r{}^i{}_s y^r y^s$ are homogeneous polynomials in (y^i) of degree three. Then we can consider the notions of Landsberg spaces and Douglas spaces as two generalizations of Berwald spaces.

The notion of weakly-Berwald spaces is the third generalization of Berwald spaces.

Definition. If a Finsler space satisfies the condition $G_{ij} = 0$, we call it a weakly-Berwald space.

In this section, we shall investigate the relation between Berwald spaces and their three generalizations, and weakly-Berwald spaces with some conditions.

By equation (2.8), it follows that any two-dimensional Finsler space F^2 is a Berwald space, if and only if the equations $I_{;1} = I_{;2} = 0$ hold. In virtue of equations (2.9), (2.6) and (2.4), it follows that two-dimensional Landsberg spaces, two-dimensional Douglas spaces and weakly-Berwald spaces are characterized by

$$I_{;1} = 0, \quad 6I_{;,1} + 2II_2 + \varepsilon I_{2.2} = 0 \quad \text{and} \quad I_2 = 0$$

respectively.

(1) Weakly–Berwald and Douglas spaces. In [7], M. Fukui and T. Yamada proved that

Berwald spaces are characterized by $G_{ij} = 0$ in Finsler spaces with vanishing Douglas tensors.

In other words, we can say that

A Finsler space F^n $(n \geq 2)$ is a weakly-Berwald and Douglas space, if and only if the space is a Berwald space.

(2) Weakly-Berwald and Landsberg spaces. Suppose that a Finsler space is a weakly-Berwald and Landsberg space.

If the dimension of the space is two, then from (3.1) we get

$$I_2 = 0 \quad \text{and} \quad I_{;1} = 0. \tag{3.2}$$

Substituting the second equation in (3.2) into the first one, we get $I_{;2} = 0$. It follows that the space F^2 is a Berwald space. Conversely, suppose that a two-dimensional Finsler space F^2 is a Berwald space. From $I_{;1} = I_{;2} = 0$, we get $I_2 = 0$. Therefore we obtain

Theorem 1. *A two-dimensional Finsler space F^2 is a weakly-Berwald space and a Landsberg space, if and only if the space is a Berwald space.*

For a Finsler space F^n $(n \geq 3)$, from the conditions $G_{ij} = 0$ and $y_r G_i{}^r{}_{jk} = 0$, we could not get the equation $G_i{}^r{}_{jk} = 0$. Namely, a Finsler space F^n which is a weakly-Berwald and Landsberg space may not be a Berwald space.

(3) Douglas and Landsberg spaces. Berwald proved ([5]) that

[B1] A two-dimensional Finsler space F^2 is a Douglas and Landsberg space, if and only if the space is a Berwald space.

The first author and M. Matsumoto proved ([1], [2])

If a Finsler space F^n $(n \geq 2)$ is a Landsberg and Douglas space, then it is a Berwald space. Conversely a Berwald space is a Landsberg and Douglas space.

(4) Weakly-Berwald and projectively flat spaces. We consider a Finsler space which is a weakly-Berwald and projectively flat space. Berwald proved ([6]) that

[B2] An n-dimensional Finsler space F^n is projectively flat, if and only if

$$n \geq 3: \quad D_i{}^h{}_{jk} = 0 \quad \textit{and} \quad W^h{}_{jk} = 0,$$

$$n = 2: \quad D_i{}^h{}_{jk} = 0 \quad \textit{and} \quad K_{jk} = 0,$$

where $K_{ij} = H_{i;j} - H_{j;i}$.

If a Finsler space F^n $(n \geq 3)$ is weakly-Berwald and projectively flat, we get

$$G_{ij} = 0, \quad D_i{}^h{}_{jk} = 0 \quad \text{and} \quad W^h{}_{jk} = 0.$$

From SZABÓ's theorem ([13]):

A Finsler space is of scalar curvature if and only if the Weyl torsion tensor $W^h{}_{ij}$ *vanishes identically.*

From this and the result of the case (1), it follows that if a Finsler space F^n $(n \geq 3)$ is weakly-Berwald and projectively flat, then the space is a Berwald space of scalar curvature. Therefore, from S. NUMATA's theorem ([8]):

If a Finsler space F^n $(n \geq 3)$ *is a Berwald space and of scalar curvature* K*, then it is a Riemannian space or a locally Minkowski space, according as* $K \neq 0$ *or* $K = 0$, we get the following

Theorem 2. *A weakly-Berwald and projectively flat Finsler space* F^n $(n \geq 3)$ *is a Riemannian space of non-zero constant curvature or a Minkowski space.*

If the dimension of the weakly-Berwald and projectively flat Finsler space is two, from Berwald's Theorem [B2] mentioned above and the formula (2.3), we get $D_i{}^h{}_{jk} = 0$ and $3R_{;2} - R_{.2;1} = 0$. In the case (3), from Berwald's Theorem [B1] it follows that the space is a Berwald space and the equation $3R_{;2} - R_{.2;1} = 0$ holds. From the Ricci formula, we get

$$S_{;1;2} - S_{;2;1} = -RS_\theta, \tag{3.3}$$

where $S_\theta = \partial S/\partial\theta$ and θ is the angle of Landsberg which satisfies the partial differential equation $L\theta|_i = m_i$ where the symbol $(\,|\,)$ stands for the v-covariant derivation with respect to the Cartan connection.

Putting $S = I$ in (3.3), we get

$$I_{;1;2} - I_{;2;1} = -RI_\theta.$$

Since the space is a Berwald space, we have $I_{;1} = I_{;2} = 0$ and get

$$RI_\theta = 0.$$

From this equation we get $R = 0$ or $I_\theta = 0$. If $R = 0$, it follows that the space is a Minkowski space. If $I_\theta = 0$, we get $I_{.2} = I_\theta = 0$. The main scalar of the space is constant. Therefore, we get

Theorem 3. *A weakly-Berwald and projectively flat Finsler space F^2 is a Minkowski space or a space whose main scalar is constant and the scalar curvature R satisfies the equation*

$$3R_{;2} - R_{.2;1} = 0.$$

(5) Weakly-Berwald spaces of scalar curvature. We consider a space which is weakly-Berwald and of scalar curvature.

We know that the equation

$$(H_{kj} - H_{jk})_{\|l} = G_{lj;k} - G_{lk;j}. \tag{3.4}$$

generally holds ([14]).

A Finsler space F^n ($n \geq 3$) is of scalar curvature K ([5]) if and only if there exists a scalar field K satisfying

$$R_{i0k} = L^2 K h_{ik}.$$

Differentiating the above equation by y^i, we get

$$R_j{}^h{}_k = K_j h^h{}_k - K_h h^h{}_j,$$

where

$$K_j = L(LK_{\|j}/3 + Kl_j).$$

Contracting h and k, we get

$$R_j{}^s{}_s = (n-1)L(LK_{\|j}/3 + Kl_j) - L(LK_{\|j}/3 + Kl_j) + LKl_j.$$

From the definition of H_{ij} and (1.2), we get

$$\begin{aligned} H_{ij} = (2n-4)Ll_iK_{\|j}/3 + (n-1)Ll_jK_{\|i} \\ + (n-1)Kl_il_j + (n-2)L^2K_{\|j\|i}/3 + (n-1)Kh_{ij}. \end{aligned}$$

Therefore we get

$$H_{ij} - H_{ji} = (n+1)L(l_jK_{\|i} - l_iK_{\|j})/3. \tag{3.5}$$

Now, supposing that a space is of constant curvature, by equation (3.5) we get $H_{ki} - H_{ik} = 0$ and from (3.4) the equations $G_{lj;k} - G_{lk;j} = 0$, that is to say, the tensor $G_{lj;k}$ is completely symmetric in i, j, k.

Conversely, suppose that the tensor $G_{lj;k}$ is completely symmetric in the indices i, j, k. Then from (3.4) we get

$$(H_{kj} - H_{jk})_{\|l} = 0. \tag{3.6}$$

From the equations (3.5) and (3.6) we get

$$g_{kl}K_{\|j} - LK_{\|k\|l}l_j = g_{jl}K_{\|k} - LK_{\|j\|l}l_k. \tag{3.7}$$

Transvecting the equation (3.7) by y^j, we obtain

$$-LK_{\|k\|l} = lK_{\|k} + K_{\|l}l_k \tag{3.8}$$

using $K_{\|j}y^j = 0$. Substituting the equation (3.8) in the equation (3.7), we get

$$g_{kl}K_{\|j} + l_l l_j K_{\|k} + l_j l_k K_{\|l} = g_{jl}K_{\|k} + l_j l_k K_{\|l} + l_l l_k K_{\|j}. \tag{3.9}$$

Transvecting the equation (3.9) by g^{kl}, we get

$$nK_{\|j} = \delta^k{}_j K_{\|k} + K_{\|j}.$$

Therefore we get

$$(n-2)K_{\|j} = 0.$$

It follows that if the dimension n is more than two, then equation $K_{\|j} = 0$ holds, that is to say, the scalar curvature is a function of the position (x^i) alone. Furthermore, we know (Proposition 26.1 in [9]) that if a Finsler space F^n $(n \geq 3)$ is of scalar curvature which is a function of the position alone, then the space is of constant curvature.

Thus we get

Theorem 4. *A Finsler space F^n $(n \geq 3)$ of scalar curvature is of constant curvature if and only if the tensor $G_{ij;k}$ is completely symmetric in the indices i, j, k.*

In particular, in a weakly-Berwald space of scalar curvature the equation $G_{lj} = 0$ holds. Therefore we get

Corollary. *A weakly-Berwald space F^n $(n \geq 3)$ of scalar curvature is of constant curvature.*

4. Examples of the weakly-Berwald spaces

Suppose that a Finsler space (M,L) is a space with (α,β)-metric. In this section, the symbol $({}_{/})$ stands for h-covariant derivation with respect to the Riemannian connection in the space (M,α) and $\gamma_j{}^i{}_k$ stand for the Christoffel symbols in the space (M,α). From [1] it is known that the G^i of the space is given by

$$(4.1)\qquad \begin{cases} 2G^i = \gamma_0{}^i{}_0 + 2B^i \\ \;B^i = (E/\alpha)y^i + (\alpha L_\beta/L_\alpha)s^i{}_0 - \\ \qquad - (\alpha L_{\alpha\alpha}/L_\alpha)C^*\{(y^i/\alpha) - (\alpha/\beta)b^i\}, \end{cases}$$

where

$$(4.2)\qquad \begin{cases} E = (\beta L_\beta/L)C^* \\ C^* = \{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)\}/\{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})\} \\ \gamma^2 = b^2\alpha^2 - \beta^2 \\ r_{ij} = (b_{i/j} + b_{j/i})/2, \quad s_{ij} = (b_{i/j} - b_{j/i})/2, \quad s_i = s_{ri}b^r. \end{cases}$$

First we suppose that $L = \alpha + \beta$, then we get

$$L_\alpha = 1, \quad L_{\alpha,\alpha} = 0 \quad \text{and} \quad L_\beta = 1.$$

Substituting the above formula in (4.2), we get

$$C^* = \alpha(r_{00} - 2\alpha s_0)/2\beta \quad \text{and} \quad E = \alpha(r_{00} - 2\alpha s_0)/2(\alpha+\beta).$$

Substituting the above equation in (4.1), we get

$$B^i = \{(r_{00} - 2\alpha s_0)/2(\alpha+\beta)\}y^i + 2\alpha s^i{}_0$$

and

$$(4.3)\qquad 2G^i = \gamma_0{}^i{}_0 + \{(r_{00} - 2\alpha s_0)/(\alpha+\beta)\}y^i + 2\alpha s^i{}_0 B^i.$$

Differentiating the equation (4.3) by y^i, we get

$$(4.4)\qquad \begin{aligned} 2G^i{}_j = 2\gamma_0{}^i{}_0 &+ \partial_j\{(r_{00} - 2\alpha s_0)/(\alpha+\beta)\}y^i \\ &+ \{(r_{00} - 2\alpha s_0)/2(\alpha+\beta)\}\delta^i{}_j + (2y_j/\alpha)s^i{}_0 + 2\alpha^i_j. \end{aligned}$$

Contracting i and j in (4.4), we obtain

$$2G^r{}_r = \gamma_0{}^r{}_r + (n+1)\{(r_{00} - 2\alpha s_0)/(\alpha+\beta)\} \tag{4.5}$$

using $S_{00} = 0$ and $s_{ij}a^{ij} = 0$.

From (4.5) it follows that the necessary and sufficient condition for the space (M, L) to be a weakly-Berwald space is that the term $(r_{00} - 2\alpha s_0)/(\alpha+\beta)$ is a homogeneous polynomial in y^i of degree one. Putting $A = (r_{00} - 2\alpha s_0)/(\alpha+\beta)$, we get

$$\alpha(A + 2s_0) + (\beta A - r_{00}) = 0.$$

Since $A + 2s_0$ and $\beta A - r_{00}$ are homogeneous polynomials in y^i of degree one and of degree two respectively, and α is irrational in y^i, we get

$$A + 2s_0 = \beta A - r_{00} = 0.$$

By the equations mentioned above, we get

$$r_{00} + 2\beta s_0 = 0, \tag{4.6}$$

that is to say,

$$(b_{i/j} + b_{j/i}) + b_i(b_{r/j} - b_{j/r})b^r + b_j(b_{r/i} - b_{i/r})b^r = 0.$$

Therefore we get

Theorem 5. *The necessary and sufficient condition for a Randers space* $(M, \alpha+\beta)$ *to be a weakly-Berwald space is that the vector* b_i *satisfies the equation* (4.6).

Secondly, we suppose that $L = \alpha^2/\beta$, then we get

$$L_\alpha = 2\alpha/\beta, \quad L_{\alpha,\alpha} = 2/\beta \quad \text{and} \quad L_\beta = -\alpha^2/\beta^2.$$

Substituting the above formulas in (4.2), we get

$$C^* = (\beta r_{00} - \alpha^2 s_0)/2\alpha b^2 \quad \text{and} \quad E = -(\beta r_{00} - \alpha^2 s_0)/2\alpha b^2.$$

Therefore we obtain

$$B^i = -\{(\beta r_{00} - \alpha^2 s_0)/\alpha^2 b^2\} y^i - (\alpha^2/2\beta) s^i_0 + \{(\beta r_{00} - \alpha^2 s_0)/\alpha b^2\} b^i$$

and

$$2G^i = \gamma_0{}^i{}_0 - \{2(\beta r_{00} - \alpha^2 s_0)/\alpha^2 b^2\} y^i - (\alpha^2/\beta) s^i_0 + \{(\beta r_{00} - \alpha^2 s_0)/\alpha b^2\} b^i. \tag{4.7}$$

Differentiating the equation (4.7) by y^i, we get

$$\begin{aligned} 2G^i{}_j = {} & 2\gamma_0{}^i{}_j - \partial_j \{2(\beta r_{00} - \alpha^2 s_0)/\alpha^2 b^2\} y^i \\ & - \{2(\beta r_{00} - \alpha^2 s_0)/\alpha^2 b^2\} \delta^i{}_j - \\ & - \{(2y_j \beta - \alpha^2 b_j)/\beta^2\} s^i{}_0 + (\alpha^2/\beta) s^i{}_j + \\ & + \{(b_j r_{00} + 2\beta r_{0j} + 2y_j s_0 + \alpha^2 s_j)/\beta b^2\} b^i. \end{aligned} \tag{4.8}$$

Contracting i and j in (4.8), we get

$$2G^r{}_r = 2\gamma_0{}^r{}_r - 2(n+1)(\beta r_{00} - \alpha^2 b^2) - 2\{n s_0 - r_{0s} b^s\}/b^2,$$

using $S_{00} = 0$, $s_r b^r$ and $s_{ij} a^{ij} = 0$.

Since the term $n s_0 - r_{0s} b^s$ is a homogeneous polynomial in (y^i) of degree one, it follows that the necessary and sufficient condition for a Kropina space $(M, \alpha^2/\beta)$ to be a weakly-Berwald space is that the term $\beta r_{00}/\alpha^2$ is a homogeneous polynomial in (y^i) of degree one, that is to say,

$$\begin{aligned} & (b_i r_{j0} + b_j r_{i0} + \beta r_{ij})\alpha^6 - (b_i r_{00} + 2\beta r_{i0}) a_{j0} \alpha^4 \\ & -(b_j r_{00} + 2\beta r_{j0}) a_{i0} \alpha^4 - \beta r_{00} a_{j0} \alpha^4 + 2\beta \alpha^2 r_{00} a_{i0} a_{j0} = 0. \end{aligned} \tag{4.9}$$

Thus we get

Theorem 6. *The necessary and sufficient condition for a Kropina space $(M, \alpha^2/\beta)$ to be a weakly-Berwald space is that the vector b_i and the tensor a_{ij} satisfy the equation* (4.9).

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