Publ. Math. Debrecen
61/3-4 (2002), 241–252

The octahedron and cube functional equations revisited

By LÁSZLÓ SZÉKELYHIDI (Debrecen)

Abstract. In [5] we considered the problem of equivalence of the n-dimensional octahedron and cube equations which was conjectured by D. Z. DJOKOVIČ and H. HARUKI. Here we prove this conjecture using results of M. LEFRANC on spectral synthesis for $\mathbb{Z}^k$. Our method leads to the proof of another conjecture which states that the solutions of these functional equations are polynomials of degree at most $2n-1$.

In [5] we studied the functional equations of mean-value type

$$\left[\sum_{i=1}^{n}\left(\tau_t^i+\tau_{-t}^i\right)\right]f=2nf, \tag{1}$$

and

$$\left[\prod_{i=1}^{n}\left(\tau_t^i+\tau_{-t}^i\right)\right]f=2^nf, \tag{2}$$

where an Abelian group G is given, n is a positive integer, $f:G^n\to\mathbb{C}$ is a function and τ_t^i denotes the partial translation operator in the i-th variable with increment t, that is,

$$\tau_t^i f(x_1,x_2,\dots,x_n)=f(x_1,x_2,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_n)$$

holds for $i=1,2,\dots,n$ and for all $x_1,x_2,\dots,x_n,t$ in G. Equation (1), respectively (2) is called *octahedron*, respectively *cube equation*. For $n=1$

Mathematics Subject Classification: 39B52.
Key words and phrases: octahedron and cube equations, spectral synthesis.
The research was supported by the OTKA, Grant No. T-031995.

they coincide and they are equivalent to the Jensen-equation. For the history of equations of the above and similar type we refer to [5]. It has been proved (see [3], [4]) that (1) implies (2) for any n, and (2) implies (1) for $n \leq 4$. It has been conjectured by D. Z. Djoković and H. Haruki (see [4]) that (1) and (2) are equivalent for all n. In [5], Section 14. we studied this conjecture. Unfortunately, our method was based on a theorem of a result in [1], which states that spectral synthesis holds for any discrete Abelian group, however, there is a gap in the proof of that theorem, as Z. Gajda pointed out in 1990 in our personal conversation: the "proof" is incomplete and actually does not prove the statement. Despite several efforts of the present author and Z. Gajda the gap has not been filled yet. Some recent considerations on this problem encourage the conjecture that even spectral analysis (a considerably weaker statement) does not hold for any discrete Abelian group. It is not the purpose of this paper to go into these details on spectral synthesis and analysis. Here we give a correct proof for the equivalence of the two functional equations, which is based on spectral synthesis for $\mathbb{Z}^k$ proved in [2]. Our present proof can also be applied in several similar cases. In [5] we also considered the characterization problem concerning the solutions of (1) and (2) in the case $G = \mathbb{R}$. We were able to prove a representation theorem for the locally integrable solutions of (1) and (2), which states that all locally integrable solutions of (1) and (2) are linear combinations of the partial derivatives of a special polynomial, which is of degree at most $2n-1$ in each variable. Our method applied in this paper allows us to generalize this theorem by proving that on an arbitrary Abelian group any solution of (1) or (2) is a polynomial of degree at most $2n-1$ in each variable. The idea is to reduce the problem to the case $G = \mathbb{Z}^k$ and to study polynomial ideals of differential operators.

In this paper $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the set of integers, the set of reals, and the set of complex numbers, respectively, and $n \geq 2$ is a fixed integer. If G is an Abelian group then homomorphisms of G into the additive group of complex numbers, resp. the multiplicative group of nonzero complex numbers are called *additive functions*, resp. *complex exponentials*. Complex polynomials of additive functions are called *complex polynomials*, or simply *polynomials* on G. Finally, the product of a polynomial and a complex exponential on G is called an *exponential monomial*.

In the sequel we shall need the following simple results, the first of which is proved in [5] and the second follows easily by induction.

Theorem 1. *Let G be an Abelian group. Then any nonzero complex exponential on G is an extremal point of the convex hull of all nonzero complex exponentials on G. (see* [5], *Lemma 14.1, p. 119.).*

Theorem 2. *Let $i_1, i_2, \ldots, i_n$ be nonnegative integers. Then we have*

$$\sum_{\varepsilon \in \{-1,1\}^n} \varepsilon_1^{i_1} \varepsilon_2^{i_2} \ldots \varepsilon_n^{i_n} = \big(1 + (-1)^{i_1}\big)\big(1 + (-1)^{i_2}\big) \ldots \big(1 + (-1)^{i_n}\big).$$

(Here we used the notation $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$.)

We note that the above statement can be reformulated as follows: the given sum is different from zero if and only if all the exponents $i_1, i_2, \ldots, i_n$ are even, and in this case it is equal to 2^n.

In the sequel we shall use multi-index notation. Multi-indices of the same dimension are added component-wise and ordered lexicographically. Similarly, we order vectors of same dimensional multi-indices lexicographically, corresponding to the ordering of their components. It is clear that both the ordering of multi-indices and that of the vectors of multi-indices are linear.

Let k be a positive integer and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ be a k-dimensional multi-index. For the k-dimensional vector $u = (u_1, u_2, \ldots, u_k)$ in $\mathbb{R}^k$ we shall write $u^\alpha = u_1^{\alpha_1} u_2^{\alpha_2} \ldots u_k^{\alpha_k}$. The *factorial* $\alpha!$ of the multi-index α is the product of the factorials of its components. The *height* of the multi-index α is equal to $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_k$. We call a multi-index *even* if its height is an even number. For any nonzero k-dimensional even multi-index α let $\Gamma_n(\alpha)$ denote the set of all vectors $(\beta_1, \beta_2, \ldots, \beta_l)$ with $1 \leq l \leq n$, where $\beta_1, \beta_2, \ldots, \beta_l$ are nonzero k-dimensional even multi-indices with $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_l$ and $\beta_1 + \beta_2 + \ldots + \beta_l = \alpha$.

For any nonzero k-dimensional even multi-index α let

$$P_n(\alpha) = \sum \frac{\alpha!}{\beta_1! \beta_2! \ldots \beta_n!} x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n},$$

where $x_1, x_2, \ldots, x_n$ are in $\mathbb{R}^k$, and the summation is extended over all k-dimensional even multi-indices $\beta_1, \beta_2, \ldots, \beta_n$ with $\beta_1 + \beta_2 + \ldots + \beta_n = \alpha$. If any component of some exponent is equal to zero, then the corresponding factor is considered to be 1.

For an arbitrary positive integer l with $1 \leq l \leq n$ and for any nonzero k-dimensional multi-indices $\beta_1, \beta_2, \ldots, \beta_l$ with $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_l$ we denote by $Q_n(\beta_1, \beta_2, \ldots, \beta_l)$ the sum of all different monomials of the form

$x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \dots x_{i_l}^{\beta_l}$, where $i_1, i_2, \dots, i_l$ are different integers between 1 and n, and $x_{i_1}, x_{i_2}, \dots, x_{i_l}$ are in $\mathbb{R}^k$. For instance

$$Q_n(\beta) = \sum_{i=1}^{n} x_i^{\beta}.$$

We remark that in the sequel we shall consider the polynomials $P_n(\alpha)$ and $Q_n(\beta_1, \beta_2, \dots, \beta_l)$ mainly as polynomial differential operators in $n \cdot k$ variables by substituting $x_i = \partial_i = (\partial_{i,1}, \partial_{i,2}, \dots, \partial_{i,k})$ and by interpreting addition and multiplication in the obvious way.

It is clear that we have the representation

$$P_n(\alpha) = \sum_{l=1}^{n} \sum_{(\beta_1, \beta_2, \dots, \beta_l) \in \Gamma_n(\alpha)} \lambda_{\beta_1, \beta_2, \dots, \beta_l} Q_n(\beta_1, \beta_2, \dots, \beta_l).$$

Here the coefficients are positive integers and the coefficient of $Q_n(\alpha)$ is 1. It is also easy to see that if we put $x_{n+1} = 0$ in $P_{n+1}(\alpha)$ then we get $P_n(\alpha)$, and if we put $x_{n+1} = 0$ in $Q_{n+1}(\beta_1, \beta_2, \dots, \beta_l)$ then we get 0 for $l = n+1$ and $Q_n(\beta_1, \beta_2, \dots, \beta_l)$ for $l \leq n$.

We have another representation for $P_n(\alpha)$.

Theorem 3. *Let n, k be positive integers and α a k-dimensional multi-index. Then we have*

$$P_n(\alpha) = 2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \left(\sum_{i=1}^{n} \varepsilon_i x_i \right)^{\alpha}.$$

PROOF. We use the Polynomial Theorem and Theorem 2 in the following computation:

$$\begin{aligned}
&2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \left(\sum_{i=1}^{n} \varepsilon_i x_i \right)^{\alpha} \\
&\quad = 2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{\beta_1 + \dots + \beta_n = \alpha} \frac{\alpha!}{\beta_1! \dots \beta_n!} \varepsilon_1^{|\beta_1|} \dots \varepsilon_n^{|\beta_n|} x_1^{\beta_1} \dots x_n^{\beta_n} \\
&\quad = 2^{-n} \sum_{\beta_1 + \dots + \beta_n = \alpha} \frac{\alpha!}{\beta_1! \dots \beta_n!} \left(\sum_{\varepsilon \in \{-1,1\}^n} \varepsilon_1^{|\beta_1|} \dots \varepsilon_n^{|\beta_n|} \right) x_1^{\beta_1} \dots x_n^{\beta_n}
\end{aligned}$$

$$= 2^{-n} \sum_{\beta_1+\ldots+\beta_n=\alpha} \frac{\alpha!}{\beta_1!\ldots\beta_n!}\left(1+(-1)^{|\beta_1|}\right)\ldots\left(1+(-1)^{|\beta_n|}\right)x_1^{\beta_1}\ldots x_n^{\beta_n}$$

$$= \sum_{\beta_1+\ldots+\beta_n=\alpha,\,\beta_i \text{ is even}} \frac{\alpha!}{\beta_1!\ldots\beta_n!}\,x_1^{\beta_1}x_2^{\beta_2}\ldots x_n^{\beta_n} = P_n(\alpha). \qquad \square$$

Obviously, any product $Q_n(\beta_1)Q_n(\beta_2)\ldots Q_n(\beta_l)$ with $(\beta_1,\beta_2,\ldots,\beta_l)$ in $\Gamma_n(\alpha)$ is a linear combination of the polynomials $Q_n(\delta_1,\delta_2,\ldots,\delta_j)$ where $1\le j\le l$ and $(\delta_1,\delta_2,\ldots,\delta_j)$ is in $\Gamma_n(\alpha)$, and in this combination $Q_n(\alpha)$ has coefficient 1, further any $Q_n(\delta_1,\delta_2,\ldots,\delta_j)$ with nonzero coefficient has the property that $(\delta_1,\delta_2,\ldots,\delta_j)\ge(\beta_1,\beta_2,\ldots,\beta_l)$. We remark, that if $j<l$, then the vector $(\delta_1,\delta_2,\ldots,\delta_j)$ is "shorter" than $(\beta_1,\beta_2,\ldots,\beta_l)$, hence in order to compare them we add zero components to it. We do the same always when ordering vectors with different numbers of components.

Now we fix a nonzero k-dimensional even multi-index α. Let $\mathcal{A}$ denote the linear hull of the polynomials $Q_n(\beta_1,\beta_2,\ldots,\beta_l)$ with $(\beta_1,\beta_2,\ldots,\beta_l)$ in $\Gamma_n(\alpha)$. Obviously this set of polynomials is linearly independent, as the functions $Q_n(\beta_1,\beta_2,\ldots,\beta_l)$ are different monomials in $n\cdot k$ variables of the same degree for different choices of $(\beta_1,\beta_2,\ldots,\beta_l)$. Let further $\mathcal{B}$ denote the linear hull of the polynomials $P_n(\alpha)$ and $Q_n(\beta_1)Q_n(\beta_2)\ldots Q_n(\beta_l)$ with $l\ge 2$ and $(\beta_1,\beta_2,\ldots,\beta_l)$ in $\Gamma_n(\alpha)$.

Theorem 4. *With the above notation $\mathcal{A}=\mathcal{B}$.*

PROOF. By the above remarks it is clear that $\mathcal{B}\subseteq\mathcal{A}$. For the converse we observe first that the cardinalities of the two given sets of polynomials generating $\mathcal{A}$ and $\mathcal{B}$ are the same, therefore it is enough to prove that the given polynomials generating $\mathcal{B}$ are linearly independent. The polynomials $Q_n(\beta_1)Q_n(\beta_2)\ldots Q_n(\beta_l)$ with $l\ge 2$ and $(\beta_1,\beta_2,\ldots,\beta_l)$ in $\Gamma_n(\alpha)$ are obviously linearly independent, hence it is enough to show that $P_n(\alpha)$ is not a linear combination of them. If we suppose that it is for some n, then according to our previous remark, by substituting $x_n=0$ we get that it is also the case for n. Hence it is enough to show that the given polynomials generating $\mathcal{B}$ are linearly independent for $n=2$. We have seen above that the polynomials $P_2(\alpha)$ and $Q_2(\beta_1)Q_2(\beta_2)\ldots Q_2(\beta_l)$ with $l\ge 2$ and $(\beta_1,\beta_2,\ldots,\beta_l)$ in $\Gamma_2(\alpha)$ can uniquely be written as linear combinations of the polynomials $Q_2(\beta_1,\beta_2,\ldots,\beta_l)$ with $1\le l\le 2$ and $(\beta_1,\beta_2,\ldots,\beta_l)$ in $\Gamma_2(\alpha)$. We show that the quadratic matrix of this linear transformation is regular. Suppose that the first row contains the coefficients of the linear

expression for $P_2(\alpha)$ in terms of the polynomials $Q_2(\beta_1, \beta_2, \ldots, \beta_l)$ corresponding to the decreasing order of $(\beta_1, \beta_2, \ldots, \beta_l)$, hence the first row has the form $(1, \lambda_1, \lambda_2, \ldots, \lambda_N)$, where $N \geq 2$ is an integer, and $\lambda_1, \lambda_2, \ldots, \lambda_N$ are positive integers. The second, third, etc., rows contain the coefficients of the linear expressions for the polynomials $Q_2(\beta_1)Q_2(\beta_2)\ldots Q_2(\beta_l)$ with $l = 2$ corresponding to the decreasing order of $(\beta_1, \beta_2, \ldots, \beta_l)$. We have two cases. If the multi-index α is not of the form $\beta + \beta$ with some (β, β) in $\Gamma_2(\alpha)$, then $Q_2(\beta_1)Q_2(\beta_2) = Q_2(\alpha) + Q_2(\beta_1, \beta_2)$ for any (β_1, β_2) in $\Gamma_2(\alpha)$, hence any row, which is different from the first has only two nonzero entries, which are equal to 1, namely, the second row is $(1, 1, 0, \ldots, 0)$, the third is $(1, 0, 1, 0, \ldots, 0)$, and the last one is $(1, 0, \ldots, 0, 1)$, where the dots represent zeros. The determinant of this matrix is $1 - \lambda_1 - \lambda_2 - \ldots - \lambda_N$, which is different from zero. In the second case $\alpha = \beta + \beta$ for some β, and in this case $Q_2(\beta)Q_2(\beta) = Q_2(\alpha) + 2Q_2(\beta, \beta)$, which means that in the corresponding row of the matrix the first entry is 1 and the other nonzero entry is 2, instead of 1. Multiplying the corresponding column of the matrix by $\frac{1}{2}$, the first row changes to $(1, \lambda_1, \ldots, \frac{\lambda_i}{2}, \ldots, \lambda_N)$ and the determinant to $1 - \lambda_1 - \ldots - \frac{\lambda_i}{2} - \ldots - \lambda_N$, which is also different from zero. This means, that the matrix of the linear transformation, which maps a basis of $\mathcal{A}$ to a generating set of $\mathcal{B}$ is regular, hence the given generating set of $\mathcal{B}$ is linearly independent and $\mathcal{A} \subseteq \mathcal{B}$. The theorem is proved. □

For any nonzero k-dimensional multi-index α let $I_n(\alpha)$, respectively $J_n(\alpha)$ denote the ideal generated by all the polynomials $Q_n(\beta)$, respectively $P_n(\beta)$ in the ring of complex polynomials in $n \cdot k$ variables, where $\beta \leq \alpha$ is nonzero and even. Obviously $I_n(\beta) \subseteq I_n(\alpha)$ and $J_n(\beta) \subseteq J_n(\alpha)$ for $\beta \leq \alpha$.

Theorem 5. *The ideals $I_n(\alpha)$ and $J_n(\alpha)$ are identical.*

PROOF. If the height of α is 2 then $I_n(\alpha)$ is generated by $Q_n(\alpha)$, and $J_n(\alpha)$ is generated by $P_n(\alpha)$, which are equal, hence $I_n(\alpha) = J_n(\alpha)$. Suppose that we have proved the theorem for all nonzero k-dimensional multi-indices with height less than $2N$ and let $|\alpha| = 2N$. We have seen above that $\mathcal{A} \subseteq \mathcal{B}$, hence $Q_n(\alpha)$ is a linear combination of the polynomials $P_n(\alpha)$ and $Q_n(\beta_1)Q_n(\beta_2)\ldots Q_n(\beta_l)$ with $2 \leq l \leq n$ and $(\beta_1, \beta_2, \ldots, \beta_l)$ in $\Gamma_n(\alpha)$. As $\beta_1 + \beta_2 + \ldots + \beta_l = \alpha$, here the height of β_i is less than $2N$, hence by our assumption $Q_n(\beta_i)$ is in $J_n(\beta_i) \subseteq J_n(\alpha)$ for $i = 1, 2, \ldots, l$. Since $P_n(\alpha)$ also belongs to $J_n(\alpha)$ we infer that $Q_n(\alpha)$ belongs to $J_n(\alpha)$, and $I_n(\alpha) \subseteq J_n(\alpha)$. Conversely, the polynomials $Q_n(\beta_1, \beta_2, \ldots, \beta_l)$ with $2 \leq l \leq n$ and $(\beta_1, \beta_2, \ldots, \beta_l)$ in $\Gamma_n(\alpha)$ are linear combinations of products

of the form $Q_n(\delta_1)Q_n(\delta_2)\dots Q_n(\delta_j)$ with $2 \le j \le n$ and $(\delta_1, \delta_2, \dots, \delta_j)$ in $\Gamma_n(\alpha)$, which all belong to $I_n(\alpha)$. Further, $P_n(\alpha)$ is a linear combination of $Q_n(\alpha)$ and $Q_n(\beta_1, \beta_2, \dots, \beta_l)$ with $2 \le l \le n$ and $(\beta_1, \beta_2, \dots, \beta_l)$ in $\Gamma_n(\alpha)$, which all belong to $I_n(\alpha)$, which implies $J_n(\alpha) \subseteq I_n(\alpha)$, and our statement is proved. □

Theorem 6. *The polynomial solutions of* (1) *and* (2) *are identical if* $G = \mathbb{Z}^k$.

PROOF. Any polynomial solution of (1) or (2) on $\mathbb{Z}^k$ is a complex polynomial in $n \cdot k$ variables. If (1) or (2) holds for a polynomial, then fixing $x_1, x_2, \dots, x_n$ in $\mathbb{Z}^k$ we have a polynomial identity in the variable t in $\mathbb{Z}^k$, which must hold for all $x_1, x_2, \dots, x_n$ and t in $\mathbb{R}^k$, too. For any fixed $x_1, x_2, \dots, x_n$ the two polynomials in $t = (t_1, t_2, \dots, t_k)$ on the two sides of (1) and (2) have the same value at $t = (0, 0, \dots, 0)$, hence they are identical if and only if their derivatives of all order are equal at $t = (0, 0, \dots, 0)$, by the Taylor-formula. Let α be any nonzero k-dimensional multi-index. Applying the differential operator $\partial_t^\alpha = \partial_{t_1}^{\alpha_1}\partial_{t_2}^{\alpha_2}\dots\partial_{t_k}^{\alpha_k}$ on both sides of (1) and then substituting $t = 0$ we have that a necessary and sufficient condition for the polynomial $f : \left(\mathbb{Z}^k\right)^n \to \mathbb{C}$ is a solution of (1) is that

$$\left(1 + (-1)^{|\alpha|}\right) \sum_{i=1}^{n} \partial_i^\alpha f = 0.$$

Here $\partial_i^\alpha = \partial_{i,1}^{\alpha_1}\partial_{i,2}^{\alpha_2}\dots\partial_{i,k}^{\alpha_k}$, where $\partial_{i,j}$ denotes partial differentiation with respect to the j-th component of the i-th variable for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, k$. This means that the polynomial $f : \left(\mathbb{Z}^k\right)^n \to \mathbb{C}$ satisfies (1) if and only if for any nonzero k-dimensional even multi-index α

$$\sum_{i=1}^{n} \partial_i^\alpha f = 0, \qquad \text{or} \qquad Q_n(\alpha) f = 0 \tag{3}$$

holds, where $Q_n(\alpha)$ is the polynomial differential operator, obtained as above with $x_i = \partial_i$.

Now we apply the differential operator ∂_t^α on both sides of (2) and substitute $t = (0, 0, \dots, 0)$. Then we have that the polynomial $f : (\mathbb{Z}^k)^n \to \mathbb{C}$ satisfies (2) if and only if for any nonzero k-dimensional even multi-index α

$$\sum_{\varepsilon \in \{-1,1\}^n} \left(\sum_{i=1}^{n} \varepsilon_i \partial_i \right)^\alpha f = 0,$$

or

$$2^n P_n(\alpha) f = 0 \tag{4}$$

holds, where $P_n(\alpha)$ is the polynomial differential operator, obtained as above with $x_i = \partial_i$. Here $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$. By Theorem 5 the ideals generated by the polynomials $Q_n(\alpha)$ and $P_n(\alpha)$ are identical, that is, the systems of partial differential equations (3) and (4) are equivalent. Hence the polynomial solutions of (1) and (2) are identical if $G = \mathbb{Z}^k$. □

Theorem 7. *The functional equations* (1) *and* (2) *are equivalent on any Abelian group G for any positive integer n.*

PROOF. By Theorem 8.13 in [5] it is enough to show that the restrictions of (1) and (2) to any finitely generated subgroup of G are equivalent, that is, (1) and (2) are equivalent on any finitely generated Abelian group. Using the result of Lefranc in [2] on spectral synthesis for $\mathbb{Z}^k$ we show that (1) and (2) are equivalent if $G = \mathbb{Z}^k$. It is enough to show, that in this case the exponential monomial solutions of (1) and (2) are identical. However, if an exponential monomial has the form pm, where $p : G^m \to \mathbb{C}$ is a polynomial and $m : G^n \to \mathbb{C}$ is an exponential and it is a solution of (1) or (2), then the exponential m is also a solution of (1) or (2), because the solution space of these equations are translation invariant linear function spaces closed under pointwise convergence. In this case Lemma 4.2 in [5], p. 40. can be applied. If $m : G^n \to \mathbb{C}$ is an exponential then it has the form $m(x_1, x_2, \ldots, x_n) = m_1(x_1)m_2(x_2)\ldots m_n(x_n)$, where $m_1, m_2, \ldots, m_n : G \to \mathbb{C}$ are exponentials. Substituting m into (1) or (2) we get easily by Theorem 1 that $m_1 = m_2 = \cdots = m_n = 1$, hence $m = 1$, that is, any exponential monomial solution of (1) or (2) is a polynomial. By Theorem 6 the polynomial solutions of (1) and (2) are identical if $G = \mathbb{Z}^k$.

Suppose now that G is an arbitrary finitely generated Abelian group and let $\varphi : \mathbb{Z}^k \to G$ be a surjective homomorphism, where k is some positive integer. The function $\Phi : \left(\mathbb{Z}^k\right)^n \to G^n$ defined by

$$\Phi(z_1, z_2, \ldots, z_n) = \big(\varphi(z_1), \varphi(z_2), \ldots, \varphi(z_n)\big)$$

for $z_1, z_2, \ldots, z_n$ in $\mathbb{Z}^k$ is a surjective homomorphism. If $f : G^n \to \mathbb{C}$ is a solution of (1) on G^n, then $f \circ \Phi : \left(\mathbb{Z}^k\right)^n \to \mathbb{C}$ is obviously a solution of (1) on $\left(\mathbb{Z}^k\right)^n$, hence $f \circ \Phi$ satisfies (2) on $\left(\mathbb{Z}^k\right)^n$, which implies that

$f : G^n \to \mathbb{C}$ is a solution of (2) on G^n. The converse follows in the same manner, hence our theorem is proved. □

Theorem 8. *If $n, k \geq$ are integers, then any polynomial solution $f : (\mathbb{Z}^k)^n \to \mathbb{C}$ of (1) satisfies*

$$\partial_i^\alpha f = 0$$

for any k-dimensional multi-index α with $|\alpha| = 2n$ $(i = 1, 2, \dots, n)$.

PROOF. We prove the statement for $i = 1$. First we show by induction on l that

$$\partial_1^\alpha \left(\sum_{1<i_1<\cdots<i_{n-l}\leq n} \partial_{i_1}^{e_p+e_q} \dots \partial_{i_{n-l}}^{e_p+e_q} \right) f = 0 \tag{5}$$

holds for any k-dimensional multi-index α with $|\alpha| = 2l$ $(l = 1, 2, \dots)$. Here e_p, respectively e_q denotes the k-dimensional multi-index, whose p-th, respectively q-th component is 1 and all the other components are 0. If $l = 1$ and $\beta_1 = \beta_2 = \cdots = \beta_n = e_p + e_q$, then

$$\partial_1^{e_p+e_q} \left(\sum_{1<i_1<\cdots<i_{n-1}\leq n} \partial_{i_1}^{e_p+e_q} \dots \partial_{i_{n-1}}^{e_p+e_q} \right) f = Q_n(\beta_1, \beta_2, \dots, \beta_n) f = 0,$$

by Theorem 4, which is (5) for $l = 1$. Suppose that (5) holds for l and let β be a k-dimensional multi-index with $|\beta| = 2(l+1)$. Then β has the form $\beta = \alpha + e_p + e_q$ with some $1 \leq p, q \leq k$, where $|\alpha| = 2l$. By Theorem 4 we have

$$Q_n(\beta_1, \beta_2, \dots, \beta_{n-l}) f = 0 \tag{6}$$

with $\beta_1 = \beta_2 = \cdots = \beta_{n-l} = e_p + e_q$, that is,

$$\partial_1^{e_p+e_q} \left(\sum_{1<i_1<\cdots<i_{n-l-1}\leq n} \partial_{i_1}^{e_p+e_q} \dots \partial_{i_{n-l-1}}^{e_p+e_q} \right) f$$
$$+ \left(\sum_{1<i_1<\cdots<i_{n-l}\leq n} \partial_{i_1}^{e_p+e_q} \dots \partial_{i_{n-l}}^{e_p+e_q} \right) f = 0.$$

(Here in the first term we collected all differential operators which contain $\partial_1^{e_p+e_q}$ as a factor, and the second term is the remaining part.) Applying

∂_1^α on both sides we have

$$\begin{aligned} 0 &= \partial_1^\beta \Bigg(\sum_{1<i_1<\cdots<i_{n-l-1}\leq n} \partial_{i_1}^{e_p+e_q} \dots \partial_{i_{n-l-1}}^{e_p+e_q} \Bigg) f \\ &\quad + \partial_1^\alpha \Bigg(\sum_{1<i_1<\cdots<i_{n-l}\leq n} \partial_{i_1}^{e_p+e_q} \dots \partial_{i_{n-l}}^{e_p+e_q} \Bigg) f \\ &= \partial_1^\beta \Bigg(\sum_{1<i_1<\cdots<i_{n-l-1}\leq n} \partial_{i_1}^{e_p+e_q} \dots \partial_{i_{n-l-1}}^{e_p+e_q} \Bigg) f, \end{aligned}$$

hence (5) holds with β in place of α. Finally, by the substitution $l = n$ in (5) we have our statement. □

Theorem 9. *Let G be an Abelian group and n a positive integer. Then any complex valued solution of* (1) *or* (2) *is a polynomial of degree at most $2n-1$ in each variable.*

Proof. By Theorem 7 equations (1) and (2) are equivalent on any Abelian group hence it is enough to deal with (1). Let k be any positive integer and let $f : \left(\mathbb{Z}^k\right)^n \to \mathbb{C}$ be a polynomial solution of (1), then by Theorem 8

$$\partial_i^\alpha f = 0$$

holds for any k-dimensional multi-index α with $|\alpha| = 2n$ and for $i = 1, 2, \dots, n$. If $\Delta_{i,t}$ denotes the partial difference operator with increment t in the i-th variable on functions $f : \left(\mathbb{Z}^k\right)^n \to \mathbb{C}$ for $i = 1, 2, \dots, n$, then this implies that

$$\Delta_{i,t}^{2n} f(z_1, z_2, \dots, z_n) = 0 \tag{7}$$

holds for any $t, z_1, z_2, \dots, z_n$ in $\mathbb{Z}^k$, that is, the function

$$z \mapsto f(z_1, z_2, \dots, z_{i-1}, z, z_{i+1}, \dots, z_n)$$

is a polynomial of degree at most $2n-1$ for fixed $z_1, z_2, \dots, z_{i-1}, z, z_{i+1}, \dots$ $\dots, z_n$ in $\mathbb{Z}^k$, and for $i = 1, 2, \dots, n$. In other words, f is a polynomial of degree at most $2n-1$ in each variable. In the proof of Theorem 7 we have seen that any exponential monomial solution of (1) is actually a polynomial, and now we know that it is of degree at most $2n-1$ in each variable. It is clear that any pointwise limit of nets of such functions is a

also polynomial of degree at most $2n-1$ in each variable, hence by the result of Lefranc in [2] on spectral synthesis for $\mathbb{Z}^k$ any complex valued solution of (1) on $\left(\mathbb{Z}^k\right)^n$ is a polynomial of degree at most $2n-1$ in each variable.

The case of finitely generated groups can be treated exactly in the same way, like in Theorem 7. Let G be any finitely generated Abelian group and let $\varphi : \mathbb{Z}^k \to G$ be a surjective homomorphism, where k is some positive integer. The function $\Phi : \left(\mathbb{Z}^k\right)^n \to G^n$ defined by

$$\Phi(z_1, z_2, \ldots, z_n) = \big(\varphi(z_1), \varphi(z_2), \ldots, \varphi(z_n)\big)$$

for $z_1, z_2, \ldots, z_n$ in $\mathbb{Z}^k$ is a surjective homomorphism. If $f : G^n \to \mathbb{C}$ is a solution of (1) on G^n, then $f \circ \Phi : \left(\mathbb{Z}^k\right)^n \to \mathbb{C}$ is obviously a solution of (1) on $\left(\mathbb{Z}^k\right)^n$, hence $f \circ \Phi$ satisfies

$$\Delta_{i,t}^{2n}(f \circ \Phi)(z_1, z_2, \ldots, z_n) = 0$$

for all $i = 1, 2, \ldots, n$ and $t, z_1, z_2, \ldots, z_n$ in $\mathbb{Z}^k$. This implies that f satisfies

$$\Delta_{i,s}^{2n} f(x_1, x_2, \ldots, x_n) = 0$$

for all $i = 1, 2, \ldots, n$ and $s, x_1, x_2, \ldots, x_n$ in G, that is, f is a polynomial of degree at most $2n-1$ in each variable.

Finally, suppose that G is an arbitrary Abelian group. Let $\{G_\gamma\}$ be the (inductive) set of all finitely generated subgroups of G. For any solution $f : G^n \to \mathbb{C}$ of (1) the function f_γ, which is equal to f on G_γ and zero outside, is a solution of (1) on the finitely generated subgroup G_γ, hence f_γ is a polynomial of degree at most $2n-1$ in each variable on G_γ. As f is the pointwise limit of the net $\{f_\gamma\}$ we infer, that f is a polynomial of degree at most $2n-1$ in each variable, and our theorem is proved. □

References

[1] R. J. Elliot, Two notes on spectral synthesis for discrete Abelian groups, *Proc. Cambridge Phil. Soc.* **61** (1965), 617–620.

[2] M. Lefranc, L'analyse harmonique dans $\mathbb{Z}^n$, *C. R. Acad. Sci. Paris* **246** (1958), 1951–1953.

[3] J. A. MacDougall and K. Ozeki, The octahedron equation implies the cube equation: an elementary proof, *Aequationes Math.* **31** (1986), 243–246.

[4] L. Sweet, On the generalized cube and octahedron functional equation, *Aequationes Math.* **22** (1981), 29–38.

[5] L. Székelyhidi, Convolution type functional equations on topological Abelian groups, *World Scientific Publishing Co. Inc., Teaneck, NJ*, 1991.

László Székelyhidi
institute of mathematics and informatics
university of debrecen
H-4010 debrecen, p.o. box 12
hungary

E-mail: szekely@math.klte.hu

(Received November 25, 1999; revised September 26, 2002)