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Higher order set-valued iterative roots of bijections

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Abstract. In the paper multifunctions are considered as generalized iterative roots. We give a construction of a set-valued iterative root of an arbitrary bijection which is single-valued whenever its ordinary iterative root exists. Moreover, this construction is universal which means that every iterative root can be obtained in this way. A version of Łojasiewicz's Theorem which gives necessary and sufficient conditions for a bijection to have an iterative root is also proved.

1. Introduction

In 1951 S. Łojasiewicz in his paper [5] gave a complete solution of the following problem. Let X be a nonempty set, $r \in \mathbb{N}$ and $f : X \to X$ be a given bijection. Find all functions $g : X \to X$ such that the r-th iterate of g is equal to f, i.e.

$$(1) \qquad g^r = f.$$

Any function which fullfils (1) is called an iterative root of order r of f or simply an r-th iterative root of f.

In the case of an arbitrary function $f : X \to X$ the problem of the general solution of (1) was solved for $r = 2$ in 1950 by R. Isaacs [2] and for arbitrary $n \in \mathbb{N}$ in 1978 by G. Zimmermann [8] (for all these results see also [7]).

Our purpose is to consider multifunctions as generalized iterative roots. The results of this article generalize those obtained in our previous paper [6] for $r = 2$. We have made some changes in the construction presented there so that it works in the case of an arbitrary r.

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Let us begin with some definitions and notations. Put

$$Z_k := \begin{cases} \mathbb{Z} & \text{for } k = 0, \\ \{0, \dots, k-1\} & \text{for } k \in \mathbb{N}. \end{cases}$$

If $p, q \in \mathbb{Z}$ then $p = q \pmod{k}$ means that k divides $p - q$ if $k \in \mathbb{N}$ and $p = q$ if $k = 0$. In the sequel, given positive integers a and b we will write $\gcd(a, b)$ for the greatest common divisor of a and b. Moreover, we put $\gcd(a, 0) := a$ and $\gcd(0, a) := a$ for every $a \in \mathbb{N}$.

The following definitions and simple facts can be found for instance in [3].

Let $f : X \to X$ be a self-mapping of a non-empty set X. By an *orbit* we mean the equivalence class under the Kuratowski relation $\sim_f$ defined by

$$x \sim_f y \iff \bigvee_{m,n \in \mathbb{N}_0} f^m(x) = f^n(y)$$

for $x, y \in X$.

The set of all orbits will be denoted by $\mathrm{Orb}(f)$.

If f is a bijection then every orbit is a k-cycle for some $k \in \mathbb{N}_0$, i.e. the set $\{x_i : i \in Z_k\}$ of distinct elements of X such that $f(x_i) = x_{i+1}$ for every $i \in Z_k$ if $k = 0$ and $f(x_0) = x_1, \dots, f(x_{k-2}) = x_{k-1}, f(x_{k-1}) = x_0$ if $k \in \mathbb{N}$. (In the literature a 0-cycle is called also a $\mathbb{Z}$-chain.) Observe also that if f is a bijection then the orbit of an $x \in X$ is simply the set $\{f^n(x) : n \in \mathbb{Z}\}$.

We denote by $\mathcal{L}_k(f)$, $k \in \mathbb{N}_0$, the set of all k-cycles of f and by $l_k \in \mathbb{N}_0 \cup \{\infty\}$ the number of elements of $\mathcal{L}_k(f)$.

We will make use of the following Łojasiewicz's Theorem ([5], also [7, Sc. 2.1], [3, Sc. XV.2] and [4, Sc. 11.1]).

Proposition. *A bijection f has an iterative root of order $r \in \mathbb{N}$ if and only if for every $k \in \mathbb{N}_0$ either $l_k = \infty$ or l_k is divisible by d_k where $d_0 = r$ and $d_k = \frac{r}{r_k}, k \in \mathbb{N}$, with r_k denoting the greatest divisor of r relatively prime to k.*

Now we give a basic definition.

Definition. Let $f : X \to X$ be a function and $r \geq 2$ be an integer. A multifunction $G : X \to 2^X$ is called a *set-valued iterative root of f of order r* if

$$f(x) \in G^r(x) \quad \text{for } x \in X,$$

where

$$G^0(x) := \{x\}$$

and

$$G^{k+1}(x) := \bigcup_{y \in G^k(x)} G(y) \quad \text{for } k \in \mathbb{N}_0.$$

Observe that every set-valued iterative root of f whose values are singletons can be identified with a function which is an iterative root of f in normal sense. The following simple example shows that if we require nothing more in the definition above then it is not hard to find set-valued roots of an arbitrary function.

Example. Let f be a self-mapping of X and $r \geq 2$ be an integer. The function $G : X \to 2^X$ defined by $G(x) := \{x, f(x)\}$ is an iterative root of order r of the function f. Indeed, for $x \in X$ we have $G^r(x) = \{x, f(x), f^2(x), \ldots, f^r(x)\}$, so $f(x) \in G^r(x)$.

Also, given an $x_0 \in X$, we can define another set-valued iterative root of f by

$$H(x) := \{x_0\} \quad \text{for } x \in X \setminus \{x_0\};\ H(x_0) := X.$$

Then $f(x) \in X = H^r(x)$ for every $x \in X$.

Unless f is the identity function the multifunction G and H can never coincide with iterative roots of f (if they exist at all). Here we are interested in multifunctions with the possibly smallest values. We present a construction of a set-valued iterative root which, in particular, is single-valued for bijections having iterative root.

Let $\mathcal{I}_r(f)$ denote the set of all functions Φ mapping $\mathrm{Orb}(f)$ onto itself and satisfying the conditions

$$\Phi(\mathcal{L}_k(f)) = \mathcal{L}_k(f) \quad \text{for every } k \in \mathbb{N}_0$$

and

$$\Phi^r = \mathrm{id}_{\mathrm{Orb}(f)}\,.$$

Since $\mathrm{id}_{\mathrm{Orb}(f)} \in \mathcal{I}_r(f)$ this set is non-empty.

Given a $\Phi \in \mathcal{I}_r(f)$ and a positive integer n denote by $\mathrm{Per}(\Phi, n)$ the set

$$\left\{C \in \mathrm{Orb}(f) : \Phi^n(C) = C \text{ and } \Phi^k(C) \neq C \text{ for } k \in Z_n \setminus \{0\}\right\}.$$

Observe that if $\Phi \in \mathcal{I}_r(f)$, $n \in \mathbb{N}$, and $\mathrm{Per}(\Phi, n) \neq \emptyset$ then $n \mid r$.

Finally, we will use the symbol $\mathcal{I}_r^*(f)$ to denote the (possibly empty) set of all $\Phi \in \mathcal{I}_r(f)$ satisfying the condition

$$\bigwedge_{k \in \mathbb{N}_0} \bigwedge_{n \in \mathbb{N}} \left(\mathcal{L}_k(f) \cap \mathrm{Per}(\Phi, n) \neq \emptyset \Rightarrow \gcd\left(\frac{r}{n}, k\right) = 1\right). \tag{2}$$

Let $f : X \to X$ be a bijection, $r \geq 2$ an integer, and $\Phi \in \mathcal{I}_r(f)$. Clearly Φ decomposes the set $\mathrm{Orb}(f)$ into at most r elementary distinct families of the form $\{C, \Phi(C), \ldots, \Phi^{r-1}(C)\}$.

Using the axiom of choice we can find a subfamily of $\mathrm{Orb}(f)$ having exactly one element in common with each of the set $\{C, \Phi(C), \ldots, \Phi^{r-1}(C)\}$. Denote by $\mathcal{S}(f, \Phi)$ the (non-void) class of all such subfamilies of $\mathrm{Orb}(f)$.

Making use of the axiom of choice again we can also find a function $a : \mathrm{Orb}(f) \to X$ (a selector) such that $a(C) \in C$; we will write a_C instead of $a(C)$. Define $\mathcal{A}(f)$ as the set of all such selectors.

Now fix $\Phi \in \mathcal{I}_r(f), \mathcal{P} \in \mathcal{S}(f, \Phi)$, and $a \in \mathcal{A}(f)$. For $x \in X$ let $C \in \mathrm{Orb}(f)$ be the orbit containing x. Then $C \in \mathcal{L}_k(f) \cap \mathrm{Per}(\Phi, n)$ for some $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$ such that $n \mid r$. Clearly C, as the orbit containing a_C, is of the form

$$C = \{f^i(a_C) : i \in Z_k\}.$$

Thus there is a unique $i \in Z_k$ such that $x = f^i(a_C)$. We will construct a key multifunction $G_{\Phi,\mathcal{P},a} : X \to 2^X$ (for the simplicity we will often omit the indices). If $C \notin \mathcal{P}$ then we put

$$G_{\Phi,\mathcal{P},a}(x) := \{f^i(a_{\Phi(C)})\}.$$

In the case $C \in \mathcal{P}$ we put

$$G_{\Phi,\mathcal{P},a}(x) := \{f^i(a_{\Phi(C)}), f^{i+1}(a_{\Phi(C)})\}$$

if $\gcd\left(\frac{r}{n}, k\right) \neq 1$ and

$$G_{\Phi,\mathcal{P},a}(x) := \{f^{i+\alpha}(a_{\Phi(C)})\}$$

whenever $\gcd\left(\frac{r}{n}, k\right) = 1$, where $\alpha \in Z_k$ is the unique solution (cf. [1], Ch. 2.6) of the equation

$$\frac{r}{n}\alpha = 1 \pmod{k}. \tag{3}$$

(Observe that if $k = 0$ then (3) means $n = r$ and $\alpha = 1$.)

In what follows $f : X \to X$ is an arbitrarily fixed bijection and $r \geq 2$ is an integer.

Theorem 1. *For every $\Phi \in \mathcal{I}_r(f)$, $\mathcal{P} \in \mathcal{S}(f, \Phi)$ and $a \in \mathcal{A}(f)$ the function $G_{\Phi,\mathcal{P},a}$ is a set-valued iterative root of order r of the function f.*

Proof. Fix $\Phi \in \mathcal{I}_r(f), \mathcal{P} \in \mathcal{S}(f, \Phi), a \in \mathcal{A}(f)$ and a point $x \in X$. Let $C \in \mathrm{Orb}(f)$ be an orbit that contains x. Then there exist numbers $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$ such that $C \in \mathcal{L}_k(f) \cap \mathrm{Per}(\Phi, n)$ and $n \mid r$. Moreover, putting

$$a_j := a_{\Phi^j(C)}$$

for $j \in Z_n$, we have $a_j \in \Phi^j(C)$ and

$$\Phi^j(C) = \{f^i(a_j) : i \in Z_k\} \quad \text{for } j \in Z_n$$

and $x = f^i(a_0)$ for some $i \in Z_k$. Put $s := \frac{r}{n}$.

Assume that $\gcd\left(\frac{r}{n}, k\right) \neq 1$ and let $j \in Z_n$ be such that $\Phi^j(C) \in \mathcal{P}$. If $p \in \mathbb{N}$ and $p \geq n$ then, by the definition of G,

$$\begin{aligned}
G^p(f^i(a_0)) &= G^{p-1}(G(f^i(a_0))) = G^{p-1}(f^i(a_1)) \\
&= G^{p-2}(f^i(a_2)) = \cdots = G^{p-j}(f^i(a_j)) \\
&= G^{p-j-1}(f^i(a_{j+1})) \cup G^{p-j-1}(f^{i+1}(a_{j+1})) \\
&= G^{p-j-2}(f^i(a_{j+2})) \cup G^{p-j-2}(f^{i+1}(a_{j+2})) \\
&= \cdots = G^{p-n+1}(f^i(a_{n-1})) \cup G^{p-n+1}(f^{i+1}(a_{n-1})) \\
&= G^{p-n}(f^i(a_0)) \cup G^{p-n}(f^{i+1}(a_0)).
\end{aligned}$$

Therefore

$$\begin{aligned}
G^r(x) &= G^{ns}(f^i(a_0)) = G^{n(s-1)}(f^i(a_0)) \cup G^{n(s-1)}(f^{i+1}(a_0)) \\
&= G^{n(s-2)}(f^i(a_0)) \cup G^{n(s-2)}(f^{i+1}(a_0)) \cup G^{n(s-2)}(f^{i+2}(a_0)) \\
&= \cdots = G^{n(s-s)}(f^i(a_0)) \cup \cdots \cup G^{n(s-s)}(f^{i+s}(a_0)) \\
&= \{f^i(a_0), \ldots, f^{i+s}(a_0)\} \ni f^{i+1}(a_0)
\end{aligned}$$

and

$$f(x) = f(f^i(a_0)) = f^{i+1}(a_0) \in G^r(x).$$

Now assume that $\gcd\left(\frac{r}{n}, k\right) = 1$. Let $\alpha \in Z_k$ be the unique solution of equation (3). Let $j \in Z_n$ be such that $\Phi^j(C) \in \mathcal{P}$. If $p \in \mathbb{N}$ and $p \geq n$ then

$$\begin{aligned} G^p(f^i(a_0)) &= G^{p-1}(G(f^i(a_0))) = G^{p-1}(f^i(a_1)) \\ &= \cdots = G^{p-j}(f^i(a_j)) = G^{p-j-1}(G(f^i(a_j))) \\ &= G^{p-j-1}(f^{i+\alpha}(a_{j+1})) = G^{p-j-2}(f^{i+\alpha}(a_{j+2})) \\ &= \cdots = G^{p-n+1}(f^{i+\alpha}(a_{n-1})) = G^{p-n}(f^{i+\alpha}(a_0)). \end{aligned}$$

Consequently, since $s\alpha = 1 \pmod{k}$ we have

$$\begin{aligned} G^r(x) &= G^{ns}(f^i(a_0)) = G^{n(s-1)}(f^{i+\alpha}(a_0)) \\ &= \cdots = G^{n(s-s)}(f^{i+s\alpha}(a_0)) \\ &= \{f^{i+1}(a_0)\} \ni f^{i+1}(a_0) \end{aligned}$$

and

$$f(x) = f(f^i(a_0)) = f^{i+1}(a_0) \in G^r(x). \qquad \square$$

Now we present a result which shows that the given construction is, in a sense, universal, that is every iterative root (if it exists) can be obtain by using it. At first, however, we will observe the following simple fact.

Lemma. *Let $g : X \to X$ be an iterative root of order r of the function f. If $C \in \mathrm{Orb}(f)$ and $g(C) = C$ then $C \in \mathrm{Orb}(g)$.*

Proof. If C_0 is the orbit of g containing a point of $C \in \mathrm{Orb}(f)$ then $C \subset C_0$ since $g^r = f$ and $C_0 \subset C$ by the equality $g(C) = C$ and the bijectivity of g. $\square$

Theorem 2. *For every iterative root $g : X \to X$ of order r of the function f there exists $\Phi \in \mathcal{I}_r^*(f)$ such that for every $\mathcal{P} \in \mathcal{S}(f, \Phi)$ there is an $a \in \mathcal{A}(f)$ for which*

$$G_{\Phi,\mathcal{P},a}(x) = \{g(x)\} \quad \text{for } x \in X.$$

Proof. Let $g : X \to X$ be an iterative root of order r of f. For every $x \in X$ denote by $C_f(x)$ the orbit of f containing x. Since f and g commute we have

$$g(f^i(x)) = f^i(g(x)) \quad \text{for } x \in X \text{ and } i \in \mathbb{Z},$$

whence

$$g(C_f(x)) = C_f(g(x)) \quad \text{for } x \in X. \tag{4}$$

Thus the formula

$$\Phi(C) := g(C)$$

defines a function $\Phi : \mathrm{Orb}(f) \to \mathrm{Orb}(f)$.

If $x \in X$ then, by (4),

$$\begin{aligned}\Phi^r(C_f(x)) &= \Phi^{r-1}(\Phi(C_f(x))) = \Phi^{r-1}(g(C_f(x)))\\ &= \Phi^{r-1}(C_f(g(x))) = \cdots = \Phi^{r-r}(C_f(g^r(x)))\\ &= C_f(f(x)) = C_f(x).\end{aligned}$$

This means that $\Phi^r = \mathrm{id}_{\mathrm{Orb}(f)}$. Moreover, since g is a bijection, we have $\operatorname{card} g(C) = \operatorname{card} C$ for every $C \in \mathrm{Orb}(f)$, so

$$\Phi(\mathcal{L}_k(f)) \subset \mathcal{L}_k(f) \quad \text{for } k \in \mathbb{N}_0.$$

Therefore

$$\mathcal{L}_k(f) = \Phi^r(\mathcal{L}_k(f)) \subset \cdots \subset \Phi(\mathcal{L}_k(f)) \subset \mathcal{L}_k(f),$$

that is $\Phi(\mathcal{L}_k(f)) = \mathcal{L}_k(f)$ for $k \in \mathbb{N}_0$. Consequently, $\Phi \in \mathcal{I}_r(f)$. We will show that $\Phi \in \mathcal{I}_r^*(f)$. Fix $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$ such that $\mathcal{L}_k(f) \cap \mathrm{Per}(\Phi, n) \neq \emptyset$ and take any $C \in \mathcal{L}_k(f) \cap \mathrm{Per}(\Phi, n)$. Clearly $n \mid r$. Put $j := \gcd\left(\frac{r}{n}, k\right)$ and fix an $x_0 \in C$. Since $C \in \mathrm{Per}(\Phi, n)$ it follows from the definition of Φ that $g^n(C) = C$. Thus $g^n(x_0) \in C$ and $g^n(x_0) = f^i(x_0)$ for an $i \in Z_k$. Then

$$f(x_0) = g^r(x_0) = g^{n\frac{r}{n}}(x_0) = f^{i\frac{r}{n}}(x_0),$$

whence $i\frac{r}{n} = 1 \pmod{k}$. Taking $l, p, q \in \mathbb{Z}$ such that $i\frac{r}{n} - 1 = lk$, $\frac{r}{n} = pj$, and $k = qj$ we obtain $ipj - 1 = lqj$, i.e. $j(ip - lq) = 1$, whence $j = 1$. Therefore $\gcd\left(\frac{r}{n}, k\right) = 1$. Consequently, $\Phi \in \mathcal{I}_r^*(f)$.

Now fix an arbitrary $\mathcal{P} \in \mathcal{S}(f, \Phi)$. We shall define a selector $a : \mathrm{Orb}(f) \to X$.

To this aim let $C \in \mathcal{P}$. Then $C \in \mathcal{L}_k(f) \cap \mathrm{Per}(\Phi, n)$ for some $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$ such that $n \mid r$. Since $\Phi \in \mathcal{I}_r^*(f)$ we have $\gcd\left(\frac{r}{n}, k\right) = 1$. So there is a unique solution $\alpha \in Z_k$ of equation (3). Let a_C be an arbitrary element of C and put

$$a_{\Phi^j(C)} := g^j(f^{k-\alpha}(a_C)) \quad \text{for } j \in Z_n \setminus \{0\}.$$

If $j \in Z_n$ then $a_{\Phi^j(C)} \in g^j(C) = \Phi^j(C)$. Observe also that $g^n(C) = \Phi^n(C) = C$, so due to the Lemma, applied for the root g^n of order $\frac{r}{n}$ of f, we have $C \in \mathcal{L}_k(g^n)$ and $g^{nk}(a_C) = a_C$. Therefore, since $n - r\alpha = 0 \pmod{nk}$ by (3),

$$\begin{aligned} g(a_{\Phi^{n-1}(C)}) &= g(g^{n-1}(f^{k-\alpha}(a_C))) = g^n(f^{k-\alpha}(a_C)) \\ &= g^n(g^{r(k-\alpha)}(a_C)) = g^{n-r\alpha+rk}(a_C) \\ &= g^{rk}(a_C) = f^k(a_C) = a_C, \end{aligned}$$

that is

$$g(a_{\Phi^{n-1}(C)}) = a_C. \tag{5}$$

Moreover, we have

$$\{C, \Phi(C), \ldots, \Phi^{n-1}(C)\} = \{C, \Phi(C), \ldots, \Phi^{n-1}(C), \ldots, \Phi^{r-1}(C)\}.$$

Thus, since the family $\mathrm{Orb}(f)$ is the sum of the disjoint subfamilies of the form

$$\{C, \Phi(C), \ldots, \Phi^{r-1}(C)\}$$

where C runs over $\mathcal{P}$, the above procedure defines a function $a : \mathrm{Orb}(f) \to X$ such that $a_C = a(C) \in C$ for every $C \in \mathrm{Orb}(f)$.

Fix an $x \in X$. Then there exist an orbit $C \in \mathcal{P}$ and numbers $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$ such that $C \in \mathcal{L}_k(f) \cap \mathrm{Per}(\Phi, n)$ and $x \in \Phi^j(C)$ for some $j \in Z_n$. Put $a_j := a_{\Phi^j(C)}$ for $j \in Z_n$. Since $\Phi^j(C) \in \mathcal{L}_k(f)$ is the orbit of f containing a_j we have

$$\Phi^j(C) = \{f^i(a_j) : i \in Z_k\}$$

Choose an $i \in Z_k$ such that $x = f^i(a_j)$. If $j = 0$ then, by the definitions of G and a, we obtain

$$\begin{aligned} G(x) &= G(f^i(a_0)) = \{f^{i+\alpha}(a_1)\} = \{f^{i+\alpha}(g(f^{k-\alpha}(a_0)))\} \\ &= \{g(f^i(a_0))\} = \{g(x)\}. \end{aligned}$$

If $j \in Z_k \setminus \{0, n-1\}$ then we have

$$\begin{aligned} G(x) &= G(f^i(a_j)) = \{f^i(a_{j+1})\} = \{f^i(g(a_j))\} \\ &= \{g(f^i(a_j))\} = \{g(x)\}. \end{aligned}$$

Finally, if $j = n-1$ then, by (5), we get

$$\begin{aligned} G(x) &= G(f^i(a_{n-1})) = \{f^i(a_0)\} = \{f^i(g(a_{n-1}))\} \\ &= \{g(f^i(a_{n-1}))\} = \{g(x)\}. \end{aligned}$$

This shows that $G(x) = \{g(x)\}$ for every $x \in X$. □

Theorem 3. *Let* $\Phi \in \mathcal{I}_r(f)$. *Then* $\Phi \in \mathcal{I}_r^*(f)$ *if and only if the function* $G_{\Phi,\mathcal{P},a}$ *is single-valued for every* $\mathcal{P} \in \mathcal{S}(f,\Phi)$ *and* $a \in \mathcal{A}(f)$.

PROOF. Assume that $\Phi \in \mathcal{I}_r^*(f)$. Fix $\mathcal{P} \in \mathcal{S}(f,\Phi)$ and $a \in \mathcal{A}(f)$. If $C \in \mathrm{Orb}(f)$ then $C \in \mathcal{L}_k(f) \cap \mathrm{Per}(\Phi, n)$ for some $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Thus, since $\Phi \in \mathcal{I}_r^*(f)$, we have $\gcd\left(\frac{r}{n}, k\right) = 1$. Consequently, it follows from the definition of $G_{\Phi,\mathcal{P},a}$ that its values on C are singletons. This means that $G_{\Phi,\mathcal{P},a}$ is single-valued.

Now assume that for every $\mathcal{P} \in \mathcal{S}(f,\Phi)$ and $a \in \mathcal{A}(f)$ the function $G_{\Phi,\mathcal{P},a}$ is single-valued. Fix numbers $k \in \mathbb{N}_0$, $n \in \mathbb{N}$ and an orbit $C \in \mathcal{L}_k(f) \cap \mathrm{Per}(\Phi, n)$. Choose a $\mathcal{P} \in \mathcal{S}(f,\Phi)$ in such a manner that $C \in \mathcal{P}$ and take an arbitrary $a \in \mathcal{A}(f)$. Since all values of $G_{\Phi,\mathcal{P},a}$ are singletons it follows from the definition of $G_{\Phi,\mathcal{P},a}$ that $\gcd\left(\frac{r}{n}, k\right) = 1$. Therefore $\Phi \in \mathcal{I}_r^*(f)$. □

The final result gives a necessary and sufficient condition for a bijection to have a real iterative root of given order, a little bit different from that proposed by Łojasiewicz, cf. the Proposition. This is a simple consequence of Theorems 2 and 3.

Theorem 4. *The bijection f has an iterative root of order r if and only if the set $\mathcal{I}_r^*(f)$ is non-empty.*

Proof. Assume that f has an iterative root of order r. According to Theorem 2 there exist $\Phi \in \mathcal{I}_r^*(f)$, $\mathcal{P} \in \mathcal{S}(f, \Phi)$ and $a \in \mathcal{A}(f)$ such that

$$G_{\Phi,\mathcal{P},a}(x) = \{g(x)\} \quad \text{for } x \in X.$$

In particular $\mathcal{I}_r^*(f) \neq \emptyset$.

Now assume that the set $\mathcal{I}_r^*(f)$ is non-empty and let $\Phi \in \mathcal{I}_r^*(f), \mathcal{P} \in \mathcal{S}(f, \Phi)$ and $a \in \mathcal{A}(f)$ be arbitrary. It follows from Theorem 3 that the function $G_{\Phi,\mathcal{P},a}$ is single-valued. Thus its only selection $g : X \to X$ is an iterative root of order r of f. □

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