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On powers of relational structures

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Abstract. We introduce a new operation of power of n-ary relational structures, which is carried by the corresponding set of homomorphisms. The introduced power combines two known powers of relational structures and all the three powers are discussed. In particular, a product of n-ary relational structures is found with respect to which the new power fulfills the first exponential law.

1. Introduction

It is well known that the category of n-ary relational structures with relational homomorphisms as morphisms does not have function spaces, i.e., well-behaved powers carried by the corresponding sets of homomorphisms. Therefore, the direct operations of product and power of n-ary relational structures do not fulfill the first exponential law. In the present paper, we introduce new operations of product and power of n-ary relational structures which satisfy the first exponential law. These new operations are obtained by modifying the direct product and the direct power in such a way that the carriers are preserved. More precisely, the power introduced is defined as the intersection of the direct power and another power of relational structures known from the literature. So, some properties of the new power follow from the properties of the two known powers, and vice versa.

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Note that, because of applications, the validity of the first exponential law, with respect to a given product, is the most important criterion for well-defined powers of mathematical structures. But we also discuss the second and third exponential laws for the introduced power. A number of examples illustrate the results.

2. Operations on relational structures

We denote by A^B the set of all mappings of a set B into a set A. It is obvious that $(A^B)^C \approx A^{B\times C}$ (where $\approx$ denotes the set equivalence) because there is a bijection $\varphi : (A^B)^C \to A^{B\times C}$, called *canonical*, and given by $\varphi(h)(b,c) = h(c)(b)$ whenever $h \in (A^B)^C$, $b \in B$ and $c \in C$.

Throughout the paper, n denotes a positive integer. By a *mono-n-ary relational structure* we understand a pair $\mathbf{A} = (A, r)$ where A is a set, the so-called *carrier* of $\mathbf{A}$, and r is an n-ary relation on A, i.e., $r \subseteq A^n$. Because only mono-n-ary relational structures will be discussed, the adjective "mono-" will be omitted. We shall write $r_{\mathbf{A}}$ instead of r to express the n-ary relational structure to which r belongs. In [Lo], the term n-dimensional structures is used for n-ary relational structures.

Let $\mathbf{A} = (A, r_{\mathbf{A}})$, $\mathbf{A'} = (A, r_{\mathbf{A'}})$ be n-ary relational structures (with the same carrier). Then we set

$$\mathbf{A} \cup \mathbf{A'} = (A, r_{\mathbf{A}} \cup r_{\mathbf{A'}}),$$

$$\mathbf{A} \cap \mathbf{A'} = (A, r_{\mathbf{A}} \cap r_{\mathbf{A'}}),$$

$$\mathbf{A} \subseteq \mathbf{A'} \text{ if and only if } r_{\mathbf{A}} \subseteq r_{\mathbf{A'}}.$$

Given a set A, we denote by e_A the n-ary relation $\{(a, a, \dots, a);\ a \in A\}$ on A and set $\mathbf{E}_A = (A, e_A)$. A relational structure $\mathbf{A}$ is called *reflexive* if $\mathbf{E}_A \subseteq \mathbf{A}$ holds.

For an arbitrary relational structure $\mathbf{A}$ with the carrier A, the *reflexive hull* of $\mathbf{A}$ is the n-ary relational structure $\overline{\mathbf{A}}$ defined by

$$\overline{\mathbf{A}} = \mathbf{A} \cup \mathbf{E}_A.$$

Let $\mathbf{A} = (A, r_{\mathbf{A}})$, $\mathbf{B} = (B, r_{\mathbf{B}})$ be n-ary relational structures. In accordance with [No], the *direct sum* of $\mathbf{A}$ and $\mathbf{B}$, defined whenever $A \cap B = \emptyset$, is the relational structure $\mathbf{A} + \mathbf{B}$ given by

$$\mathbf{A} + \mathbf{B} = (A \cup B, r_{\mathbf{A}} \cup r_{\mathbf{B}}),$$

and the *direct product* of $\mathbf{A}$ and $\mathbf{B}$ is the relational structure $\mathbf{A} \times \mathbf{B}$ given by

$$\mathbf{A} \times \mathbf{B} = (A \times B, r_{\mathbf{A}\times\mathbf{B}})$$

where $r_{\mathbf{A}\times\mathbf{B}} = \{(a_1, b_1), \dots, (a_n, b_n)); \ (a_1, \dots, a_n) \in r_{\mathbf{A}}, \ (b_1, \dots, b_n) \in r_{\mathbf{B}}\}$.

Remark 1. a) In the category of n-ary relational structures with homomorphisms as morphisms, the direct sum and product are categorical sum and product, respectively (but not vice versa). The direct product has been introduced also in [Lo]. On the other hand, the cardinal sum from [Lo] coincides with the categorical sum and so is more general than the direct sum.

b) In the case of ordered sets, the direct sum and direct product coincide with the well-known cardinal sum and cardinal product in the sense of Birkhoff [Bi1].

The next operation $\mathbf{A} \circ \mathbf{B}$ is called the *combined product* of $\mathbf{A} = (A, r_{\mathbf{A}})$ and $\mathbf{B} = (B, r_{\mathbf{B}})$ and is defined by combining some of the above operations:

$$\mathbf{A} \circ \mathbf{B} = (\overline{\mathbf{A}} \times \mathbf{B}) \cup (\mathbf{A} \times \overline{\mathbf{B}}).$$

So, we have

$$\mathbf{A} \circ \mathbf{B} = (A \times B, r_{\mathbf{A}\circ\mathbf{B}})$$

where $r_{\mathbf{A}\circ\mathbf{B}} = r_{\mathbf{A}\times\mathbf{B}} \cup r_{\mathbf{E}_A\times\mathbf{B}} \cup r_{\mathbf{A}\times\mathbf{E}_B}$, i.e., for any $(a_1, b_1), \dots, (a_n, b_n) \in A \times B$ the condition $((a_1, b_1), \dots, (a_n, b_n)) \in r_{\mathbf{A}\circ\mathbf{B}}$ is satisfied if and only if one of the following three cases occurs:

(i) $(a_1, \dots, a_n) \in r_{\mathbf{A}}$ and $(b_1, \dots, b_n) \in r_{\mathbf{B}}$,

(ii) $a_1 = \dots = a_n$ and $(b_1, \dots, b_n) \in r_{\mathbf{B}}$,

(iii) $(a_1, \dots, a_n) \in r_{\mathbf{A}}$ and $b_1 = \dots = b_n$.

Remark 2. Clearly, $\mathbf{A} \circ \mathbf{B}$ is reflexive whenever $\mathbf{A}$ or $\mathbf{B}$ is reflexive. Further, since $\mathbf{A}\circ\mathbf{B} = (\mathbf{A}\times\mathbf{B}) \cup (\mathbf{E}_A\times\mathbf{B}) \cup (\mathbf{A}\times\mathbf{E}_B)$, we have $\mathbf{A}\circ\mathbf{B} = \mathbf{A}\times\mathbf{B}$ whenever $\mathbf{A}$ and $\mathbf{B}$ are reflexive.

A mapping $h : B \to A$ is said to be a *homomorphism* of $\mathbf{B} = (B, r_{\mathbf{B}})$ into $\mathbf{A} = (A, r_{\mathbf{A}})$ if $(b_1, \dots, b_n) \in r_{\mathbf{B}}$ implies $(h(b_1), \dots, h(b_n)) \in r_{\mathbf{A}}$. We denote by $\mathrm{Hom}(\mathbf{B}, \mathbf{A})$ the system of all homomorphisms of $\mathbf{B}$ into $\mathbf{A}$. If h is a bijective homomorphism of $\mathbf{B}$ into $\mathbf{A}$ such that, whenever $b_1, \dots, b_n \in B$, $(h(b_1), \dots, h(b_n)) \in r_{\mathbf{A}}$ implies $(b_1, \dots, b_n) \in r_{\mathbf{B}}$, then h is said to be an *isomorphism* of $\mathbf{B}$ onto $\mathbf{A}$. If there exists an isomorphism of $\mathbf{B}$ onto $\mathbf{A}$, we say that $\mathbf{B}$ and $\mathbf{A}$ are *isomorphic*, in symbols $\mathbf{B} \cong \mathbf{A}$.

Remark 3. a) The homomorphism of relational structures introduced in [Lo] is more special than that defined above.

b) It is obvious that for n-ary relational structures the distributive laws $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) \cong (\mathbf{A}\times\mathbf{B})+(\mathbf{A}\times\mathbf{C})$ and $\mathbf{A}\circ(\mathbf{B}+\mathbf{C}) \cong (\mathbf{A}\circ\mathbf{B})+(\mathbf{A}\circ\mathbf{C})$ are valid.

In what follows, we present mono-algebras (i.e., universal algebras with just one operation), called briefly *algebras*, as particular cases of relational structures. Indeed, if (A, o) is an algebra with an n-ary operation o, then $r = \{(x_1, \dots, x_n, o(x_1, \dots, x_n));\ (x_1, \dots, x_n) \in A^n\}$ is an $(n+1)$-ary relation on A. We get an $(n+1)$-ary relational structure $\mathbf{A} = (A, r_{\mathbf{A}})$ where $r_{\mathbf{A}} = r$. Clearly, it is possible to reconstruct the operation o from $\mathbf{A} = (A, r_{\mathbf{A}})$. Thus, we can treat algebras as relational structures (see also [Lo]).

Example 1. If (A, o) is an idempotent groupoid, i.e., a groupoid with $o(x, x) = x$ for any $x \in A$, then $(x, x, x) \in r_{\mathbf{A}}$ for any $x \in A$, so that the relational structure $\mathbf{A} = (A, r_{\mathbf{A}})$ is reflexive. Thus, if $\mathbf{A}$, $\mathbf{B}$ are ternary relational structures corresponding to idempotent groupoids, then $\mathbf{A}\circ\mathbf{B} = \mathbf{A}\times\mathbf{B}$ by Remark 2.

Example 2. Let (A, o) be the unary algebra with $A = \{1, 2\}$, $o(2) = o(1) = 1$. The corresponding binary relation $r_{\mathbf{A}}$ is defined by $r_{\mathbf{A}} = \{(1, 1), (2, 1)\}$. Thus, $\mathbf{A} = (A, r_{\mathbf{A}})$ is given and we can construct the relational structure $\mathbf{A}\circ\mathbf{A}$. This is the set $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ provided with a binary relation $r_{\mathbf{A}\circ\mathbf{A}}$.

We describe the relation $r_{\mathbf{A}\circ\mathbf{A}}$. First, it contains the pairs $((a_1, b_1), (a_2, b_2))$ where $(a_1, a_2) \in r_{\mathbf{A}}$, $(b_1, b_2) \in r_{\mathbf{A}}$, i.e., the pairs $((1, 1), (1, 1))$, $((1, 2), (1, 1))$, $((2, 1), (1, 1))$, $((2, 2), (1, 1))$. Second, it contains the pairs $((a_1, b_1), (a_2, b_2))$ where $a_1 = a_2 \in A$, $(b_1, b_2) \in r_{\mathbf{A}}$, hence $((1, 1), (1, 1))$, $((1, 2), (1, 1))$, $((2, 1), (2, 1))$, $((2, 2), (2, 1))$. The last pairs in $r_{\mathbf{A}\circ\mathbf{A}}$ are of the form $((a_1, b_1), (a_2, b_2))$ where $b_1 = b_2 \in A$, $(a_1, a_2) \in r_{\mathbf{A}}$. These are the pairs $((1, 1), (1, 1))$, $((1, 2), (1, 2))$, $((2, 1), (1, 1))$, $((2, 2), (1, 2))$. Thus, $r_{\mathbf{A}\circ\mathbf{A}} = \{((1, 1), (1, 1)), ((1, 2), (1, 1)), ((2, 1,)(1, 1)), ((2, 2), (1, 1)), ((2, 1), (2, 1)), ((2, 2), (2, 1)), ((1, 2), (1, 2)), ((2, 2), (1, 2))\}$.

It is easy to see that $\mathbf{A}\circ\mathbf{A}$ is not an algebra. For the reader's convenience, we present the graphs of $\mathbf{A}$ and of $\mathbf{A}\circ\mathbf{A}$.

Let $\mathbf{A} = (A, r_{\mathbf{A}})$ and $\mathbf{B} = (B, r_{\mathbf{B}})$ be n-ary relational structures. According to [No], we define the *direct power* $\mathbf{A}^\wedge\mathbf{B}$ of n-ary relational structures $\mathbf{A}$ and $\mathbf{B}$ by

$$\mathbf{A}^\wedge\mathbf{B} = (\mathrm{Hom}(\mathbf{B}, \mathbf{A}), r_{\mathbf{A}^\wedge\mathbf{B}})$$

where $r_{\mathbf{A}^\wedge\mathbf{B}} = \{(h_1, \dots, h_n) \in (\mathrm{Hom}(\mathbf{B}, \mathbf{A}))^n;\ (h_1(b), \dots, h_n(b)) \in r_{\mathbf{A}}$ whenever $b \in B\}$.

Next, we define the *structural power* $\mathbf{A}^\sim\mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$ by

$$\mathbf{A}^\sim\mathbf{B} = (\mathrm{Hom}(\mathbf{B}, \mathbf{A}), r_{\mathbf{A}^\sim\mathbf{B}})$$

where $r_{\mathbf{A}^\sim\mathbf{B}} = \{(h_1, \dots, h_n) \in (\mathrm{Hom}(\mathbf{B}, \mathbf{A}))^n;\ (h_1(b_1), \dots, h_n(b_n)) \in r_{\mathbf{A}}$ whenever $(b_1, \dots, b_n) \in r_{\mathbf{B}}\}$.

Remark 4. a) The direct power has been studied, for instance, in [No], [Sl1], [Sl3], and [Sl4]. For ordered sets the direct power coincides with the well-known Birkhoff's cardinal power [Bi2]. One can easily show that, given ordered sets $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, the following three laws hold for the direct power:

the *second exponential law* $(\mathbf{A} \times \mathbf{B})^\wedge\mathbf{C} \cong (\mathbf{A}^\wedge\mathbf{C}) \times (\mathbf{B}^\wedge\mathbf{C})$,
the *third exponential law* $\mathbf{A}^\wedge(\mathbf{B} + \mathbf{C}) \cong (\mathbf{A}^\wedge\mathbf{B}) \times (\mathbf{A}^\wedge\mathbf{C})$,
and the *first exponential law* $(\mathbf{A}^\wedge\mathbf{B})^\wedge\mathbf{C} \cong \mathbf{A}^\wedge(\mathbf{B} \times \mathbf{C})$.

The validity of the three laws follows from the fact that the category of ordered sets with homomorphisms as morphisms is cartesian closed and its function spaces are just the direct powers.

b) The structural power is also known from the literature. Namely, in the cartesian closed category of all reflexive n-ary relational structures with homomorphisms as morphisms, the function spaces are just the structural powers – see [Sl2]. It follows that, given reflexive n-ary relational structures $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, all the three exponential laws hold for the structural power, i.e.,

$$(\mathbf{A} \times \mathbf{B})^\sim \mathbf{C} \cong (\mathbf{A}^\sim \mathbf{C}) \times (\mathbf{B}^\sim \mathbf{C}),$$

$$\mathbf{A}^\sim (\mathbf{B} + \mathbf{C}) \cong (\mathbf{A}^\sim \mathbf{B}) \times (\mathbf{A}^\sim \mathbf{C}),$$

$$(\mathbf{A}^\sim \mathbf{B})^\sim \mathbf{C} \cong \mathbf{A}^\sim (\mathbf{B} \times \mathbf{C}).$$

We show in Lemmas 1 and 2 that the first two of these laws are valid for arbitrary n-ary relational structures, not only for the reflexive ones, and that the last law is valid even in the case when $\mathbf{A}$ is not reflexive.

c) In the definition of the structural power, when replacing $\mathrm{Hom}(\mathbf{A}, \mathbf{B})$ with B^A, we obtain the power studied in [Lo]. It is shown in [Lo] that also this power fulfills all the three exponential laws.

Let $\mathbf{A}$, $\mathbf{B}$ be n-ary relational structures. The *combined power* $\mathbf{A}^{\mathbf{B}}$ of $\mathbf{A}$ and $\mathbf{B}$ is also defined by combining some of the above operations:

$$\mathbf{A}^{\mathbf{B}} = (\mathbf{A}^\wedge \mathbf{B}) \cap (\mathbf{A}^\sim \mathbf{B}).$$

Thus, we have

$$\mathbf{A}^{\mathbf{B}} = (\mathrm{Hom}(\mathbf{B}, \mathbf{A}), r_{\mathbf{A}^{\mathbf{B}}})$$

where $r_{\mathbf{A}^{\mathbf{B}}} = \{(h_1, \dots, h_n) \in (\mathrm{Hom}(\mathbf{B}, \mathbf{A}))^n;\ (h_1(b_1), \dots, h_n(b_n)) \in r_{\mathbf{A}}$ whenever $(b_1, \dots, b_n) \in r_{\overline{\mathbf{B}}}\}$.

Remark 5. Let $\mathbf{A}, \mathbf{B}$ be n-ary relational structures. If $\mathbf{B}$ is reflexive, then clearly $\mathbf{A}^\sim \mathbf{B} \subseteq \mathbf{A}^\wedge \mathbf{B}$, and thus $\mathbf{A}^{\mathbf{B}} = \mathbf{A}^\sim \mathbf{B}$. Let us recall ([No]) that $\mathbf{A}$ is said to be *diagonal* if any $n \times n$-matrix $M = (a_{ij})$ over A has the property $(a_{11}, a_{22}, \dots, a_{nn}) \in r_{\mathbf{A}}$ whenever all rows and columns of M belong to $r_{\mathbf{A}}$. If $\mathbf{A}$ is diagonal, then $\mathbf{A}^\wedge \mathbf{B} \subseteq \mathbf{A}^\sim \mathbf{B}$ by [Sl2], and thus $\mathbf{A}^{\mathbf{B}} = \mathbf{A}^\wedge \mathbf{B}$. Consequently, if $\mathbf{A}$ is diagonal and $\mathbf{B}$ is reflexive, then $\mathbf{A}^\wedge \mathbf{B} = \mathbf{A}^\sim \mathbf{B} = \mathbf{A}^{\mathbf{B}}$.

In some of the following examples, we will present – among others – homomorphisms of algebras where these algebras will be treated as relational structures. Let (A, o) and (A', o') be algebras where o and o' are n-ary operations. Let $\mathbf{A} = (A, r_{\mathbf{A}})$ and $\mathbf{A'} = (A', r_{\mathbf{A'}})$ be the corresponding $(n+1)$-ary relational structures and let h be a mapping of the set A into A'.

If h is a homomorphism of the algebra (A, o) into (A', o') and if $(x_1, \ldots, x_n, x_{n+1}) \in r_{\mathbf{A}}$, then $x_{n+1} = o(x_1, \ldots, x_n)$ and $h(x_{n+1}) = o'(h(x_1), \ldots, h(x_n))$ which means $(h(x_1), \ldots, h(x_n), h(x_{n+1})) \in r_{\mathbf{A'}}$. Thus, h is a homomorphism of $(A, r_{\mathbf{A}})$ into $(A', r_{\mathbf{A'}})$.

Conversely, if h is a homomorphism of $(A, r_{\mathbf{A}})$ into $(A', r_{\mathbf{A'}})$ and $(x_1, \ldots, x_n) \in A^n$ is arbitrary, then $(x_1, \ldots, x_n, o(x_1, \ldots, x_n)) \in r_{\mathbf{A}}$ which implies $(h(x_1), \ldots, h(x_n), h(o(x_1, \ldots, x_n))) \in r_{\mathbf{A'}}$; this means $h(o(x_1, \ldots, x_n)) = o'(h(x_1), \ldots, h(x_n))$ by the definition of $r_{\mathbf{A'}}$. Hence h is a homomorphism of the algebra (A, o) into (A', o').

It follows that in the following examples we may replace homomorphisms of relational structures by homomorphisms of algebras. This simplifies the situation because we are able to find easily the homomorphisms of these algebras – see [Ny1], [Ny2].

Example 3. Let $\mathbf{A} = (A, r_{\mathbf{A}})$ be the relational system from Example 2. We construct $\mathbf{A}^{\mathbf{A}} = (\mathrm{Hom}(\mathbf{A}, \mathbf{A}), r_{\mathbf{A}^{\mathbf{A}}})$.

There are only two possibilities for $h \in \mathrm{Hom}(\mathbf{A}, \mathbf{A})$: either $h(2) = 2$ or $h(2) = 1$. The construction of homomorphisms of unary algebras provides $h(1) = 1$ in either case. It follows that $\mathrm{Hom}(\mathbf{A}, \mathbf{A}) = \{\alpha, \beta\}$ where $\alpha(1) = \alpha(2) = 1$, $\beta(1) = 1$, $\beta(2) = 2$.

Since $(\beta(2), \beta(2)) = (2, 2) \notin r_{\mathbf{A}}$, we obtain $(\beta, \beta) \notin r_{\mathbf{A}^{\mathbf{A}}}$. Similarly, $(\alpha(2), \beta(2)) = (1, 2) \notin r_{\mathbf{A}}$ entails $(\alpha, \beta) \notin r_{\mathbf{A}^{\mathbf{A}}}$. Furthermore, $(\beta(x), \alpha(y)) = (\beta(x), 1) \in r_{\mathbf{A}}$ for any $x \in A$, $y \in A$ which implies $(\beta, \alpha) \in r_{\mathbf{A}^{\mathbf{A}}}$. Finally, $(\alpha(x), \alpha(y)) = (1, 1) \in r_{\mathbf{A}}$ for any $x \in A$, $y \in A$ and, therefore, $(\alpha, \alpha) \in r_{\mathbf{A}^{\mathbf{A}}}$.

We have proved that $r_{\mathbf{A}^{\mathbf{A}}} = \{(\alpha, \alpha), (\beta, \alpha)\}$. Hence $\mathbf{A}^{\mathbf{A}} \cong \mathbf{A}$.

Example 4. Let (A, o) be the unary algebra from Example 2 and let (A', o') be the unary algebra with $A' = \{a, b, c, d, e\}$ and the operation o' given by the following table.

x	a	b	c	d	e
$o'(x)$	b	c	d	a	a

Then there are only two homomorphisms of the algebra $\mathbf{A'}$ into $\mathbf{A}$: $\alpha(x) = 1$ for any $x \in A'$ and $\beta(x) = 1$ for $x \in \{a, b, c, d\}$, $\beta(e) = 2$ (see [Ny1], [Ny2]). Let $\mathbf{A} = (A, r_{\mathbf{A}})$ and $\mathbf{A'} = (A', r_{A'})$ be the corresponding binary relational structures. We have $r_{\mathbf{A}} = \{(1,1), (2,1)\}$, $r_{\mathbf{A'}} = \{(a,b), (b,c), (c,d), (d,a), (e,a)\}$. The binary relation $r_{\mathbf{A}^{\mathbf{A'}}}$ is defined on the set $\mathrm{Hom}(\mathbf{A'}, \mathbf{A}) = \{\alpha, \beta\}$.

Clearly $(\alpha(x), \alpha(y)) = (1,1) \in r_{\mathbf{A}}$ for any $x \in A'$, $y \in A'$. It follows that $(\alpha, \alpha) \in r_{\mathbf{A}^{\mathbf{A'}}}$. Furthermore, $(\alpha(e), \beta(e)) = (1,2) \notin r_{\mathbf{A}}$ which implies $(\alpha, \beta) \notin r_{\mathbf{A}^{\mathbf{A'}}}$. Similarly, $(\beta(e), \beta(e)) = (2,2) \notin r_{\mathbf{A}}$ and, thus, $(\beta, \beta) \notin r_{\mathbf{A}^{\mathbf{A'}}}$. Finally, $(\beta(x), \alpha(y)) = (1,1) \in r_{\mathbf{A}}$ for $x \neq e$ and $y \in A'$ and $(\beta(x), \alpha(y)) = (2,1) \in r_{\mathbf{A}}$ for $x = e$ and $y \in A'$. It follows that $(\beta, \alpha) \in r_{\mathbf{A}^{\mathbf{A'}}}$. Hence $r_{\mathbf{A}^{\mathbf{A'}}} = \{(\alpha, \alpha), (\beta, \alpha)\}$. It means that the relational structure $\mathbf{A}^{\mathbf{A'}}$ is an algebra isomorphic to $\mathbf{A}$.

Comparing Examples 3 and 4 we see that $\mathbf{A}^{\mathbf{B}} \cong \mathbf{A}^{\mathbf{C}}$ need not imply $\mathbf{B} \cong \mathbf{C}$.

Example 5. Let (A, o) be the unary algebra with $A = \{1, 2\}$, $o(1) = 2$, $o(2) = 1$ and let (A', o') be the same as in Example 4. Then there exist precisely two homomorphisms α and β of the algebra (A', o') into (A, o), which are defined as follows:

x	a	b	c	d	e
$\alpha(x)$	1	2	1	2	2
$\beta(x)$	2	1	2	1	1

Again, let $\mathbf{A} = (A, r_{\mathbf{A}})$ and $\mathbf{A'} = (A', r_{\mathbf{A'}})$ be the corresponding binary relational structures. Clearly, $r_{\mathbf{A}} = \{(1,2), (2,1)\}$.

Since $(a, b) \in r_{\mathbf{A'}}$, $(\alpha(a), \beta(b)) = (1,1) \notin r_{\mathbf{A}}$, $(\beta(a), \alpha(b)) = (2,2) \notin r_{\mathbf{A}}$, we obtain $(\alpha, \beta) \notin r_{\mathbf{A}^{\mathbf{A'}}}$, $(\beta, \alpha) \notin r_{\mathbf{A}^{\mathbf{A'}}}$. Moreover, $(\alpha(a), \alpha(a)) =$

$(1,1) \notin r_{\mathbf{A}}$, $(\beta(a), \beta(a)) = (2,2) \notin r_{\mathbf{A}}$; it follows that $(\alpha, \alpha) \notin r_{\mathbf{A}^{\mathbf{A}'}}$, $(\beta, \beta) \notin r_{\mathbf{A}^{\mathbf{A}'}}$. Consequently, $r_{\mathbf{A}^{\mathbf{A}'}} = \emptyset$.

Example 6. Let $G = \{0,1\}$ and let $+$ denote the mod 2 addition on G. The groupoid $(G,+)$ defines the ternary relational structure $\mathbf{G} = (G, r_{\mathbf{G}})$ where $r_{\mathbf{G}} = \{(0,0,0), (0,1,1), (1,0,1), (1,1,0)\}$. It is easy to see that $(G,+)$ has only two endomorphisms α, β where $\beta = \mathrm{id}_G$ and $\alpha(x) = 0$ for any $x \in G$.

By definition, $r_{\mathbf{G}^{\mathbf{G}}}$ is a ternary relation on the set $\{\alpha, \beta\}$. Since $(\alpha(x), \alpha(y), \alpha(z)) = (0,0,0)$ for any $(x,y,z) \in G^3$, we obtain $(\alpha,\alpha,\alpha) \in r_{\mathbf{G}^{\mathbf{G}}}$.

Clearly, $(1,0,1) \in r_{\mathbf{G}}$ and $(\alpha(1), \alpha(0), \beta(1)) = (0,0,1) \notin r_{\mathbf{G}}$, $(\alpha(1), \beta(0), \beta(1)) = (0,0,1) \notin r_{\mathbf{G}}$ and, therefore, $(\alpha,\alpha,\beta) \notin r_{\mathbf{G}^{\mathbf{G}}}$, $(\alpha,\beta,\beta) \notin r_{\mathbf{G}^{\mathbf{G}}}$. Similarly, $(1,1,0) \in r_{\mathbf{G}}$ and $(\alpha(1), \beta(1), \alpha(0)) = (0,1,0) \notin r_{\mathbf{G}}$ which implies $(\alpha,\beta,\alpha) \notin r_{\mathbf{G}^{\mathbf{G}}}$.

Furthermore, $(\beta(1), \alpha(1), \alpha(1)) = (1,0,0) \notin r_{\mathbf{G}}$ and, hence, $(\beta,\alpha,\alpha) \notin r_{\mathbf{G}^{\mathbf{G}}}$. Clearly, $(0,1,1) \in r_{\mathbf{G}}$ and $(\beta(0), \alpha(1), \beta(1)) = (0,0,1) \notin r_{\mathbf{G}}$, $(\beta(0), \beta(1), \alpha(1)) = (0,1,0) \notin r_{\mathbf{G}}$ and, thus, $(\beta,\alpha,\beta) \notin r_{\mathbf{G}^{\mathbf{G}}}$, $(\beta,\beta,\alpha) \notin r_{\mathbf{G}^{\mathbf{G}}}$. Finally, $(\beta(1), \beta(1), \beta(1)) = (1,1,1) \notin r_{\mathbf{G}}$ which implies $(\beta,\beta,\beta) \notin r_{\mathbf{G}^{\mathbf{G}}}$.

Thus, $r_{\mathbf{G}^{\mathbf{G}}} = \{(\alpha,\alpha,\alpha)\}$. This relation can be regarded as a partial binary operation $+$ on $\{\alpha, \beta\}$ such that $\alpha + \alpha = \alpha$ while $\alpha + \beta$, $\beta + \alpha$, $\beta + \beta$ are not defined.

Example 7. Let A denote the set of nonnegative integers and let $\mathbf{A} = (A, r_{\mathbf{A}})$ be the ternary relational structure with $r_{\mathbf{B}} = \{(x,y,z) \in A^3;\ x < y < z\}$.

If h is an endomorphism of the relational structure $\mathbf{A} = (A, r_{\mathbf{A}})$, then for any $x \in A$, $y \in A$ with $x < y$ we obtain $(x, y, y+1) \in r_{\mathbf{A}}$ which implies $(h(x), h(y), h(y+1)) \in r_{\mathbf{A}}$ and, therefore, $h(x) < h(y)$. Thus, any endomorphism of the relational structure $\mathbf{A}$ is an increasing function. Conversely, any increasing selfmap of $(A, <)$ is an endomorphism of $\mathbf{A}$.

Let h_1, h_2 and h_3 be endomorphisms of the relational structure $\mathbf{A}$. Then $(h_1, h_2, h_3) \in r_{\mathbf{A}^{\mathbf{A}}}$ implies $(h_1(x), h_2(x), h_3(x)) \in r_{\mathbf{A}}$, i.e., $h_1(x) < h_2(x) < h_3(x)$ for any $x \in A$. If the last condition is satisfied and $(x,y,z) \in r_{\mathbf{A}}$, then $x < y < z$. It follows that $h_1(x) < h_2(x) < h_2(y)$ because h_2 is increasing. Furthermore, $h_1(x) < h_2(y) < h_3(y)$ and $h_3(y) <$

$h_3(z)$ as h_3 is increasing. Thus, $h_1(x) < h_2(y) < h_3(z)$ and, therefore, $(h_1(x), h_2(y), h_3(z)) \in r_{\mathbf{A}}$, i.e., $(h_1, h_2, h_3) \in r_{\mathbf{A}^{\mathbf{A}}}$.

Hence, $(h_1, h_2, h_3) \in r_{\mathbf{A}^{\mathbf{A}}}$ if and only if $h_1(x) < h_2(x) < h_3(x)$ holds for any $x \in A$. In other words, we have $\mathbf{A}^{\mathbf{A}} = \mathbf{A}^{\sim}\mathbf{A}$. But this equality follows also from Remark 5 because $r_{\mathbf{A}}$ is clearly diagonal.

Example 8. Let A, A' be sets and let $\mathbf{A}$, $\mathbf{A'}$ be the ternary relational structures given by $\mathbf{A} = \mathbf{E}_A$, $\mathbf{A'} = \mathbf{E}_{A'}$. Let $h \in A^{A'}$. Then $(x', y', z') \in e_{A'}$ implies $x' = y' = z'$ and, therefore, $h(x') = h(y') = h(z')$ which means $(h(x'), h(y'), h(z')) \in e_A$. Thus $\mathrm{Hom}(\mathbf{A'}, \mathbf{A}) = A^{A'}$.

Suppose $(h_1, h_2, h_3) \in r_{\mathbf{A}^{\mathbf{A'}}}$. Then $(h_1(x'), h_2(x'), h_3(x')) \in e_A$ for any $x' \in A'$ which means $h_1(x') = h_2(x') = h_3(x')$ for any $x' \in A'$, i.e., $h_1 = h_2 = h_3$. Thus $\mathbf{A}^{\mathbf{A'}} \subseteq \mathbf{E}_{A^{A'}}$. Conversely, let $(h_1, h_2, h_3) \in e_{A^{A'}}$. Then $h_1 = h_2 = h_3$ and consequently $(h_1(x'), h_2(x'), h_3(x')) \in e_A$ whenever $x' \in A'$. Furthermore, we have $(h_1(x'_1), h_2(x'_2), h_3(x'_3)) \in e_A$ for any $(x'_1, x'_2, x'_3) \in e_{A'}$ because the last condition means $x'_1 = x'_2 = x'_3$ and, therefore, $h_1(x'_1) = h_2(x'_2) = h_3(x'_3)$. Thus, $\mathbf{E}_{A^{A'}} \subseteq \mathbf{A}^{\mathbf{A'}}$ and hence $\mathbf{A}^{\mathbf{A'}} = \mathbf{E}_{A^{A'}}$. (Note that $\mathbf{E}_A$, $\mathbf{E}_{A'}$ are both reflexive and diagonal, so we have $\mathbf{A}^{\mathbf{A'}} = \mathbf{A}^{\wedge}\mathbf{A'} = \mathbf{A}^{\sim}\mathbf{A'}$.)

3. Exponential laws

Lemma 1. *Let $\mathbf{A} = (A, r_{\mathbf{A}})$, $\mathbf{B} = (B, r_{\mathbf{B}})$, $\mathbf{C} = (C, r_{\mathbf{C}})$ be n-ary relational structures such that $B \cap C = \emptyset$. Then the third exponential law holds for both the direct power and the structural power:*

$$(1) \quad \mathbf{A}^{\wedge}(\mathbf{B} + \mathbf{C}) \cong (\mathbf{A}^{\wedge}\mathbf{B}) \times (\mathbf{A}^{\wedge}\mathbf{C}),$$

and

$$(2) \quad \mathbf{A}^{\sim}(\mathbf{B} + \mathbf{C}) \cong (\mathbf{A}^{\sim}\mathbf{B}) \times (\mathbf{A}^{\sim}\mathbf{C}).$$

Proof. As (1) is proved in [Sl1], Theorem 8, we only prove (2). By definition, $\mathbf{A}^{\sim}(\mathbf{B} + \mathbf{C}) = (\mathrm{Hom}(\mathbf{B} + \mathbf{C}, \mathbf{A}), r_{\mathbf{A}^{\sim}(\mathbf{B}+\mathbf{C})})$. Thus, if $h \in \mathrm{Hom}(\mathbf{B} + \mathbf{C}, \mathbf{A})$, then h is a mapping of the set $B \cup C$ into A. Put $h^1 = h\lceil B$, $h^2 = h\lceil C$, $\varphi(h) = (h^1, h^2)$. It is easy to see that h^1 is a homomorphism of $\mathbf{B}$ into $\mathbf{A}$ and h^2 is a homomorphism of $\mathbf{C}$ into $\mathbf{A}$. It follows that $\varphi(h) \in \mathrm{Hom}(\mathbf{B},\mathbf{A}) \times \mathrm{Hom}(\mathbf{C},\mathbf{A})$. Clearly, φ is a bijection of the set $\mathrm{Hom}(\mathbf{B}+\mathbf{C},\mathbf{A})$ onto $\mathrm{Hom}(\mathbf{B},\mathbf{A}) \times \mathrm{Hom}(\mathbf{C},\mathbf{A})$.

Suppose that $h_1, \dots h_n$ are in $\mathrm{Hom}(\mathbf{B}+\mathbf{C},\mathbf{A})$. Then $(h_1,\dots,h_n) \in r_{\mathbf{A}^\sim(\mathbf{B}+\mathbf{C})}$ if and only if $(h_1(x_1),\dots,h_n(x_n)) \in r_{\mathbf{A}}$ for any $(x_1,\dots,x_n) \in r_{\mathbf{B}} \cup r_{\mathbf{C}}$. This is equivalent to the simultaneous validity of the conditions $(h_1^1(b_1),\dots,h_n^1(b_n)) \in r_{\mathbf{A}}$ for any $(b_1,\dots,b_n) \in r_{\mathbf{B}}$ and $(h_1^2(c_1),\dots,h_n^2(c_n)) \in r_{\mathbf{A}}$ for any $(c_1,\dots,c_n) \in r_{\mathbf{C}}$. These conditions hold if and only if the conditions $(h_1^1,\dots,h_n^1) \in r_{\mathbf{A}^\sim\mathbf{B}}$, $(h_1^2,\dots,h_n^2) \in r_{\mathbf{A}^\sim\mathbf{C}}$ are satisfied. Hence $(\varphi(h_1),\dots,\varphi(h_n)) = ((h_1^1,h_1^2),\dots,(h_n^1,h_n^2)) \in r_{(\mathbf{A}^\sim\mathbf{B})\times(\mathbf{A}^\sim\mathbf{C})}$. Thus, φ is an isomorphism of the relational structure $\mathbf{A}^\sim(\mathbf{B}+\mathbf{C}) = (\mathrm{Hom}(\mathbf{B}+\mathbf{C},\mathbf{A}), r_{\mathbf{A}^\sim(\mathbf{B}+\mathbf{C})})$ onto the relational structure $(\mathrm{Hom}(\mathbf{B},\mathbf{A}) \times \mathrm{Hom}(\mathbf{C},\mathbf{A}), r_{(\mathbf{A}^\sim\mathbf{B})\times(\mathbf{A}^\sim\mathbf{C})}) = (\mathbf{A}^\sim\mathbf{B}) \times (\mathbf{A}^\sim\mathbf{C})$. This gives the condition (2). $\square$

We get the third exponential law also for the combined power:

Theorem 1. *Let $\mathbf{A} = (A, r_{\mathbf{A}})$, $\mathbf{B} = (B, r_{\mathbf{B}})$, $\mathbf{C} = (C, r_{\mathbf{C}})$ be n-ary relational structures such that $B \cap C = \emptyset$. Then*

$$\mathbf{A}^{\mathbf{B}+\mathbf{C}} \cong \mathbf{A}^{\mathbf{B}} \times \mathbf{A}^{\mathbf{C}}.$$

Proof. The statement follows from Lemma 1 because $\mathbf{A}^{\mathbf{B}+\mathbf{C}} = \mathbf{A}^\wedge(\mathbf{B}+\mathbf{C}) \cap \mathbf{A}^\sim(\mathbf{B}+\mathbf{C}) \cong (\mathbf{A}^\wedge\mathbf{B} \times \mathbf{A}^\wedge\mathbf{C}) \cap (\mathbf{A}^\sim\mathbf{B} \times \mathbf{A}^\sim\mathbf{C}) = (\mathbf{A}^\wedge\mathbf{B} \cap \mathbf{A}^\sim\mathbf{B}) \times (\mathbf{A}^\wedge\mathbf{C} \cap \mathbf{A}^\sim\mathbf{C}) = \mathbf{A}^{\mathbf{B}} \times \mathbf{A}^{\mathbf{C}}$. $\square$

Lemma 2. *Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be n-ary relational structures. Then the second exponential law holds for both the direct power and the structural power:*

$$(1) \quad (\mathbf{A} \times \mathbf{B})^\wedge\mathbf{C} \cong (\mathbf{A}^\wedge\mathbf{C}) \times (\mathbf{B}^\wedge\mathbf{C}),$$

and

$$(2) \quad (\mathbf{A} \times \mathbf{B})^\sim\mathbf{C} \cong (\mathbf{A}^\sim\mathbf{C}) \times (\mathbf{B}^\sim\mathbf{C}).$$

Proof. As (1) is proved in [Sl1], Theorem 7, we only prove (2). Let A, B and C be the underlying sets of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, respectively, and put $s = r_{\mathbf{A}\times\mathbf{B}}$, $t = r_{\mathbf{A}^\sim\mathbf{C}}$, $u = r_{\mathbf{B}^\sim\mathbf{C}}$, $v = r_{(\mathbf{A}\times\mathbf{B})^\sim\mathbf{C}}$, and $w = r_{(\mathbf{A}^\sim\mathbf{C})\times(\mathbf{B}^\sim\mathbf{C})}$. For an arbitrary $h \in \mathrm{Hom}(\mathbf{C}, \mathbf{A}\times\mathbf{B})$ set $\varphi(h) = (\mathrm{pr}_A h, \mathrm{pr}_B h)$. It is easy to see that φ is a bijection of the set $\mathrm{Hom}(\mathbf{C}, \mathbf{A} \times \mathbf{B})$ onto $\mathrm{Hom}(\mathbf{C}, \mathbf{A}) \times \mathrm{Hom}(\mathbf{C}, \mathbf{B})$.

Let $h_1, \dots h_n \in \mathrm{Hom}(\mathbf{C}, \mathbf{A} \times \mathbf{B})$. Clearly, the following conditions satisfy (a) $\iff$ (b) $\iff$ (c) $\iff$ (d) $\iff$ (e) $\iff$ (f).

(a) $(h_1, \dots, h_n) \in v$.

(b) $(h_1(c_1), \dots, h_n(c_n)) \in s$ whenever $(c_1, \dots, c_n) \in r_{\mathbf{C}}$.

(c) $(\mathrm{pr}_A\, h_1(c_1), \dots, \mathrm{pr}_A\, h_n(c_n)) \in r_{\mathbf{A}}$ and $(\mathrm{pr}_B\, h_1(c_1), \dots, \mathrm{pr}_B\, h_n(c_n)) \in r_{\mathbf{B}}$ whenever $(c_1, \dots, c_n) \in r_{\mathbf{C}}$.

(d) $(\mathrm{pr}_{\mathrm{Hom}(\mathbf{C},\mathbf{A})}\, \varphi(h_1)(c_1), \dots, \mathrm{pr}_{\mathrm{Hom}(\mathbf{C},\mathbf{A})}\, \varphi(h_n(c_n)) \in r_{\mathbf{A}}$ whenever $(c_1, \dots, c_n) \in r_{\mathbf{C}}$ and $(\mathrm{pr}_{\mathrm{Hom}(\mathbf{C},\mathbf{B})}\, \varphi(h_1(c_1), \dots, \mathrm{pr}_{\mathrm{Hom}(\mathbf{C},\mathbf{B})}\, \varphi(h_n(c_n)) \in r_{\mathbf{B}}$ whenever $(c_1, \dots, c_n) \in r_{\mathbf{C}}$.

(e) $(\mathrm{pr}_{\mathrm{Hom}(\mathbf{C},\mathbf{A})}\, \varphi(h_1), \dots, \mathrm{pr}_{\mathrm{Hom}(\mathbf{C},\mathbf{A})}\, \varphi(h_n)) \in t$ and $(\mathrm{pr}_{\mathrm{Hom}(\mathbf{C},\mathbf{B})}\, \varphi(h_1), \dots, \mathrm{pr}_{\mathrm{Hom}(\mathbf{C},\mathbf{B})} \varphi(h_n)) \in u$.

(f) $(\varphi(h_1), \dots, \varphi(h_n)) \in w$.

Hence (a) is equivalent to (f) which means that φ is an isomorphism of $(\mathbf{A} \times \mathbf{B})^{\sim}\mathbf{C}$ onto $(\mathbf{A}^{\sim}\mathbf{C}) \times (\mathbf{B}^{\sim}\mathbf{C})$. This gives the condition (2). □

We get the second exponential law also for the combined power:

Theorem 2. *Let* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ *be* n*-ary relational structures. Then*

$$(\mathbf{A} \times \mathbf{B})^{\mathbf{C}} \cong \mathbf{A}^{\mathbf{C}} \times \mathbf{B}^{\mathbf{C}}.$$

Proof. The statement follows from Lemma 2 because $(\mathbf{A} \times \mathbf{B})^{\mathbf{C}} = (\mathbf{A} \times \mathbf{B})^{\wedge}\mathbf{C} \cap (\mathbf{A} \times \mathbf{B})^{\sim}\mathbf{C} \cong (\mathbf{A}^{\wedge}\mathbf{C} \times \mathbf{B}^{\wedge}\mathbf{C}) \cap (\mathbf{A}^{\sim}\mathbf{C} \times \mathbf{B}^{\sim}\mathbf{C}) = (\mathbf{A}^{\wedge}\mathbf{C} \cap \mathbf{A}^{\sim}\mathbf{C}) \times (\mathbf{B}^{\wedge}\mathbf{C} \cap \mathbf{B}^{\sim}\mathbf{C}) = \mathbf{A}^{\mathbf{C}} \times \mathbf{B}^{\mathbf{C}}$. □

The first exponential law for the direct power

$$(\mathbf{A}^{\wedge}\mathbf{B})^{\wedge}\mathbf{C} \cong \mathbf{A}^{\wedge}(\mathbf{B} \times \mathbf{C})$$

need not be satisfied and the same is true for both the structural power and the combined power – see the following example:

Example 9. Let $\mathbf{A} = (A, r_{\mathbf{A}})$ and $\mathbf{B} = (B, r_{\mathbf{B}})$ be the binary relational structures with $A = \{a, b\}$, $r_{\mathbf{A}} = \{(a, b)\}$, $B = \{c\}$, $r_{\mathbf{B}} = \{(c, c)\}$. One can easily see that $\mathrm{Hom}(\mathbf{B}, \mathbf{A}) = \emptyset$, hence $\mathrm{Hom}(\mathbf{A}, \mathbf{A}^\wedge\mathbf{B}) = \emptyset$. On the other hand, we clearly have $\mathrm{Hom}(\mathbf{B} \times \mathbf{A}, \mathbf{A}) = \{\mathrm{pr}_A\}$. Consequently, $(\mathbf{A}^\wedge\mathbf{B})^\wedge\mathbf{A}$ is not isomorphic to $\mathbf{A}^\wedge(\mathbf{B} \times \mathbf{A})$, $(\mathbf{A}^\sim\mathbf{B})^\sim\mathbf{A}$ is not isomorphic to $\mathbf{A}^\sim(\mathbf{B} \times \mathbf{A})$, and $(\mathbf{A}^\mathbf{B})^\mathbf{A}$ is not isomorphic to $\mathbf{A}^{\mathbf{B}\times\mathbf{A}}$.

We shall show that the first exponential law is valid for $\mathbf{A}^\mathbf{B}$ when replacing $\times$ with $\circ$, i.e., that there holds

$$(\mathbf{A}^\mathbf{B})^\mathbf{C} \cong \mathbf{A}^{\mathbf{B}\circ\mathbf{C}}.$$

Lemma 3. *Let $\mathbf{A} = (A, r_{\mathbf{A}})$, $\mathbf{B} = (B, r_{\mathbf{B}})$, $\mathbf{C} = (C, r_{\mathbf{C}})$ be n-ary relational structures. Then the canonical bijection $\varphi : (A^B)^C \to A^{B\times C}$ restricted to* $\mathrm{Hom}(\mathbf{C}, \mathbf{A}^\mathbf{B})$ *is a bijection of* $\mathrm{Hom}(\mathbf{C}, \mathbf{A}^\mathbf{B})$ *onto* $\mathrm{Hom}(\mathbf{B} \circ \mathbf{C}, \mathbf{A})$.

Proof. Let $h \in (A^B)^C$. It is easy to see that the following conditions satisfy (a) $\iff$ (b) $\iff$ (c) $\iff$ (d) $\iff$ (e) $\iff$ (f).

(a) $h \in \mathrm{Hom}(\mathbf{C}, \mathbf{A}^\mathbf{B})$.

(b) $h(c) \in \mathrm{Hom}(\mathbf{B}, \mathbf{A})$ for any $c \in C$;
$(h(c_1), \dots, h(c_n)) \in r_{\mathbf{A}^\mathbf{B}}$ for any $(c_1, \dots, c_n) \in r_\mathbf{C}$.

(c) $(h(c_1)(b_1), \dots, h(c_n)(b_n)) \in r_\mathbf{A}$ whenever $(c_1, \dots, c_n) \in r_{\overline{\mathbf{C}}}$ and $(b_1, \dots, b_n) \in r_\mathbf{B}$ or $(c_1, \dots, c_n) \in r_\mathbf{C}$ and $(b_1, \dots, b_n) \in r_{\overline{\mathbf{B}}}$.

(d) $(\varphi(h)(b_1, c_1), \dots, \varphi(h)(b_n, c_n)) \in r_\mathbf{A}$ whenever $(c_1, \dots, c_n) \in r_{\overline{\mathbf{C}}}$ and $(b_1, \dots, b_n) \in r_\mathbf{B}$ $(b_1, \dots, b_n) \in r_\mathbf{B}$ or $(c_1, \dots, c_n) \in r_\mathbf{C}$ and $(b_1, \dots, b_n) \in r_{\overline{\mathbf{B}}}$.

(e) $(\varphi(h)(b_1, c_1), \dots, \varphi(h)(b_n, c_n)) \in r_\mathbf{A}$ for any
$((b_1, c_1), \dots, (b_n, c_n)) \in r_{\mathbf{B}\circ\mathbf{C}}$.

(f) $\varphi(h) \in \mathrm{Hom}(\mathbf{B} \circ \mathbf{C}, \mathbf{A})$.

Hence (a) is equivalent to (f). This proves the statement. $\square$

We get the following form of the first exponential law for the combined power:

Theorem 3. *Let* $\mathbf{A} = (A, r_{\mathbf{A}})$, $\mathbf{B} = (B, r_{\mathbf{B}})$, $\mathbf{C} = (C, r_{\mathbf{C}})$ *be* n*-ary relational structures. Then* $(\mathbf{A}^{\mathbf{B}})^{\mathbf{C}} \cong \mathbf{A}^{\mathbf{B}\circ\mathbf{C}}$.

Proof. Denote by φ the canonical bijection $(A^B)^C \to A^{B\times C}$ restricted to $\mathrm{Hom}(\mathbf{C}, \mathbf{A}^{\mathbf{B}})$ and let $h_1, \dots, h_n \in (A^B)^C$. From the definitions of the relations $r_{\mathbf{A}^{\mathbf{B}}}$, $r_{\mathbf{B}\circ\mathbf{C}}$ and from Lemma 3 it follows that the following conditions satisfy (a) $\iff$ (b) $\iff$ (c) $\iff$ (d) $\iff$ (e) $\iff$ (f).

(a) $h_1, \dots, h_n \in \mathrm{Hom}(\mathbf{C}, \mathbf{A}^{\mathbf{B}})$;
$(h_1, \dots, h_n) \in r_{(\mathbf{A}^{\mathbf{B}})^{\mathbf{C}}}$.

(b) $h_1, \dots, h_n \in \mathrm{Hom}(\mathbf{C}, \mathbf{A}^{\mathbf{B}})$;
$(h_1(c_1), \dots, h_n(c_n)) \in r_{\mathbf{A}^{\mathbf{B}}}$ for any $(c_1, \dots, c_n) \in r_{\overline{\mathbf{C}}}$.

(c) $h_1, \dots, h_n \in \mathrm{Hom}(\mathbf{C}, \mathbf{A}^{\mathbf{B}})$;
$(h_1(c_1)(b_1), \dots, h_n(c_n)(b_n)) \in r_{\mathbf{A}}$ for any $(b_1, \dots, b_n) \in r_{\overline{\mathbf{B}}}$ and any $(c_1, \dots, c_n) \in r_{\overline{\mathbf{C}}}$.

(d) $\varphi(h_1), \dots, \varphi(h_n) \in \mathrm{Hom}(\mathbf{B}\circ\mathbf{C}, \mathbf{A})$;
$(\varphi(h_1)(b_1, c_1), \dots, \varphi(h_n)(b_n, c_n)) \in r_{\mathbf{A}}$ for any $(b_1, \dots, b_n) \in r_{\overline{\mathbf{B}}}$ and any $(c_1, \dots, c_n) \in r_{\overline{\mathbf{C}}}$.

(e) $\varphi(h_1), \dots, \varphi(h_n) \in \mathrm{Hom}(\mathbf{B}\circ\mathbf{C}, \mathbf{A})$;
$(\varphi(h_1)(b_1, c_1), \dots, \varphi(h_n)(b_n, c_n)) \in r_{\mathbf{A}}$ for any $((b_1, c_1), \dots, (b_n, c_n)) \in r_{\overline{\mathbf{B}\circ\mathbf{C}}}$.

(f) $\varphi(h_1), \dots, \varphi(h_n)$ are in $\mathrm{Hom}(\mathbf{B}\circ\mathbf{C}, \mathbf{A})$;
$(\varphi(h_1), \dots, \varphi(h_n)) \in r_{\mathbf{A}^{\mathbf{B}\circ\mathbf{C}}}$.

Hence (a) is equivalent to (f) which means that φ is an isomorphism of $(\mathbf{A}^{\mathbf{B}})^{\mathbf{C}}$ onto $\mathbf{A}^{\mathbf{B}\circ\mathbf{C}}$. The statement is proved. □

Corollary 1. *Let* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}$ *be* n*-ary relational structures. If* $\mathbf{B}$ *and* $\mathbf{C}$ *are reflexive, then* $(\mathbf{A}{}^\sim\mathbf{B}){}^\sim\mathbf{C} \cong \mathbf{A}{}^\sim(\mathbf{B}\times\mathbf{C})$.

Proof. Clearly, if $\mathbf{B}$, $\mathbf{C}$ are reflexive, $\mathbf{B}\times\mathbf{C}$ is reflexive too. Thus, by Remark 5, we have $(\mathbf{A}{}^\sim\mathbf{B}){}^\sim\mathbf{C} = (\mathbf{A}^{\mathbf{B}})^{\mathbf{C}}$ and $\mathbf{A}{}^\sim(\mathbf{B}\times\mathbf{C}) = \mathbf{A}^{\mathbf{B}\times\mathbf{C}}$. Since $\mathbf{B}\times\mathbf{C} = \mathbf{B}\circ\mathbf{C}$, the assertion follows from Theorem 3. □

Corollary 2. *Let* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}$ *be n-ary relational structures. If* $\mathbf{A}$ *is diagonal and* $\mathbf{B}$ *and* $\mathbf{C}$ *are reflexive, then* $(\mathbf{A}\hat{}\,\mathbf{B})\hat{}\,\mathbf{C} \cong \mathbf{A}\hat{}\,(\mathbf{B}\times\mathbf{C})$.

Proof. Clearly, if $\mathbf{A}$ is diagonal, $\mathbf{A}\hat{}\,\mathbf{B}$ is diagonal too. Thus, by Remark 5, we have $(\mathbf{A}\hat{}\,\mathbf{B})\hat{}\,\mathbf{C} = (\mathbf{A}^{\mathbf{B}})^{\mathbf{C}}$ and $\mathbf{A}\hat{}\,(\mathbf{B}\times\mathbf{C}) = \mathbf{A}^{\mathbf{B}\times\mathbf{C}}$. Since $\mathbf{B}\times\mathbf{C} = \mathbf{B}\circ\mathbf{C}$, the assertion follows from Theorem 3. $\square$

Remark 6. The statement of Corollary 2 is well known – it is proved, e.g., in [No].

Example 10. Let $\mathbf{A}$ be the relational structure from Example 2. We construct $\mathbf{A}^{\mathbf{A}\circ\mathbf{A}} = (\mathrm{Hom}(\mathbf{A}\circ\mathbf{A},\mathbf{A}), r_{\mathbf{A}^{\mathbf{A}\circ\mathbf{A}}})$. The relational structure $\mathbf{A}\circ\mathbf{A}$ has been constructed in Example 2.

Suppose $h \in \mathrm{Hom}(\mathbf{A}\circ\mathbf{A},\mathbf{A})$. For any $(x,y) \in A\times A$, $(x,y) \neq (2,2)$, we have $((2,2),(x,y)) \in r_{\mathbf{A}\circ\mathbf{A}}$ which implies $(h(2,2),h(x,y)) \in r_{\mathbf{A}}$, i.e., either $(h(2,2),h(x,y)) = (1,1)$ or $(h(2,2),h(x,y)) = (2,1)$. Thus, $h(x,y) = 1$ and either $h(2,2) = 1$ or $h(2,2) = 2$. Hence $\mathrm{Hom}(\mathbf{A}\circ\mathbf{A},\mathbf{A})$ has only two elements α, β where $\alpha(x,y) = 1$ for any

$(x,y) \in A\times A$ and $\beta(2,2) = 2$, $\beta(x,y) = 1$ for any $(x,y) \in A\times A$, $(x,y) \neq (2,2)$.

Since $(\beta(2,2),\beta(2,2)) = (2,2) \notin r_{\mathbf{A}}$, we have $(\beta,\beta) \notin r_{\mathbf{A}^{\mathbf{A}\circ\mathbf{A}}}$. Similarly, $(\alpha(2,2),\beta(2,2)) = (1,2) \notin r_{\mathbf{A}}$ implies $(\alpha,\beta) \notin r_{\mathbf{A}^{\mathbf{A}\circ\mathbf{A}}}$. Furthermore, $(\beta(x,y),\alpha(u,v)) = (\beta(x,y),1) \in r_{\mathbf{A}}$ holds for any $(x,y) \in A\times A$, $(u,v) \in A\times A$ which entails $(\beta,\alpha) \in r_{\mathbf{A}^{\mathbf{A}\circ\mathbf{A}}}$. Finally, $(\alpha(x,y),\alpha(u,v)) = (1,1) \in r_{\mathbf{A}}$ for any $x,y,u,v \in A$ and, therefore, $(\alpha,\alpha) \in r_{\mathbf{A}^{\mathbf{A}\circ\mathbf{A}}}$.

Hence, $r_{\mathbf{A}^{\mathbf{A}\circ\mathbf{A}}} = \{(\alpha,\alpha),(\beta,\alpha)\}$. It follows that $\mathbf{A}^{\mathbf{A}\circ\mathbf{A}}$ is isomorphic to $\mathbf{A}$.

In Example 3, we have proved that $\mathbf{A}^{\mathbf{A}} \cong \mathbf{A}$. It follows that $(\mathbf{A}^{\mathbf{A}})^{\mathbf{A}} \cong \mathbf{A}^{\mathbf{A}} \cong \mathbf{A}$. Since $\mathbf{A}^{\mathbf{A}\circ\mathbf{A}} \cong \mathbf{A}$ as we have seen now, we obtain $(\mathbf{A}^{\mathbf{A}})^{\mathbf{A}} \cong \mathbf{A}^{\mathbf{A}\circ\mathbf{A}}$ which illustrates Theorem 3.

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